
Digital
Communications
John G. Proakis

10

COMMUNICATION THROUGH BAND-LIMITED LINEAR FILTER CHANNELS

In Chapter 9, we focused on the design of the modulator and demodulator filters for band-limited channels. The design procedure was based on the assumption that the (ideal or non-ideal) channel response characteristic $C(f)$ was known a priori. However, in practical digital communications systems that are designed to transmit at high speed through band-limited channels, the frequency response $C(f)$ of the channel is not known with sufficient precision to design optimum filters for the modulator and demodulator. For example, in digital communication over the dial-up telephone network, the communication channel will be different every time we dial a number, because the channel route will be different. This is an example of a channel whose characteristics are unknown a priori. There are other types of channels, e.g., wireless channels such as radio channels and underwater acoustic channels, whose frequency response characteristics are time-variant. For such channels, it is not possible to design optimum fixed demodulation filters.

In this chapter, we consider the problem of receiver design in the presence of channel distortion, which is not known a priori, and AWGN. The channel distortion results in intersymbol interference, which, if left uncompensated, causes high error rates. The solution to the ISI problem is to design a receiver that employs a means for compensating or reducing the ISI in the received signal. The compensator for the ISI is called an *equalizer*.

Three types of equalization methods are treated in this chapter. One is based on the maximum-likelihood (ML) sequence detection criterion, which is optimum from a probability of error viewpoint. A second equalization method is based on the use of a linear filter with adjustable coefficients. The third equalization method that is described exploits the use of previous detected symbols to suppress the ISI in the present symbol being detected, and it is called *decision-feedback equalization*. We begin with the derivation of the optimum detector for channels with ISI.

10-1 OPTIMUM RECEIVER FOR CHANNELS WITH ISI AND AWGN

In this section, we derive the structure of the optimum demodulator and detector for digital transmission through a nonideal, band-limited channel with additive gaussian noise. We begin with the transmitted (equivalent lowpass) signal given by (9-2-1). The received (equivalent lowpass) signal is expressed as

$$r_l(t) = \sum_n I_n h(t - nT) + z(t) \tag{10-1-1}$$

where $h(t)$ represents the response of the channel to the input signal pulse $g(t)$ and $z(t)$ represents the additive white gaussian noise.

First we demonstrate that the optimum demodulator can be realized as a filter matched to $h(t)$, followed by a sampler operating at the symbol rate $1/T$ and a subsequent processing algorithm for estimating the information sequence $\{I_n\}$ from the sample values. Consequently, the samples at the output of the matched filter are sufficient for the estimation of the sequence $\{I_n\}$.

10-1-1 Optimum Maximum-Likelihood Receiver

Let us expand the received signal $r_l(t)$ in the series

$$r_l(t) = \lim_{N\to\infty} \sum_{k=1}^{N} r_k f_k(t) \tag{10-1-2}$$

where $\{f_k(t)\}$ is a complete set of orthonormal functions and $\{r_k\}$ are the observable random variables obtained by projecting $r_l(t)$ onto the set $\{f_k(t)\}$. It is easily shown that

$$r_k = \sum_n I_n h_{kn} + z_k, \qquad k = 1, 2, \ldots \tag{10-1-3}$$

where h_{kn} is the value obtained from projecting $h(t - nT)$ onto $f_k(t)$, and z_k is the value obtained from projecting $z(t)$ onto $f_k(t)$. The sequence $\{z_k\}$ is gaussian with zero mean and covariance

$$\tfrac{1}{2}E(z_k^* z_m) = N_0 \delta_{km} \tag{10-1-4}$$

The joint probability density function of the random variables

$\mathbf{r}_N \equiv [r_1 \ r_2 \ \dots \ r_N]$ conditioned on the transmitted sequence $\mathbf{I}_p \equiv [I_1 \ I_2 \ \dots \ I_p]$, where $p \leq N$, is

$$p(\mathbf{r}_N \, 3\mathbf{I}_p) = \left(\frac{1}{2\pi N_0}\right)^N \exp\left(-\frac{1}{2N_0}\sum_{k=1}^{N}\left| r_k - \sum_n I_n h_{kn}\right|^2\right) \tag{10-1-5}$$

In the limit as the number N of observable random variables approaches infinity, the logarithm of $p(\mathbf{r}_N \, 3\mathbf{I}_p)$ is proportional to the metrics $PM(\mathbf{I}_p)$, defined as

$$\begin{aligned} PM(\mathbf{I}_p) &= -\int_{-\infty}^{\infty}\left| r_l(t) - \sum_n I_n h(t-nT)\right|^2 dt \\ &= -\int_{-\infty}^{\infty} |r_l(t)|^2\,dt + 2\,\mathrm{Re}\sum_n\left[I_n^*\int_{-\infty}^{\infty} r_l(t)h^*(t-nT)\,dt\right] \\ &\quad -\sum_n\sum_m I_n^* I_m \int_{-\infty}^{\infty} h^*(t-nT)h(t-mT)\,dt \end{aligned} \tag{10-1-6}$$

The maximum-likelihood estimates of the symbols $I_1, I_2, \dots, I_p$ are those that maximize this quantity. Note, however, that the integral of $|r_l(t)|^2$ is common to all metrics, and, hence, it may be discarded. The other integral involving $r(t)$ gives rise to the variables

$$y_n \equiv y(nT) = \int_{-\infty}^{\infty} r_l(t)h^*(t-nT)\,dt \tag{10-1-7}$$

These variables can be generated by passing $r(t)$ through a filter matched to $h(t)$ and sampling the output at the symbol rate $1/T$. The samples $\{y_n\}$ form a set of sufficient statistics for the computation of $PM(\mathbf{I}_p)$ or, equivalently, of the correlation metrics

$$CM(\mathbf{I}_p) = 2\,\mathrm{Re}\left(\sum_n I_n^* y_n\right) - \sum_n\sum_m I_n^* I_m x_{n-m} \tag{10-1-8}$$

where, by definition, $x(t)$ is the response of the matched filter to $h(t)$ and

$$x_n \equiv x(nT) = \int_{-\infty}^{\infty} h^*(t)h(t+nT)\,dt \tag{10-1-9}$$

Hence, $x(t)$ represents the output of a filter having an impulse response $h^*(-t)$ and an excitation $h(t)$. In other words, $x(t)$ represents the autocorrelation function of $h(t)$. Consequently, $\{x_n\}$ represents the samples of the autocorrelation function of $h(t)$, taken periodically at $1/T$. We are not particularly concerned with the noncausal characteristic of the filter matched to $h(t)$, since, in practice, we can introduce a sufficiently large delay to ensure causality of the matched filter.

If we substitute for $r_l(t)$ in (10-1-7) using (10-1-1), we obtain

$$y_k = \sum_n I_n x_{k-n} + v_k \tag{10-1-10}$$

where v_k denotes the additive noise sequence of the output of the matched filter, i.e.,

$$v_k = \int_{-\infty}^{\infty} z(t)h^*(t-kT)\,dt \tag{10-1-11}$$

The output of the demodulator (matched filter) at the sampling instants is corrupted by ISI as indicated by (10-1-10). In any practical system, it is reasonable to assume that the ISI affects a finite number of symbols. Hence, we may assume that $x_n = 0$ for $|n| > L$. Consequently, the ISI observed at the output of the demodulator may be viewed as the output of a finite state machine. This implies that the channel output with ISI may be represented by a trellis diagram, and the maximum-likelihood estimate of the information sequence $(I_1, I_2, \ldots, I_p)$ is simply the most probable path through the trellis given the received demodulator output sequence $\{y_n\}$. Clearly, the Viterbi algorithm provides an efficient means for performing the trellis search.

The metrics that are computed for the MLSE of the sequence $\{I_k\}$ are given by (10-1-8). It can be seen that these metrics can be computed recursively in the Viterbi algorithm, according to the relation

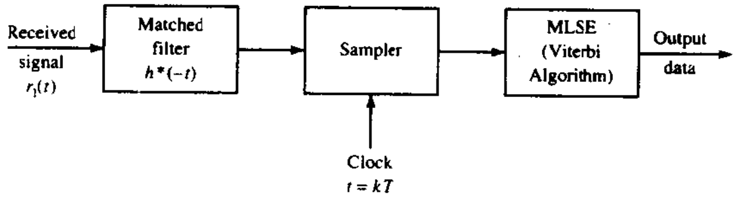
$$CM_n(\mathbf{I}_n) = CM_{n-1}(\mathbf{I}_{n-1}) + \mathrm{Re}\left[I_n^*\left(2y_n - x_0 I_n - 2\sum_{m=1}^{L} x_m I_{n-m}\right)\right] \tag{10-1-12}$$

Figure 10-1-1 illustrates the block diagram of the optimum receiver for an AWGN channel with ISI.

10-1-2 A Discrete-Time Model for a Channel with ISI

In dealing with band-limited channels that result in ISI, it is convenient to develop an equivalent discrete-time model for the analog (continuous-time) system. Since the transmitter sends discrete-time symbols at a rate $1/T$ symbols/s and the sampled output of the matched filter at the receiver is also a discrete-time signal with samples occurring at a rate $1/T$ per second, it follows that the cascade of the analog filter at the transmitter with impulse response $g(t)$, the channel with impulse response $c(t)$, the matched filter at the receiver with impulse response $h^*(-t)$, and the sampler can be represented by

FIGURE 10-1-1 Optimum receiver for an AWGN channel with ISI.



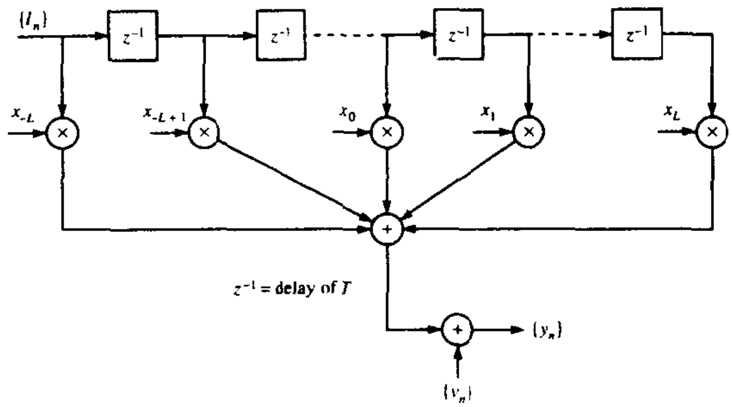


FIGURE 10-1-2 Equivalent discrete-time model of channel with intersymbol interference.

an equivalent discrete-time transversal filter having tap gain coefficients $\{x_k\}$. Consequently, we have an equivalent discrete-time transversal filter that spans a time interval of $2LT$ seconds. Its input is the sequence of information symbols $\{I_k\}$ and its output is the discrete-time sequence $\{y_k\}$ given by (10-1-10). The equivalent discrete-time model is shown in Fig. 10-1-2.

The major difficulty with this discrete-time model occurs in the evaluation of performance of the various equalization or estimation techniques that are discussed in the following sections. The difficulty is caused by the correlations in the noise sequence $\{v_k\}$ at the output of the matched filter. That is, the set of noise variables $\{v_k\}$ is a gaussian-distributed sequence with zero mean and autocorrelation function (see Problem 10-5)

$$\tfrac{1}{2}E(v_k^* v_j) = \begin{cases} N_0 x_{k-j} & (|k-j| \le L) \\ 0 & \text{(otherwise)} \end{cases} \tag{10-1-13}$$

Hence, the noise sequence is correlated unless $x_k = 0$, $k \neq 0$. Since it is more convenient to deal with the white noise sequence when calculating the error rate performance, it is desirable to whiten the noise sequence by further filtering the sequence $\{y_k\}$. A discrete-time noise-whitening filter is determined as follows.

Let $X(z)$ denote the (two-sided) z transform of the sampled autocorrelation function $\{x_k\}$, i.e.,

$$X(z) = \sum_{k=-L}^{L} x_k z^{-k} \tag{10-1-14}$$

Since $x_k = x_{-k}^*$, it follows that $X(z) = X^*(z^{-1})$ and the $2L$ roots of $X(z)$ have the symmetry that if ρ is a root, $1/\rho^*$ is also a root. Hence, $X(z)$ can be factored and expressed as

$$X(z) = F(z)F^*(z^{-1}) \tag{10-1-15}$$

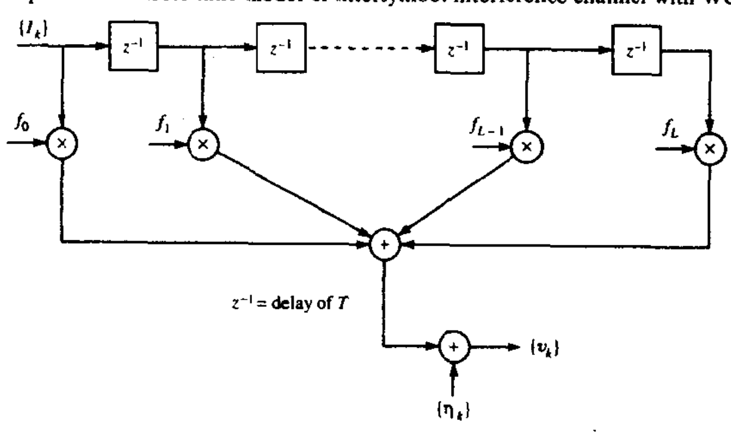
where $F(z)$ is a polynomial of degree L having the roots $\rho_1, \rho_2, \ldots, \rho_L$ and $F^*(z^{-1})$ is a polynomial of degree L having the roots $1/\rho_1^*, 1/\rho_2^*, \ldots, 1/\rho_L^*$. Then an appropriate noise-whitening filter has a z transform $1/F^*(z^{-1})$. Since there are 2^L possible choices for the roots of $F^*(z^{-1})$, each choice resulting in a filter characteristic that is identical in magnitude but different in phase from other choices of the roots, we propose to choose the unique $F^*(z^{-1})$ having minimum phase, i.e., the polynomial having all its roots inside the unit circle. Thus, when all the roots of $F^*(z^{-1})$ are inside the unit circle, $1/F^*(z^{-1})$ is a physically realizable, stable, recursive discrete-time filter.† Consequently, passage of the sequence $\{y_k\}$ through the digital filter $1/F^*(z^{-1})$ results in an output sequence $\{v_k\}$ that can be expressed as

$$v_k = \sum_{n=0}^{L} f_n I_{k-n} + \eta_k \tag{10-1-16}$$

where $\{\eta_k\}$ is a white gaussian noise sequence and $\{f_k\}$ is a set of tap coefficients of an equivalent discrete-time transversal filter having a transfer function $F(z)$. In general, the sequence $\{v_k\}$ is complex-valued.

In summary, the cascade of the transmitting filter $g(t)$, the channel $c(t)$, the matched filter $h^*(-t)$, the sampler, and the discrete-time noise-whitening filter $1/F^*(z^{-1})$ can be represented as an equivalent discrete-time transversal filter having the set $\{f_k\}$ as its tap coefficients. The additive noise sequence $\{\eta_k\}$ corrupting the output of the discrete-time transversal filter is a white gaussian noise sequence having zero mean and variance N_0. Figure 10-1-3 illustrates the model of the equivalent discrete-time system with white noise. We refer to this model as the *equivalent discrete-time white noise filter model.*

FIGURE 10-1-3 Equivalent discrete-time model of intersymbol interference channel with WGN.



†By removing the stability condition, we can also show $F^*(z^{-1})$ to have roots on the unit circle.

Example 10-1-1

Suppose that the transmitter signal pulse $g(t)$ has duration T and unit energy and the received signal pulse is $h(t) = g(t) + ag(t - T)$. Let us determine the equivalent discrete-time white-noise filter model. The sample autocorrelation function is given by

$$x_k = \begin{cases} a^* & (k = -1) \\ 1 + |a|^2 & (k = 0) \\ a & (k = 1) \end{cases} \tag{10-1-17}$$

The z transform of x_k is

$$\begin{aligned} X(z) &= \sum_{k=-1}^{1} x_k z^{-k} \\ &= a^*z + (1 + |a|^2) + az^{-1} \\ &= (az^{-1} + 1)(a^*z + 1) \end{aligned} \tag{10-1-18}$$

Under the assumption that $|a| > 1$, one chooses $F(z) = az^{-1} + 1$, so that the equivalent transversal filter consists of two taps having tap gain coefficients $f_0 = 1$, $f_1 = a$. Note that the correlation sequence $\{x_k\}$ may be expressed in terms of the $\{f_n\}$ as

$$x_k = \sum_{n=0}^{L-k} f_n^* f_{n+k}, \qquad k = 0, 1, 2, \ldots, L \tag{10-1-19}$$

When the channel impulse response is changing slowly with time, the matched filter at the receiver becomes a time-variable filter. In this case, the time variations of the channel/matched-filter pair result in a discrete-time filter with time-variable coefficients. As a consequence, we have time-variable intersymbol interference effects, which can be modeled by the filter illustrated in Fig. 10-1-3, where the tap coefficients are slowly varying with time.

The discrete-time white noise linear filter model for the intersymbol interference effects that arise in high-speed digital transmission over nonideal band-limited channels will be used throughout the remainder of this chapter in our discussion of compensation techniques for the interference. In general, the compensation methods are called *equalization techniques* or *equalization algorithms*.

10-1-3 The Viterbi Algorithm for the Discrete-Time White Noise Filter Model

MLSE of the information sequence $\{I_k\}$ is most easily described in terms of the received sequence $\{v_k\}$ at the output of the whitening filter. In the presence of intersymbol interference that spans $L + 1$ symbols (L interfering components), the MLSE criterion is equivalent to the problem of estimating the state of a discrete-time finite-state machine. The finite-state machine in this case is the equivalent discrete-time channel with coefficients $\{f_k\}$, and its state at any instant in time is given by the L most recent inputs, i.e., the state at time k is

$$S_k = (I_{k-1}, I_{k-2}, \ldots, I_{k-L}) \tag{10-1-20}$$

where $I_k = 0$ for $k \leq 0$. Hence, if the information symbols are M-ary, the channel filter has M^L states. Consequently, the channel is described by an M^L-state trellis and the Viterbi algorithm may be used to determine the most probable path through the trellis.

The metrics used in the trellis search are akin to the metrics used in soft-decision decoding of convolutional codes. In brief, we begin with the samples $v_1, v_2, \ldots, v_{L+1}$, from which we compute the M^{L+1} metrics

$$\sum_{k=1}^{L+1} \ln p(v_k \mid I_k, I_{k-1}, \ldots, I_{k-L}) \tag{10-1-21}$$

The M^{L+1} possible sequences of $I_{L+1}, I_L, \ldots, I_2, I_1$ are subdivided into M^L groups corresponding to the M^L states $(I_{L+1}, I_L, \ldots, I_2)$. Note that the M sequences in each group (state) differ in I_1 and correspond to the paths through the trellis that merge at a single node. From the M sequences in each of the M^L states, we select the sequence with the largest probability (with respect to I_1) and assign to the surviving sequence the metric

$$\begin{aligned} PM_1(\mathbf{I}_{L+1}) &\equiv PM_1(I_{L+1}, I_L, \ldots, I_2) \\ &= \max_{I_1} \sum_{k=1}^{L+1} \ln p(v_k \mid I_k, I_{k-1}, \ldots, I_{k-L}) \end{aligned} \tag{10-1-22}$$

The $M - 1$ remaining sequences from each of the M^L groups are discarded. Thus, we are left with M^L surviving sequences and their metrics.

Upon reception of v_{L+2}, the M^L surviving sequences are extended by one stage, and the corresponding M^{L+1} probabilities for the extended sequences are computed using the previous metrics and the new increment, which is $\ln p(v_{L+2} \mid I_{L+2}, I_{L+1}, \ldots, I_2)$. Again, the M^{L+1} sequences are subdivided into M^L groups corresponding to the M^L possible states $(I_{L+2}, \ldots, I_3)$ and the most probable sequence from each group is selected, while the other $M - 1$ sequences are discarded.

The procedure described continues with the reception of subsequent signal samples. In general, upon reception of v_{L+k}, the metrics†

$$PM_k(\mathbf{I}_{L+k}) = \max_{I_k} [\ln p(v_{L+k} \mid I_{L+k}, \ldots, I_k) + PM_{k-1}(\mathbf{I}_{L+k-1})] \tag{10-1-23}$$

†We observe that the metrics $PM_k(\mathbf{I})$ are simply related to the euclidean distance metrics $DM_k(\mathbf{I})$ when the additive noise is gaussian.

that are computed give the probabilities of the M^L surviving sequences. Thus, as each signal sample is received, the Viterbi algorithm involves first the computation of the M^{L+1} probabilities

$$\ln p(v_{L+k} \mid I_{L+k}, \ldots, I_k) + PM_{k-1}(\mathbf{I}_{L+k-1}) \tag{10-1-24}$$

corresponding to the M^{L+1} sequences that form the continuations of the M^L surviving sequences from the previous stage of the process. Then the M^{L+1} sequences are subdivided into M^L groups, with each group containing M sequences that terminate in the same set of symbols $I_{L+k}, \ldots, I_{k+1}$ and differ in the symbol I_k. From each group of M sequences, we select the one having the largest probability as indicated by (10-1-23), while the remaining $M-1$ sequences are discarded. Thus, we are left again with M^L sequences having the metrics $PM_k(\mathbf{I}_{L+k})$.

As indicated previously, the delay in detecting each information symbol is variable. In practice, the variable delay is avoided by truncating the surviving sequences to the q most recent symbols, where $q \gg L$, thus achieving a fixed delay. In the case that the M^L surviving sequences at time k disagree on the symbol I_{k-q}, the symbol in the most probable sequence may be chosen. The loss in performance resulting from this suboptimum decision procedure is negligible if $q \geq 5L$.

Example 10-1-2

For illustrative purposes, suppose that a duobinary signal pulse is employed to transmit four-level ($M=4$) PAM. Thus, each symbol is a number selected from the set $\{-3, -1, 1, 3\}$. The controlled intersymbol interference in this partial response signal is represented by the equivalent discrete-time channel model shown in Fig. 10-1-4. Suppose we have received v_1 and v_2, where

$$\begin{aligned} v_1 &= I_1 + \eta_1 \\ v_2 &= I_2 + I_1 + \eta_2 \end{aligned} \tag{10-1-25}$$

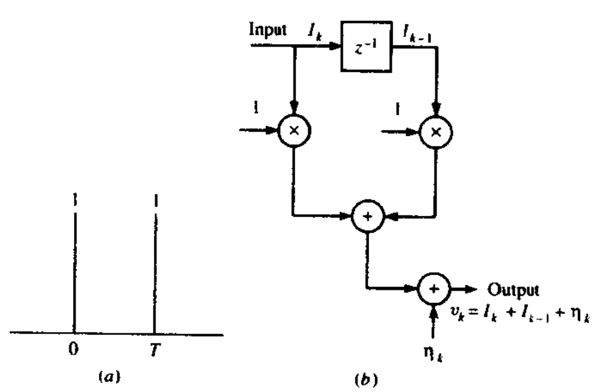


FIGURE 10-1-4 Equivalent discrete-time model for intersymbol interference resulting from a duobinary pulse.

and $\{\eta_i\}$ is a sequence of statistically independent zero-mean gaussian noise. We may now compute the 16 metrics

$$PM_1(I_2, I_1) = -\sum_{k=1}^{2}\left(v_k - \sum_{j=0}^{1} I_{k-j}\right)^2, \qquad I_1, I_2 = \pm 1, \pm 3 \qquad (10\text{-}1\text{-}26)$$

where $I_k = 0$ for $k \leq 0$.

Note that any subsequently received signals $\{v_i\}$ do not involve I_1. Hence, at this stage, we may discard 12 of the 16 possible pairs $\{I_1, I_2\}$. This step is illustrated by the tree diagram shown in Fig. 10-1-5. In other words, after computing the 16 metrics corresponding to the 16 paths in the tree diagram,

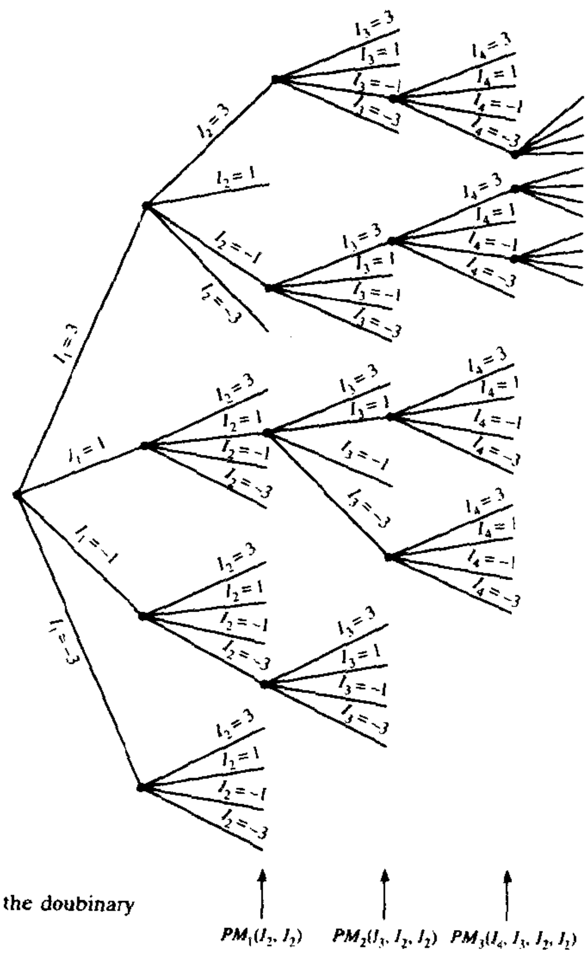


FIGURE 10-1-5 Tree diagram for Viterbi decoding of the doubinary pulse.

we discard three out of the four paths that terminate with $I_2 = 3$ and save the most probable of these four. Thus, the metric for the surviving path is

$$PM_1(I_2 = 3, I_1) = \max_{I_1}\left[-\sum_{k=1}^{2}\left(v_k - \sum_{j=0}^{1} I_{k-j}\right)^2\right]$$

The process is repeated for each set of four paths terminating with $I_2 = 1$, $I_2 = -1$, and $I_2 = -3$. Thus four paths and their corresponding metrics survive after v_1 and v_2 are received.

When v_3 is received, the four paths are extended as shown in Fig. 10-1-5, to yield 16 paths and 16 corresponding metrics, given by

$$PM_2(I_3, I_2, I_1) = PM_1(I_2, I_1) - \left(v_3 - \sum_{j=0}^{1} I_{3-j}\right)^2 \tag{10-1-27}$$

Of the four paths terminating with the $I_3 = 3$, we save the most probable. This procedure is again repeated for $I_3 = 1$, $I_3 = -1$, and $I_3 = -3$. Consequently, only four paths survive at this stage. The procedure is then repeated for each subsequently received signal v_k for $k > 3$.

10-1-4 Performance of MLSE for Channels with ISI

We shall now determine the probability of error for MLSE of the received information sequence when the information is transmitted via PAM and the additive noise is gaussian. The similarity between a convolutional code and a finite-duration intersymbol interference channel implies that the method for computing the error probability for the latter carries over from the former. In particular, the method for computing the performance of soft-decision decoding of a convolutional code by means of the Viterbi algorithm, described in Section 8-2-3, applies with some modification.

In PAM signaling with additive gaussian noise and intersymbol interference, the metrics used in the Viterbi algorithm may be expressed as in (10-1-23), or equivalently, as

$$PM_{k-L}(\mathbf{I}_k) = PM_{k-L-1}(\mathbf{I}_{k-1}) - \left(v_k - \sum_{j=0}^{L} f_j I_{k-j}\right)^2 \tag{10-1-28}$$

where the symbols $\{I_n\}$ may take the values $\pm d, \pm 3d, \ldots, \pm(M-1)d$, and $2d$ is the distance between successive levels. The trellis has M^L states, defined at time k as

$$S_k(I_{k-1}, I_{k-2}, \ldots, I_{k-L}) \tag{10-1-29}$$

Let the estimated symbols from the Viterbi algorithm be denoted by $\{\tilde{I}_n\}$ and the corresponding estimated state at time k by

$$\tilde{S}_k = (\tilde{I}_{k-1}, \tilde{I}_{k-2}, \ldots, \tilde{I}_{k-L}) \tag{10-1-30}$$

Now suppose that the estimated path through the trellis diverges from the correct path at time k and remerges with the correct path at time $k+l$. Thus, $\tilde{S}_k = S_k$ and $\tilde{S}_{k+l} = S_{k+l}$, but $\tilde{S}_m \neq S_m$ for $k < m < k+l$. As in a convolutional code, we call this an *error event*. Since the channel spans $L+1$ symbols, it follows that $l \geq L+1$.

For such an error event, we have $\tilde{I}_k \neq I_k$ and $\tilde{I}_{k+l-L-1} \neq I_{k+l-L-1}$, but $\tilde{I}_m = I_m$ for $k-L \leq m \leq k-1$ and $k+l-L \leq m \leq k+l-1$. It is convenient to define an error vector $\boldsymbol{\varepsilon}$ corresponding to this error event as

$$\boldsymbol{\varepsilon} = [\varepsilon_k \quad \varepsilon_{k+1} \quad \ldots \quad \varepsilon_{k+l-L-1}] \tag{10-1-31}$$

where the components of $\boldsymbol{\varepsilon}$ are defined as

$$\varepsilon_j = \frac{1}{2d}(I_j - \tilde{I}_j), \qquad j = k, k+1, \ldots, k+l-L-1 \tag{10-1-32}$$

The normalization factor of $2d$ in (10-1-32) results in elements ε_j that take on the values $\pm 1, \pm 2, \pm 3, \ldots, \pm(M-1)$. Moreover, the error vector is characterized by the properties that $\varepsilon_k \neq 0$, $\varepsilon_{k+l-L-1} \neq 0$, and there is no sequence of L consecutive elements that are zero. Associated with the error vector in (10-1-31) is the polynomial of degree $l-L-1$,

$$\varepsilon(z) = \varepsilon_k + \varepsilon_{k+1} z^{-1} + \varepsilon_{k+2} z^{-2} + \ldots + \varepsilon_{k+l-L-1} z^{-(l-L-1)} \tag{10-1-33}$$

We wish to determine the probability of occurrence of the error event that begins at time k and is characterized by the error vector $\boldsymbol{\varepsilon}$ given in (10-1-31), or, equivalently, by the polynomial given in (10-1-33). To accomplish this, we follow the procedure developed by Forney (1972). Specifically, for the error event $\boldsymbol{\varepsilon}$ to occur, the following three subevents E_1, E_2, and E_3 must occur:

- E_1: at time k, $\tilde{S}_k = S_k$;
- E_2: the information symbols $I_k, I_{k+1}, \ldots, I_{k+l-L-1}$ when added to the scaled error sequence $2d(\varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_{k+1-L-1})$ must result in an allowable sequence, i.e., the sequence $\tilde{I}_k, \tilde{I}_{k+1}, \ldots, \tilde{I}_{k+l-L-1}$ must have values selected from $\pm d, \pm 3d, \pm \cdots \pm (M-1)d$;
- E_3: for $k \leq m < k+l$, the sum of the branch metrics of the estimated path exceed the sum of the branch metrics of the correct path.

The probability of occurrence of E_3 is

$$P(E_3) = P\left[\sum_{i=k}^{k+l-1}\left(v_i - \sum_{j=0}^{L} f_j \tilde{I}_{i-j}\right)^2 < \sum_{i=k}^{k+l-1}\left(v_i - \sum_{j=0}^{L} f_j I_{i-j}\right)^2\right] \tag{10-1-34}$$

But

$$v_i = \sum_{j=0}^{L} f_j I_{i-j} + \eta_i \tag{10-1-35}$$

where $\{\eta_i\}$ is a real-valued white gaussian noise sequence. Substitution of (10-1-35) into (10-1-34) yields

$$
\begin{aligned}
P(E_3) &= P\left[\sum_{i=k}^{k+l-1}\left(\eta_i + 2d\sum_{j=0}^{L} f_j\varepsilon_{i-j}\right)^2 < \sum_{i=k}^{k+l-1}\eta_i^2\right] \\
&= P\left[4d\sum_{i=k}^{k+l-1}\eta_i\left(\sum_{j=0}^{L} f_j\varepsilon_{i-j}\right) < -4d^2\sum_{i=k}^{k+l-1}\left(\sum_{j=0}^{L} f_j\varepsilon_{i-j}\right)^2\right]
\end{aligned}
\tag{10-1-36}
$$

where $\varepsilon_j = 0$ for $j < k$ and $j > k + l - L - 1$. If we define

$$\alpha_i = \sum_{j=0}^{L} f_j\varepsilon_{i-j} \tag{10-1-37}$$

then (10-1-36) may be expressed as

$$P(E_3) = P\left(\sum_{i=k}^{k+l-1}\alpha_i\eta_i < -d\sum_{i=k}^{k+l-1}\alpha_i^2\right) \tag{10-1-38}$$

where the factor of $4d$ common to both terms has been dropped. Now (10-1-38) is just the probability that a linear combination of statistically independent guassian random variables is less than some negative number. Thus

$$P(E_3) = Q\left(\sqrt{\frac{2d^2}{N_0}\sum_{i=k}^{k+l-1}\alpha_i^2}\right) \tag{10-1-39}$$

For convenience, we define

$$\delta^2(\boldsymbol{\varepsilon}) = \sum_{i=k}^{k+l-1}\alpha_i^2 = \sum_{i=k}^{k+l-1}\left(\sum_{j=0}^{L} f_j\varepsilon_{i-j}\right)^2 \tag{10-1-40}$$

where $\varepsilon_j = 0$ for $j < k$ and $j > k + l - L - 1$. Note that the $\{\alpha_i\}$ resulting from the convolution of $\{f_i\}$ with $\{\varepsilon_j\}$ are the coefficients of the polynomial

$$
\begin{aligned}
\alpha(z) &= F(z)\varepsilon(z) \\
&= \alpha_k + \alpha_{k+1}z^{-1} + \ldots + \alpha_{k+l-1}z^{-(l-1)}
\end{aligned}
\tag{10-1-41}
$$

Furthermore, $\delta^2(\boldsymbol{\varepsilon})$ is simply equal to the coefficient of z^0 in the polynomial

$$
\begin{aligned}
\alpha(z)\alpha(z^{-1}) &= F(z)F(z^{-1})\varepsilon(z)\varepsilon(z^{-1}) \\
&= X(z)\varepsilon(z)\varepsilon(z^{-1})
\end{aligned}
\tag{10-1-42}
$$

We call $\delta^2(\boldsymbol{\varepsilon})$ the *euclidean weight* of the error event $\boldsymbol{\varepsilon}$.

An alternative method for representing the result of convolving $\{f_j\}$ with $\{\varepsilon_j\}$ is the matrix form

$$\boldsymbol{\alpha} = \mathbf{e}\mathbf{f}$$

where $\boldsymbol{\alpha}$ is an l-dimensional vector, $\mathbf{f}$ is an $(L+1)$-dimensional vector, and $\mathbf{e}$ is an $l \times (L+1)$ matrix, defined as

$$\boldsymbol{\alpha} = \begin{bmatrix} \alpha_k \\ \alpha_{k+1} \\ \vdots \\ \alpha_{k+l-1} \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_L \end{bmatrix}$$

$$\mathbf{e} = \begin{bmatrix} \varepsilon_k & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \varepsilon_{k+1} & \varepsilon_k & 0 & \cdots & 0 & \cdots & 0 \\ \varepsilon_{k+2} & \varepsilon_{k+1} & \varepsilon_k & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ \varepsilon_{k+l-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \varepsilon_{k+l-L-1} \end{bmatrix} \tag{10-1-43}$$

Then

$$\begin{aligned} \delta^2(\boldsymbol{\varepsilon}) &= \boldsymbol{\alpha}'\boldsymbol{\alpha} \\ &= \mathbf{f}'\mathbf{e}'\mathbf{e}\mathbf{f} \\ &= \mathbf{f}'\mathbf{A}\mathbf{f} \end{aligned} \tag{10-1-44}$$

where $\mathbf{A}$ is an $(L+1) \times (L+1)$ matrix of the form

$$\mathbf{A} = \mathbf{e}'\mathbf{e} = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_L \\ \beta_1 & \beta_0 & \beta_1 & \cdots & \beta_{L-1} \\ \beta_2 & \beta_1 & \beta_0 & \beta_1 & \beta_{L-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_L & \cdots & \cdots & \cdots & \beta_0 \end{bmatrix} \tag{10-1-45}$$

and

$$\beta_m = \sum_{i=k}^{k+l-1-m} \varepsilon_i \varepsilon_{i+m} \tag{10-1-46}$$

We may use either (10-1-40) and (10-1-41) or (10-1-45)–(10-1-46) in evaluating the error rate performance. We consider these computations later. For now we conclude that the probability of the subevent E_3, given by (10–1–39), may be expressed as

$$\begin{aligned} P(E_3) &= Q\left(\sqrt{\frac{2d^2}{N_0}\,\delta^2(\boldsymbol{\varepsilon})}\right) \\ &= Q\left(\sqrt{\frac{6}{M^2-1}\,\gamma_{\text{av}}\delta^2(\boldsymbol{\varepsilon})}\right) \end{aligned} \tag{10-1-47}$$

where we have used the relation

$$d^2 = \frac{3}{M^2-1}\,TP_{\text{av}} \tag{10-1-48}$$

to eliminate d^2 and $\gamma_{\text{av}} = TP_{\text{av}}/N_0$. Note that, in the absence of intersymbol interference, $\delta^2(\boldsymbol{\varepsilon}) = 1$ and $P(E_3)$ is proportional to the symbol error probability of M-ary PAM.

The probability of the subevent E_2 depends only on the statistical properties of the input sequence. We assume that the information symbols are equally probable and that the symbols in the transmitted sequence are statistically independent. Then, for an error of the form $|\varepsilon_i| = j$, $j = 1, 2, \ldots, M-1$, there are $M - j$ possible values of I_i such that

$$I_i = \tilde{I}_i + 2d\varepsilon_i$$

Hence

$$P(E_2) = \prod_{i=0}^{l-L-1} \frac{M - |i|}{M} \tag{10-1-49}$$

The probability of the subevent E_1 is much more difficult to compute exactly because of its dependence on the subevent E_3. That is, we must compute $P(E_1 \mid E_3)$. However, $P(E_1 \mid E_3) = 1 - P_M$, where P_M is the symbol error probability. Hence $P(E_1 \mid E_3)$ is well approximated (and upper-bounded) by unity for reasonably low symbol error probabilities. Therefore, the probability of the error event $\boldsymbol{\varepsilon}$ is well approximated and upper-bounded as

$$P(\boldsymbol{\varepsilon}) \le Q\left(\sqrt{\frac{6}{M^2-1}\gamma_{av}\delta^2(\boldsymbol{\varepsilon})}\right)\prod_{i=0}^{l-L-1} \frac{M - |i|}{M} \tag{10-1-50}$$

Let E be the set of all error events $\boldsymbol{\varepsilon}$ starting at time k and let $w(\boldsymbol{\varepsilon})$ be the corresponding number of nonzero components (Hamming weight or number of symbol errors) in each error event $\boldsymbol{\varepsilon}$. Then the probability of a symbol error is upper-bounded (union bound) as

$$\begin{aligned} P_M &\le \sum_{\boldsymbol{\varepsilon} \in E} w(\boldsymbol{\varepsilon}) P(\boldsymbol{\varepsilon}) \\ &\le \sum_{\boldsymbol{\varepsilon} \in E} w(\boldsymbol{\varepsilon}) Q\left(\sqrt{\frac{6}{M^2-1}\gamma_{av}\delta^2(\boldsymbol{\varepsilon})}\right)\prod_{i=0}^{l-L-1} \frac{M - |i|}{M} \end{aligned} \tag{10-1-51}$$

Now let D be the set of all $\delta(\boldsymbol{\varepsilon})$. For each $\delta \in D$, let E_δ be the subset of error events for which $\delta(\boldsymbol{\varepsilon}) = \delta$. Then (10-1-51) may be expressed as

$$\begin{aligned} P_M &\le \sum_{\delta \in D} Q\left(\sqrt{\frac{6}{M^2-1}\gamma_{av}\delta^2}\right)\left[\sum_{\boldsymbol{\varepsilon} \in E_\delta} w(\boldsymbol{\varepsilon}) \prod_{i=0}^{l-L-1} \frac{M - |i|}{M}\right] \\ &\le \sum_{\delta \in D} K_\delta Q\left(\sqrt{\frac{6}{M^2-1}\gamma_{av}\delta^2}\right) \end{aligned} \tag{10-1-52}$$

where

$$K_\delta = \sum_{\boldsymbol{\varepsilon} \in E_\delta} w(\boldsymbol{\varepsilon}) \prod_{i=0}^{l-L-1} \frac{M - |i|}{M} \tag{10-1-53}$$

The expression for the error probability in (10-1-52) is similar to the form of the error probability for a convolutional code with soft-decision decoding given by (8-2-26). The weighting factors $\{K_\delta\}$ may be determined by means of the error state diagram, which is akin to the state diagram of a convolutional encoder. This approach has been illustrated by Forney (1972) and Viterbi and Omura (1979).

In general, however, the use of the error state diagram for computing P_M is tedious. Instead, we may simplify the computation of P_M by focusing on the dominant term in the summation of (10-1-52). Due to the exponential dependence of each term in the sum, the expression P_M is dominated by the term corresponding to the minimum value of δ, denoted as δ_{min}. Hence the symbol error probability may be approximated as

$$P_M \approx K_{\delta_{min}} Q\left(\sqrt{\frac{6}{M^2-1}\gamma_{av}\delta_{min}^2}\right) \tag{10-1-54}$$

where

$$K_{\delta_{min}} = \sum_{\varepsilon \in E_{\delta_{min}}} w(\varepsilon) \prod_{i=0}^{l-L-1} \frac{M-|i|}{M} \tag{10-1-55}$$

In general, $\delta_{min}^2 \leqslant 1$. Hence, $10 \log \delta_{min}^2$ represents the loss in SNR due to intersymbol interference.

The minimum value of δ may be determined either from (10-1-40) or from evaluation of the quadratic form in (10-1-44) for different error sequences. In the following two examples we use (10-1-40).

Example 10-1-3

Consider a two-path channel ($L=1$) with arbitrary coefficients f_0 and f_1 satisfying the constraint $f_0^2+f_1^2=1$. The channel characteristic is

$$F(z) = f_0 + f_1 z^{-1} \tag{10-1-56}$$

For an error event of length n,

$$\varepsilon(z) = \varepsilon_0 + \varepsilon_1 z^{-1} + \ldots + \varepsilon_{n-1} z^{-(n-1)}, \qquad n \geqslant 1 \tag{10-1-57}$$

The product $\alpha(z) = F(z)\varepsilon(z)$ may be expressed as

$$\alpha(z) = \alpha_0 + \alpha_1 z^{-1} + \ldots + \alpha_n z^{-n} \tag{10-1-58}$$

where $\alpha_0 = \varepsilon_0 f_0$ and $\alpha_n = f_1 \varepsilon_{n-1}$. Since $\varepsilon_0 \neq 0$, $\varepsilon_{n-1} \neq 0$, and

$$\delta^2(\varepsilon) = \sum_{k=0}^{n} \alpha_k^2 \tag{10-1-59}$$

it follows that

$$\delta_{min}^2 \geqslant f_0^2 + f_1^2 = 1$$

Indeed, $\delta_{min}^2 = 1$ when a single error occurs, i.e. $\varepsilon(z) = \varepsilon_0$. Thus, we conclude that there is no loss in SNR in maximum-likelihood sequence estimation of the information symbols when the channel dispersion has length 2.

Example 10-1-4

The controlled intersymbol interference in a partial response signal may be viewed as having been generated by a time-dispersive channel. Thus, the intersymbol interference from a duobinary pulse may be represented by the (normalized) channel characteristic

$$F(z) = \sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{1}{2}}z^{-1} \tag{10-1-60}$$

Similarly, the representation for a modified duobinary pulse is

$$F(z) = \sqrt{\tfrac{1}{2}} - \sqrt{\tfrac{1}{2}}z^{-2} \tag{10-1-61}$$

The minimum distance $\delta^2_{\min} = 1$ for any error event of the form

$$\varepsilon(z) = \pm(1 - z^{-1} - z^{-2} \ldots - z^{-(n-1)}), \qquad n \geq 1 \tag{10-1-62}$$

for the channel given by (10-1-60) since

$$\alpha(z) = \pm\sqrt{\tfrac{1}{2}} \mp \sqrt{\tfrac{1}{2}}z^{-n}$$

Similarly, when

$$\varepsilon(z) = \pm(1 + z^{-2} - z^{-4} + \ldots + z^{-2(n-1)}), \qquad n \geq 1 \tag{10-1-63}$$

$\delta^2_{\min} = 1$ for the channel given by (10-1-61), since

$$\alpha(z) = \pm\sqrt{\tfrac{1}{2}} \mp \sqrt{\tfrac{1}{2}}z^{-2n}$$

Hence MLSE of these two partial response signals results in no loss in SNR. In contrast, the suboptimum symbol-by-symbol detection described previously resulted in a 2.1 dB loss.

The constant $K_{\delta_{\min}}$ is easily evaluated for these two signals. With precoding, the number of output symbol errors (Hamming weight) associated with the error events in (10-1-62) and (10-1-63) is two. Hence,

$$K_{\delta_{\min}} = 2\sum_{n=1}^{\infty}\left(\frac{M-1}{M}\right)^n = 2(M-1) \tag{10-1-64}$$

On the other hand, without precoding, these error events result in n symbol errors, and, hence,

$$K_{\delta_{\min}} = 2\sum_{n=1}^{\infty} n\left(\frac{M-1}{M}\right)^n = 2M(M-1) \tag{10-1-65}$$

As a final exercise, we consider the evaluation of $\delta^2_{\min}$ from the quadratic form in (10-1-44). The matrix $\mathbf{A}$ of the quadratic form is positive-definite; hence, all its eigenvalues are positive. If $\{\mu_k(\boldsymbol{\varepsilon})\}$ are the eigenvalues and $\{\mathbf{v}_k(\boldsymbol{\varepsilon})\}$ are the corresponding orthonormal eigenvectors of $\mathbf{A}$ for an error event $\boldsymbol{\varepsilon}$ then the quadratic form in (10-1-44) can be expressed as

$$\delta^2(\boldsymbol{\varepsilon}) = \sum_{k=1}^{L+1} \mu_k(\boldsymbol{\varepsilon})[\mathbf{f}'\mathbf{v}_k(\boldsymbol{\varepsilon})]^2 \tag{10-1-66}$$

In other words, $\delta^2(\boldsymbol{\varepsilon})$ is expressed as a linear combination of the squared projections of the channel vector $\mathbf{f}$ onto the eigenvectors of $\mathbf{A}$. Each squared projection in the sum is weighted by the corresponding eigenvalue $\mu_k(\boldsymbol{\varepsilon})$, $k = 1, 2, \ldots, L+1$. Then

$$\delta^2_{\min} = \min_{\boldsymbol{\varepsilon}} \delta^2(\boldsymbol{\varepsilon}) \tag{10-1-67}$$

It is interesting to note that the worst channel characteristic of a given length $L+1$ can be obtained by finding the eigenvector corresponding to the minimum eigenvalue. Thus, if $\mu_{\min}(\boldsymbol{\varepsilon})$ is the minimum eigenvalue for a given error event $\boldsymbol{\varepsilon}$ and $\mathbf{v}_{\min}(\boldsymbol{\varepsilon})$ is the corresponding eigenvector then

$$\mu_{min} = \min_{\boldsymbol{\varepsilon}} \mu_{min}(\boldsymbol{\varepsilon})$$

$$\mathbf{f} = \min_{\boldsymbol{\varepsilon}} \mathbf{v}_{\min}(\boldsymbol{\varepsilon})$$

and

$$\delta^2_{\min} = \mu_{\min}$$

Example 10-1-5

Let us determine the worst time-dispersive channel of length 3 ($L = 2$) by finding the minimum eigenvalue of $\mathbf{A}$ for different error events. Thus,

$$F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2}$$

where f_0, f_1, and f_2 are the components of the eigenvector of $\mathbf{A}$ corresponding to the minimum eigenvalue. An error event of the form

$$\varepsilon(z) = 1 - z^{-1}$$

results in a matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

which has the eigenvalues $\mu_1 = 2$, $\mu_2 = 2 + \sqrt{2}$, $\mu_3 = 2 - \sqrt{2}$. The eigenvector corresponding to μ_3 is

$$\mathbf{v}_3' = [\tfrac{1}{2} \quad \sqrt{\tfrac{1}{2}} \quad \tfrac{1}{2}] \tag{10-1-68}$$

We may also consider the dual error event

$$\epsilon(z) = 1 + z^{-1}$$

which results in the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

This matrix has eigenvalues identical to those of the one for $\varepsilon(z) = 1 - z^{-1}$ The corresponding eigenvector for $\mu_3 = 2 - \sqrt{2}$ is

$$\mathbf{v}_3' = [\, -\tfrac{1}{2} \quad \sqrt{\tfrac{1}{2}} \quad -\tfrac{1}{2}] \tag{10-1-69}$$

Any other error events lead to larger values for $\mu_{\min}$. Hence, $\mu_{\min} = 2 - \sqrt{2}$ and the worst-case channel is either

$$[\tfrac{1}{2} \quad \sqrt{\tfrac{1}{2}} \quad \tfrac{1}{2}] \quad \text{or} \quad [-\tfrac{1}{2} \quad \sqrt{\tfrac{1}{2}} \quad -\tfrac{1}{2}]$$

The loss in SNR from the channel is

$$-10 \log \delta_{\min}^2 = -10 \log \mu_{\min} = 2.3 \text{ dB}$$

Repetitions of the above computation for channels with $L = 3$, 4, and 5 yield the results given in Table 10-1-1.

10-2 LINEAR EQUALIZATION

The MLSE for a channel with ISI has a computational complexity that grows exponentially with the length of the channel time dispersion. If the size of the symbol alphabet is M and the number of interfering symbols contributing to ISI is L, the Viterbi algorithm computes M^{L+1} metrics for each new received symbol. In most channels of practical interest, such a large computational complexity is prohibitively expensive to implement.

In this and the following sections, we describe two suboptimum channel equalization approaches to compensate for the ISI. One approach employs a linear transversal filter, which is described in this section. These filter

TABLE 10-1-1 MAXIMUM PERFORMANCE LOSS AND CORRESPONDING CHANNEL CHARACTERISTICS

Channel length $L+1$	Performance loss $-10 \log \delta_{\min}^2$ (dB)	Minimum-distance channel
3	2.3	0.50, 0.71, 0.50
4	4.2	0.38, 0.60, 0.60, 0.38
5	5.7	0.29, 0.50, 0.58, 0.50, 0.29
6	7.0	0.23, 0.42, 0.52, 0.52, 0.42, 0.23

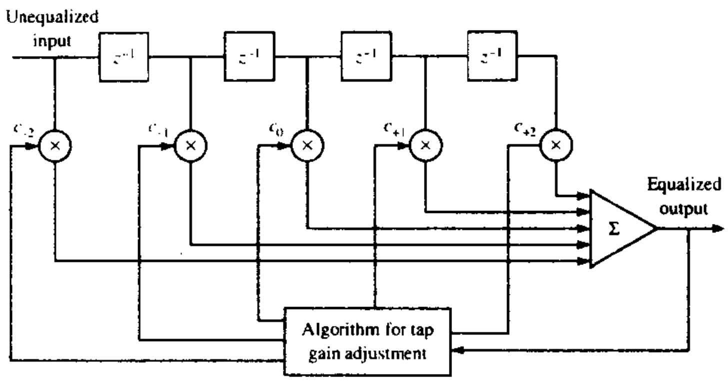


FIGURE 10-2-1 Linear transversal filter.

structures have a computational complexity that is a linear function of the channel dispersion length L.

The linear filter most often used for equalization is the transversal filter shown in Fig. 10-2-1. Its input is the sequence $\{v_k\}$ given in (10-1-16) and its output is the estimate of the information sequence $\{I_k\}$. The estimate of the k th symbol may be expressed as

$$\hat{I}_k = \sum_{j=-K}^{K} c_j v_{k-j} \tag{10-2-1}$$

where $\{c_j\}$ are the $2K + 1$ complex-valued tap weight coefficients of the filter. The estimate $\hat{I}_k$ is quantized to the nearest (in distance) information symbol to form the decision $\tilde{I}_k$. If $\tilde{I}_k$ is not identical to the transmitted information symbol I_k, an error has been made.

Considerable research has been performed on the criterion for optimizing the filter coefficients $\{c_k\}$. Since the most meaningful measure of performance for a digital communications system is the average probability of error, it is desirable to choose the coefficients to minimize this performance index. However, the probability of error is a highly nonlinear function of $\{c_j\}$. Consequently, the probability of error as a performance index for optimizing the tap weight coefficients of the equalizer is impractical.

Two criteria have found widespread use in optimizing the equalizer coefficients $\{c_j\}$. One is the peak distortion criterion and the other is the mean square error criterion.

10-2-1 Peak Distortion Criterion

The peak distortion is simply defined as the worst-case intersymbol interference at the output of the equalizer. The minimization of this performance index is called the *peak distortion criterion*. First we consider the minimization of the peak distortion assuming that the equalizer has an infinite number of taps. Then we shall discuss the case in which the transversal equalizer spans a finite time duration.

We observe that the cascade of the discrete-time linear filter model having an impulse response $\{f_n\}$ and an equalizer having an impulse response $\{c_n\}$ can be represented by a single equivalent filter having the impulse response

$$q_n = \sum_{j=-\infty}^{\infty} c_j f_{n-j} \tag{10-2-2}$$

That is, $\{q_n\}$ is simply the convolution of $\{c_n\}$ and $\{f_n\}$. The equalizer is assumed to have an infinite number of taps. Its output at the kth sampling instant can be expressed in the form

$$\hat{I}_k = q_0 I_k + \sum_{n \neq k} I_n q_{k-n} + \sum_{j=-\infty}^{\infty} c_j \eta_{k-j} \tag{10-2-3}$$

The first term in (10-2-3) represents a scaled version of the desired symbol. For convenience, we normalize q_0 to unity. The second term is the intersymbol interference. The peak value of this interference, which is called the *peak distortion*, is

$$\begin{aligned} \mathscr{D}(\mathbf{c}) &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} |q_n| \\ &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left| \sum_{j=-\infty}^{\infty} c_j f_{n-j} \right| \end{aligned} \tag{10-2-4}$$

Thus, $\mathscr{D}(\mathbf{c})$ is a function of the equalizer tap weights.

With an equalizer having an infinite number of taps, it is possible to select the tap weights so that $\mathscr{D}(\mathbf{c}) = 0$, i.e., $q_n = 0$ for all n except $n = 0$. That is, the intersymbol interference can be completely eliminated. The values of the tap weights for accomplishing this goal are determined from the condition

$$q_n = \sum_{j=-\infty}^{\infty} c_j f_{n-j} = \begin{cases} 1 & (n = 0) \\ 0 & (n \neq 0) \end{cases} \tag{10-2-5}$$

By taking the z transform of (10-2-5), we obtain

$$Q(z) = C(z)F(z) = 1 \tag{10-2-6}$$

or, simply,

$$C(z) = \frac{1}{F(z)} \tag{10-2-7}$$

where $C(z)$ denotes the z transform of the $\{c_j\}$. Note that the equalizer, with transfer function $C(z)$, is simply the inverse filter to the linear filter model $F(z)$. In other words, complete elimination of the intersymbol interference requires the use of an inverse filter to $F(z)$. We call such a filter a *zero-forcing*

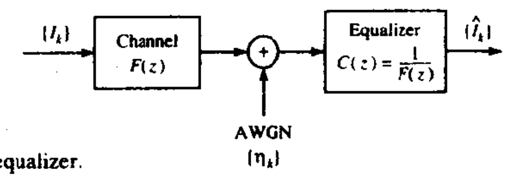


FIGURE 10-2-2 Block diagram of channel with zero-forcing equalizer.

filter. Figure 10-2-2 illustrates in block diagram the equivalent discrete-time channel and equalizer.

The cascade of the noise-whitening filter having the transfer function $1/F^*(z^{-1})$ and the zero-forcing equalizer having the transfer function $1/F(z)$ results in an equivalent zero-forcing equalizer having the transfer function

$$C'(z) = \frac{1}{F(z)F^*(z^{-1})} = \frac{1}{X(z)} \tag{10-2-8}$$

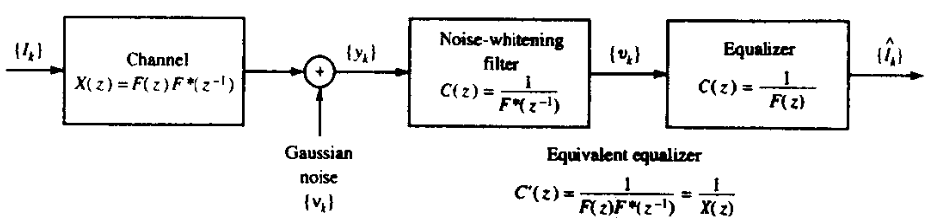
as shown in Fig. 10-2-3. This combined filter has as its input the sequence $\{y_k\}$ of samples from the matched filter, given by (10-1-10). Its output consists of the desired symbols corrupted only by additive zero-mean gaussian noise. The impulse response of the combined filter is

$$\begin{aligned} c'_k &= \frac{1}{2\pi j} \oint C'(z) z^{k-1}\, dz \\ &= \frac{1}{2\pi j} \oint \frac{z^{k-1}}{X(z)}\, dz \end{aligned} \tag{10-2-9}$$

where the integration is performed on a closed contour that lies within the region of convergence of $C'(z)$. Since $X(z)$ is a polynomial with $2L$ roots $(\rho_1, \rho_2, \ldots, \rho_L, 1/\rho_1^*, 1/\rho_2^*, \ldots, 1/\rho_L^*)$, it follows that $C'(z)$ must converge in an annular region in the z plane that includes the unit circle ($z = e^{j\theta}$). Consequently, the closed contour in the integral can be the unit circle.

The performance of the infinite-tap equalizer that completely eliminates the intersymbol interference can be expressed in terms of the signal-to-noise ratio (SNR) at its output. For mathematical convenience, we normalize the received

FIGURE 10-2-3 Block of channel with equivalent zero-forcing equalizer.



signal energy to unity.† This implies that $q_0 = 1$ and that the expected value of $|I_k|^2$ is also unity. Then the SNR is simply the reciprocal of the noise variance σ_n^2 at the output of the equalizer.

The value of σ_n^2 can be simply determined by observing that the noise sequence $\{v_k\}$ at the input to the equivalent zero-forcing equalizer $C'(z)$ has zero mean and a power spectral density

$$\Phi_{vv}(\omega) = N_0 X(e^{j\omega T}), \qquad |\omega| \le \frac{\pi}{T} \tag{10-2-10}$$

where $X(e^{j\omega T})$ is obtained from $X(z)$ by the substitution $z = e^{j\omega T}$. Since $C'(z) = 1/X(z)$, it follows that the noise sequence at the output of the equalizer has a power spectral density

$$\Phi_{nn}(\omega) = \frac{N_0}{X(e^{j\omega T})}, \qquad |\omega| \le \frac{\pi}{T} \tag{10-2-11}$$

Consequently, the variance of the noise variable at the output of the equalizer is

$$\begin{aligned} \sigma_n^2 &= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \Phi_{nn}(\omega)\, d\omega \\ &= \frac{TN_0}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{d\omega}{X(e^{j\omega T})} \end{aligned} \tag{10-2-12}$$

and the SNR for the zero-forcing equalizer is

$$\begin{aligned} \gamma_\infty &= 1/\sigma_n^2 \\ &= \left[\frac{TN_0}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{d\omega}{X(e^{j\omega T})} \right]^{-1} \end{aligned} \tag{10-2-13}$$

where the subscript on γ indicates that the equalizer has an infinite number of taps.

The spectral characteristics $X(e^{j\omega T})$ corresponding to the Fourier transform of the sampled sequence $\{x_n\}$ has an interesting relationship to the analog filter $H(\omega)$ used at the receiver. Since

$$x_k = \int_{-\infty}^{\infty} h^*(t) h(t + kT)\, dt$$

use of Parseval's theorem yields

$$x_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 e^{j\omega kT}\, d\omega \tag{10-2-14}$$

where $H(\omega)$ is the Fourier transform of $h(t)$. But the integral in (10-2-14) can be expressed in the form

$$x_k = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left[\sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right) \right|^2 \right] e^{j\omega kT}\, d\omega \tag{10-2-15}$$

† This normalization is used throughout this chapter for mathematical convenience.

Now, the Fourier transform of $\{x_k\}$ is

$$X(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} x_k e^{-j\omega kT} \tag{10-2-16}$$

and the inverse transform yields

$$x_k = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} X(e^{j\omega T}) e^{j\omega kT} \, d\omega \tag{10-2-17}$$

From a comparison of (10-2-15) and (10-2-17), we obtain the desired relationship between $X(e^{j\omega T})$ and $H(\omega)$. That is,

$$X(e^{j\omega T}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right) \right|^2, \qquad |\omega| \leq \frac{\pi}{T} \tag{10-2-18}$$

where the right-hand side of (10-2-18) is called the *folded spectrum* of $|H(\omega)|^2$. We also observe that $|H(\omega)|^2 = X(\omega)$, where $X(\omega)$ is the Fourier transform of the waveform $x(t)$ and $x(t)$ is the response of the matched filter to the input $h(t)$. Therefore the right-hand side of (10-2-18) can also be expressed in terms of $X(\omega)$.

Substitution for $X(e^{j\omega T})$ in (10-2-13) using the result in (10-2-18) yields the desired expression for the SNR in the form

$$\gamma_\infty = \left[\frac{T^2 N_0}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{d\omega}{\sum_{n=-\infty}^{\infty} |H(\omega + 2\pi n/T)|^2} \right]^{-1} \tag{10-2-19}$$

We observe that if the folded spectral characteristic of $H(\omega)$ possesses any zeros, the integrand becomes infinite and the SNR goes to zero. In other words, the performance of the equalizer is poor whenever the folded spectral characteristic possesses nulls or takes on small values. This behavior occurs primarily because the equalizer, in eliminating the intersymbol interference, enhances the additive noise. For example, if the channel contains a spectral null in its frequency response, the linear zero-forcing equalizer attempts to compensate for this by introducing an infinite gain at that frequency. But this compensates for the channel distortion at the expense of enhancing the additive noise. On the other hand, an ideal channel coupled with an appropriate signal design that results in no intersymbol interference will have a folded spectrum that satisfies the condition

$$\sum_{n=-\infty}^{\infty} \left| H\left(\omega + \frac{2\pi n}{T}\right) \right|^2 = T, \qquad |\omega| \leq \frac{\pi}{T} \tag{10-2-20}$$

In this case, the SNR achieves its maximum value, namely,

$$\gamma_\infty = \frac{1}{N_0} \tag{10-2-21}$$

Finite-Length Equalizer Let us now turn our attention to an equalizer having $2K+1$ taps. Since $c_j = 0$ for $|j| > K$, the convolution of $\{f_n\}$ with $\{c_n\}$ is zero outside the range $-K \le n \le K+L-1$. That is, $q_n = 0$ for $n < -K$ and $n > K+L-1$. With q_0 normalized to unity, the peak distortion is

$$\mathscr{D}(\mathbf{c}) = \sum_{\substack{n=-K \\ n\neq 0}}^{K+L-1} |q_n| = \sum_{\substack{n=-K \\ n\neq 0}}^{K+L-1} \left| \sum_j c_j f_{n-j} \right| \tag{10-2-22}$$

Although the equalizer has $2K+1$ adjustable parameters, there are $2K+L$ nonzero values in the response $\{q_n\}$. Therefore, it is generally impossible to completely eliminate the intersymbol interference at the output of the equalizer. There is always some residual interference when the optimum coefficients are used. The problem is to minimize $\mathscr{D}(\mathbf{c})$ with respect to the coefficients $\{c_j\}$.

The peak distortion given by (10-2-22) has been shown by Lucky (1965) to be a convex function of the coefficients $\{c_j\}$. That is, it possesses a global minimum and no relative minima. Its minimization can be carried out numerically using, for example, the method of steepest descent. Little more can be said for the general solution to this minimization problem. However, for one special but important case, the solution for the minimization of $\mathscr{D}(\mathbf{c})$ is known. This is the case in which the distortion at the input to the equalizer, defined as

$$D_0 = \frac{1}{|f_0|} \sum_{n=1}^{L} |f_n| \tag{10-2-23}$$

is less than unity. This condition is equivalent to having the eye open prior to equalization. That is, the intersymbol interference is not severe enough to close the eye. Under this condition, the peak distortion $\mathscr{D}(\mathbf{c})$ is minimized by selecting the equalizer coefficients to force $q_n = 0$ for $1 \le |n| \le K$ and $q_0 = 1$. In other words, the general solution to the minimization of $\mathscr{D}(\mathbf{c})$, when $D_0 < 1$, is the zero-forcing solution for $\{q_n\}$ in the range $1 \le |n| \le K$. However, the values of $\{q_n\}$ for $K+1 \le n \le K+L-1$ are nonzero, in general. These nonzero values constitute the residual intersymbol interference at the output of the equalizer.

10-2-2 Mean Square Error (MSE) Criterion

In the MSE criterion, the tap weight coefficients $\{c_j\}$ of the equalizer are adjusted to minimize the mean square value of the error

$$\varepsilon_k = I_k - \hat{I}_k \tag{10-2-24}$$

where I_k is the information symbol transmitted in the kth signaling interval and $\hat{I}_k$ is the estimate of that symbol at the output of the equalizer, defined in

(10-2-1). When the information symbols $\{I_k\}$ are complex-valued, the performance index for the MSE criterion, denoted by J, is defined as

$$\begin{aligned} J &= E\,|\varepsilon_k|^2 \\ &= E\,|I_k - \hat{I}_k|^2 \end{aligned} \tag{10-2-25}$$

On the other hand, when the information symbols are real-valued, the performance index is simply the square of the real part of ε_k. In either case, J is a quadratic function of the equalizer coefficients $\{c_j\}$. In the following discussion, we consider the minimization of the complex-valued form given in (10-2-25).

Infinite-Length Equalizer First, we shall derive the tap weight coefficients that minimize J when the equalizer has an infinite number of taps. In this case, the estimate $\hat{I}_k$ is expressed as

$$\hat{I}_k = \sum_{j=-\infty}^{\infty} c_j v_{k-j} \tag{10-2-26}$$

Substitution of (10-2-26) into the expression for J given in (10-2-25) and expansion of the result yields a quadratic function of the coefficients $\{c_j\}$. This function can be easily minimized with respect to the $\{c_j\}$ to yield a set (infinite in number) of linear equations for the $\{c_j\}$. Alternatively, the set of linear equations can be obtained by invoking the orthogonality principle in mean square estimation. That is, we select the coefficients $\{c_j\}$ to render the error ε_k orthogonal to the signal sequence $\{v_{k-l}^*\}$ for $-\infty < l < \infty$. Thus,

$$E(\varepsilon_k v_{k-l}^*) = 0, \qquad -\infty < l < \infty \tag{10-2-27}$$

Substitution for ε_k in (10-2-27) yields

$$E\left[\left(I_k - \sum_{j=-\infty}^{\infty} c_j v_{k-j}\right) v_{k-l}^*\right] = 0$$

or, equivalently,

$$\sum_{j=-\infty}^{\infty} c_j E(v_{k-j} v_{k-l}^*) = E(I_k v_{k-l}^*), \qquad -\infty < l < \infty \tag{10-2-28}$$

To evaluate the moments in (10-2-28), we use the expression for v_k given in (10-1-16). Thus, we obtain

$$\begin{aligned} E(v_{k-j} v_{k-l}^*) &= \sum_{n=0}^{L} f_n^* f_{n+l-j} + N_0 \delta_{lj} \\ &= \begin{cases} x_{l-j} + N_0 \delta_{lj} & (|l-j| \le L) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned} \tag{10-2-29}$$

and

$$E(I_k v_{k-l}^*) = \begin{cases} f_{-l}^* & (-L \le l \le 0) \\ 0 & (\text{otherwise}) \end{cases} \tag{10-2-30}$$

Now, if we substitute (10-2-29) and (10-2-30) into (10-2-28) and take the z transform of both sides of the resulting equation, we obtain

$$C(z)[F(z)F^*(z^{-1}) + N_0] = F^*(z^{-1}) \tag{10-2-31}$$

Therefore, the transfer function of the equalizer based on the MSE criterion is

$$C(z) = \frac{F^*(z^{-1})}{F(z)F^*(z^{-1}) + N_0} \tag{10-2-32}$$

When the noise-whitening filter is incorporated into $C(z)$, we obtain an equivalent equalizer having the transfer function

$$\begin{aligned} C'(z) &= \frac{1}{F(z)F^*(z^{-1}) + N_0} \\ &= \frac{1}{X(z) + N_0} \end{aligned} \tag{10-2-33}$$

We observe that the only difference between this expression for $C'(z)$ and the one based on the peak distortion criterion is the noise spectral density factor N_0 that appears in (10-2-33). When N_0 is very small in comparison with the signal, the coefficients that minimize the peak distortion $\mathcal{D}(\mathbf{c})$ are approximately equal to the coefficients that minimize the MSE performance index J. That is, in the limit as $N_0 \to 0$, the two criteria yield the same solution for the tap weights. Consequently, when $N_0 = 0$, the minimization of the MSE results in complete elimination of the intersymbol interference. On the other hand, that is not the case when $N_0 \neq 0$. In general, when $N_0 \neq 0$, there is both residual intersymbol interference and additive noise at the output of the equalizer.

A measure of the residual intersymbol interference and additive noise is obtained by evaluating the minimum value of J, denoted by $J_{\min}$, when the transfer function $C(z)$ of the equalizer is given by (10-2-32). Since $J = E\,|\varepsilon_k|^2 = E(\varepsilon_k I_k^*) - E(\varepsilon_k \hat{I}_k^*)$, and since $E(\varepsilon_k \hat{I}_k^*) = 0$ by virtue of the orthogonality conditions given in (10-2-27), it follows that

$$\begin{aligned} J_{\min} &= E(\varepsilon_k I_k^*) \\ &= E\,|I_k|^2 - \sum_{j=-\infty}^{\infty} c_j E(v_{k-j} I_k^*) \\ &= 1 - \sum_{j=-\infty}^{\infty} c_j f_{-j} \end{aligned} \tag{10-2-34}$$

This particular form for $J_{\min}$ is not very informative. More insight on the performance of the equalizer as a function of the channel characteristics is obtained when the summation in (10-2-34) is transformed into the frequency domain. This can be accomplished by first noting that the summation in (10-2-34) is the convolution of $\{c_j\}$ with $\{f_j\}$, evaluated at a shift of zero. Thus, if $\{b_k\}$ denotes the convolution of these two sequences, the summation in (10-2-34) is simply equal to b_0. Since the z transform of the sequence $\{b_k\}$ is

$$\begin{aligned} B(z) &= C(z)F(z) \\ &= \frac{F(z)F^*(z^{-1})}{F(z)F^*(z^{-1}) + N_0} \\ &= \frac{X(z)}{X(z) + N_0} \end{aligned} \tag{10-2-35}$$

the term b_0 is

$$\begin{aligned} b_0 &= \frac{1}{2\pi j} \oint \frac{B(z)}{z}\, dz \\ &= \frac{1}{2\pi j} \oint \frac{X(z)}{z[X(z) + N_0]}\, dz \end{aligned} \tag{10-2-36}$$

The contour integral in (10-2-36) can be transformed into an equivalent line integral by the change of variable $z = e^{j\omega T}$. The result of this change of variable is

$$b_0 = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{X(e^{j\omega T})}{X(e^{j\omega T}) + N_0}\, d\omega \tag{10-2-37}$$

Finally, substitution of the result in (10-2-37) for the summation in (10-2-34) yields the desired expression for the minimum MSE in the form

$$\begin{aligned} J_{\min} &= 1 - \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{X(e^{j\omega T})}{X(e^{j\omega T}) + N_0}\, d\omega \\ &= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N_0}{X(e^{j\omega T}) + N_0}\, d\omega \\ &= \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N_0}{T^{-1} \sum_{n=-\infty}^{\infty} |H(\omega + 2\pi n/T)|^2 + N_0}\, d\omega \end{aligned} \tag{10-2-38}$$

In the absence of intersymbol interference, $X(e^{j\omega T}) = 1$ and, hence,

$$J_{\min} = \frac{N_0}{1 + N_0} \tag{10-2-39}$$

We observe that $0 \leq J_{\min} \leq 1$. Furthermore, the relationship between the output (normalized by the signal energy) SNR γ_∞ and $J_{\min}$ must be

$$\gamma_\infty = \frac{1 - J_{\min}}{J_{\min}} \tag{10-2-40}$$

More importantly, this relation between γ_∞ and $J_{\min}$ also holds when there is residual intersymbol interference in addition to the noise.

Finite-Length Equalizer Let us now turn our attention to the case in which the transversal equalizer spans a finite time duration. The output of the equalizer in the kth signaling interval is

$$\hat{I}_k = \sum_{j=-K}^{K} c_j v_{k-j} \tag{10-2-41}$$

The MSE for the equalizer having $2K + 1$ taps, denoted by $J(K)$, is

$$J(K) = E\,|I_k - \hat{I}_k|^2 = E\left|I_k - \sum_{j=-K}^{K} c_j v_{k-j}\right|^2 \tag{10-2-42}$$

Minimization of $J(K)$ with respect to the tap weights $\{c_j\}$ or, equivalently, forcing the error $\varepsilon_k = I_k - \hat{I}_k$ to be orthogonal to the signal samples v_{j-l}^*, $|l| \leq K$, yields the following set of simultaneous equations:

$$\sum_{j=-K}^{K} c_j \Gamma_{lj} = \xi_l, \qquad l = -K, \ldots, -1, 0, 1, \ldots, K \tag{10-2-43}$$

where

$$\Gamma_{lj} = \begin{cases} x_{l-j} + N_0 \delta_{lj} & (|l-j| \leq L) \\ 0 & (\text{otherwise}) \end{cases} \tag{10-2-44}$$

and

$$\xi_l = \begin{cases} f_{-l}^* & (-L \leq l \leq 0) \\ 0 & (\text{otherwise}) \end{cases} \tag{10-2-45}$$

It is convenient to express the set of linear equations in matrix form. Thus,

$$\mathbf{\Gamma C} = \boldsymbol{\xi} \tag{10-2-46}$$

where $\mathbf{C}$ denotes the column vector of $2K + 1$ tap weight coefficients, $\mathbf{\Gamma}$ denotes the $(2K + 1) \times (2K + 1)$ Hermitian covariance matrix with elements Γ_{ij}, and $\boldsymbol{\xi}$ is a $(2K + 1)$-dimensional column vector with elements ξ_l. The solution of (10-2-46) is

$$\mathbf{C}_{\text{opt}} = \mathbf{\Gamma}^{-1} \boldsymbol{\xi} \tag{10-2-47}$$

Thus, the solution for $\mathbf{C}_{\text{opt}}$ involves inverting the matrix $\mathbf{\Gamma}$. The optimum tap weight coefficients given by (10-2-47) minimize the performance index $J(K)$, with the result that the minimum value of $J(K)$ is

$$\begin{aligned} J_{\min}(K) &= 1 - \sum_{j=-K}^{0} c_j f_{-j} \\ &= 1 - \boldsymbol{\xi}'^{*} \mathbf{\Gamma}^{-1} \boldsymbol{\xi} \end{aligned} \tag{10-2-48}$$

where $\boldsymbol{\xi}'$ represents the transpose of the column vector $\boldsymbol{\xi}$. $J_{\min}(K)$ may be used in (10-2-40) to compute the output SNR for the linear equalizer with $2K + 1$ tap coefficients.

10-2-3 Performance Characteristics of the MSE Equalizer

In this section, we consider the performance characteristics of the linear equalizer that is optimized by using the MSE criterion. Both the minimum MSE and the probability of error are considered as performance measures for some specific channels. We begin by evaluating the minimum MSE $J_{\min}$ and the output SNR γ_∞ for two specific channels. Then, we consider the evaluation of the probability of error.

Example 10-2-1

First, we consider an equivalent discrete-time channel model consisting of two components f_0 and f_1, which are normalized to $|f_0|^2 + |f_1|^2 = 1$. Then

$$F(z) = f_0 + f_1 z^{-1} \tag{10-2-49}$$

and

$$X(z) = f_0 f_1^* z + 1 + f_0^* f_1 z^{-1} \tag{10-2-50}$$

The corresponding frequency response is

$$\begin{aligned} X(e^{j\omega T}) &= f_0 f_1^* e^{j\omega T} + 1 + f_0^* f_1 e^{-j\omega T} \\ &= 1 + 2|f_0|\,|f_1| \cos(\omega T + \theta) \end{aligned} \tag{10-2-51}$$

where θ is the angle of $f_0 f_1^*$. We note that this channel characteristic possesses a null at $\omega = \pi/T$ when $f_0 = f_1 = \sqrt{\tfrac{1}{2}}$.

A linear equalizer with an infinite number of taps, adjusted on the basis of the MSE criterion, will have the minimum MSE given by (10-2-38). Evaluation of the integral in (10-2-38) for the $X(e^{j\omega T})$ given in (10-2-51) yields the result

$$\begin{aligned} J_{\min} &= \frac{N_0}{\sqrt{N_0^2 + 2N_0(|f_0|^2 + |f_1|^2) + (|f_0|^2 - |f_1|^2)^2}} \\ &= \frac{N_0}{\sqrt{N_0^2 + 2N_0 + (|f_0|^2 - |f_1|^2)^2}} \end{aligned} \tag{10-2-52}$$

Let us consider the special case in which $f_0 = f_1 = \sqrt{\tfrac{1}{2}}$. The minimum MSE is $J_{\min} = N_0/\sqrt{N_0^2 + 2N_0}$ and the corresponding output SNR is

$$\begin{aligned} \gamma_\infty &= \sqrt{1 + \frac{2}{N_0}} - 1 \\ &\approx \left(\frac{2}{N_0}\right)^{1/2}, \quad N_0 \ll 1 \end{aligned} \tag{10-2-53}$$

This result should be compared with the output SNR of $1/N_0$ obtained in the case of no intersymbol interference. A significant loss in SNR occurs from this channel.

Example 10-2-2

As a second example, we consider an exponentially decaying characteristic of the form

$$f_k = \sqrt{1-a^2}\, a^k, \qquad k = 0, 1, \ldots$$

where $a < 1$. The Fourier transform of this sequence is

$$X(e^{j\omega T}) = \frac{1-a^2}{1+a^2-2a\cos\omega T} \tag{10-2-54}$$

which is a function that contains a minimum at $\omega = \pi/T$.

The output SNR for this channel is

$$\begin{aligned} \gamma_\infty &= \left(\sqrt{1+2N_0\frac{1+a^2}{1-a^2}+N_0^2}-1\right)^{-1} \\ &\approx \frac{1-a^2}{(1+a^2)N_0}, \qquad N_0 \ll 1 \end{aligned} \tag{10-2-55}$$

Therefore, the loss in SNR due to the presence of the interference is

$$10\log_{10}\left(\frac{1-a^2}{1+a^2}\right)$$

Probability of Error Performance of Linear MSE Equalizer Above, we discussed the performance of the linear equalizer in terms of the minimum achievable MSE $J_{\min}$ and the output SNR γ that is related to $J_{\min}$ through the formula in (10-2-40). Unfortunately, there is no simple relationship between these quantities and the probability of error. The reason is that the linear MSE equalizer contains some residual intersymbol interference at its output. This situation is unlike that of the infinitely long zero-forcing equalizer, for which there is no residual interference, but only gaussian noise. The residual interference at the output of the MSE equalizer is not well characterized as an additional gaussian noise term, and, hence, the output SNR does not translate easily into an equivalent error probability.

One approach to computing the error probability is a brute force method that yields an exact result. To illustrate this method, let us consider a PAM signal in which the information symbols are selected from the set of values $2n - M - 1$, $n = 1, 2, \ldots, M$, with equal probability. Now consider the decision on the symbol I_n. The estimate of I_n is

$$\hat{I}_n = q_0 I_n + \sum_{k\neq n} I_k q_{n-k} + \sum_{j=-K}^{K} c_j \eta_{n-j} \tag{10-2-56}$$

where $\{q_n\}$ represent the convolution of the impulse response of the equalizer and equivalent channel, i.e.,

$$q_n = \sum_{k=-K}^{K} c_k f_{n-k} \tag{10-2-57}$$

and the input signal to the equalizer is

$$v_k = \sum_{j=0}^{L} f_j I_{k-j} + \eta_k \tag{10-2-58}$$

The first term in the right-hand side of (10-2-56) is the desired symbol, the middle term is the intersymbol interference, and the last term is the gaussian noise. The variance of the noise is

$$\sigma_n^2 = N_0 \sum_{j=-K}^{K} c_j^2 \tag{10-2-59}$$

For an equalizer with $2K + 1$ taps and a channel response that spans $L + 1$ symbols, the number of symbols involved in the intersymbol interference is $2K + L$.

Define

$$\mathscr{D} = \sum_{k \neq n} I_k q_{n-k} \tag{10-2-60}$$

For a particular sequence of $2K + L$ information symbols, say the sequence $\mathbf{I}_J$, the intersymbol interference term $\mathscr{D} \equiv D_J$ is fixed. The probability of error for a fixed D_J is

$$\begin{aligned} P_M(D_J) &= 2\frac{(M-1)}{M} P(N + D_J > q_0) \\ &= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{(q_0 - D_J)^2}{\sigma_n^2}}\right) \end{aligned} \tag{10-2-61}$$

where N denotes the additive noise term. The average probability of error is obtained by averaging $P_M(D_J)$ over all possible sequences $\mathbf{I}_J$. That is,

$$\begin{aligned} P_M &= \sum_{\mathbf{I}_J} P_M(D_J) P(\mathbf{I}_J) \\ &= \frac{2(M-1)}{M} \sum_{\mathbf{I}_J} Q\left(\sqrt{\frac{(q_0 - D_J)^2}{\sigma_n^2}}\right) P(\mathbf{I}_J) \end{aligned} \tag{10-2-62}$$

When all the sequences are equally likely,

$$P(\mathbf{I}_J) = \frac{1}{M^{2K+L}} \tag{10-2-63}$$

The conditional error probability terms $P_M(D_J)$ are dominated by the sequence that yields the largest value of D_J. This occurs when $I_n = \pm(M-1)$

and the signs of the information symbols match the signs of the corresponding $\{q_n\}$. Then,

$$D_J^* = (M-1)\sum_{k\neq 0}|q_k|$$

and

$$P_M(D_J^*) = \frac{2(M-1)}{M}Q\left(\sqrt{\frac{q_0^2}{\sigma_n^2}\left(1-\frac{M-1}{q_0}\sum_{k\neq 0}|q_k|\right)^2}\right) \qquad (10\text{-}2\text{-}64)$$

Thus, an upper bound on the average probability of error for equally likely symbol sequences is

$$P_M \leq P_M(D_J^*) \qquad (10\text{-}2\text{-}65)$$

If the computation of the exact error probability in (10-2-62) proves to be too cumbersome and too time consuming because of the large number of terms in the sum and if the upper bound is too loose, one can resort to one of a number of different approximate methods that have been devised, which are known to yield tight bounds on P_M. A discussion of these different approaches would take us too far afield. The interested reader is referred to the papers by Saltzberg (1968), Lugannani (1969), Ho and Yeh (1970), Shimbo and Celebiler (1971), Glave (1972), Yao (1972), and Yao and Tobin (1976).

As an illustration of the performance limitations of a linear equalizer in the presence of severe intersymbol interference, we show in Fig. 10-2-4 the probability of error for binary (antipodal) signaling, as measured by Monte Carlo simulation, for the three discrete-time channel characteristic shown in

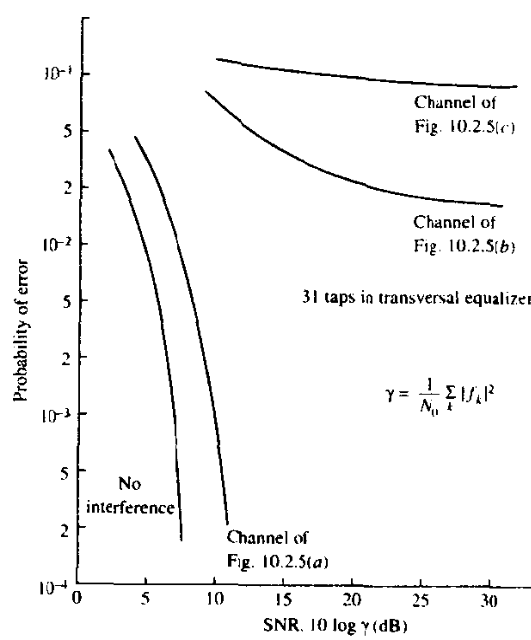


FIGURE 10-2-4 Error rate performance of linear MSE equalizer.

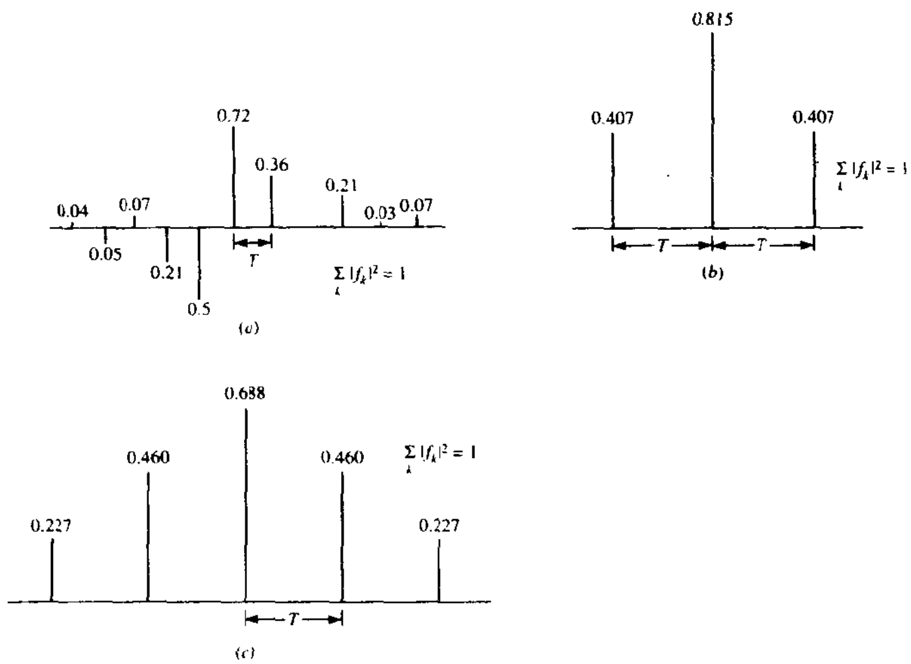


FIGURE 10-2-5 Three discrete-time channel characteristics.

Fig. 10-2-5. For purposes of comparison, the performance obtained for a channel with no intersymbol interference is also illustrated in Fig. 10-2-4. The equivalent discrete-time channel shown in Fig. 10-2-5(a) is typical of the response of a good quality telephone channel. In contrast, the equivalent discrete-time channel characteristics shown in Fig. 10-2-5(b) and (c) result in severe intersymbol interference. The spectral characteristics $|X(e^{j\omega})|$ for the three channels, illustrated in Fig. 10-2-6, clearly show that the channel in Fig. 10-2-5(c) has the worst spectral characteristic. Hence the performance of the linear equalizer for this channel is the poorest of the three cases. Next in performance is the channel shown in Fig. 10-2-5(b), and finally, the best performance is obtained with the channel shown in Fig. 10-2-5(a). In fact, the error rate of the latter is within 3 dB of the error rate achieved with no interference.

One conclusion reached from the results on output SNR γ_∞ and the limited probability of error results illustrated in Fig. 10-2-4 is that a linear equalizer yields good performance on channels such as telephone lines, where the spectral characteristics of the channels are well behaved and do not exhibit spectral nulls. On the other hand, a linear equalizer is inadequate as a compensator for the intersymbol interference on channels with spectral nulls, which may be encountered in radio transmission.

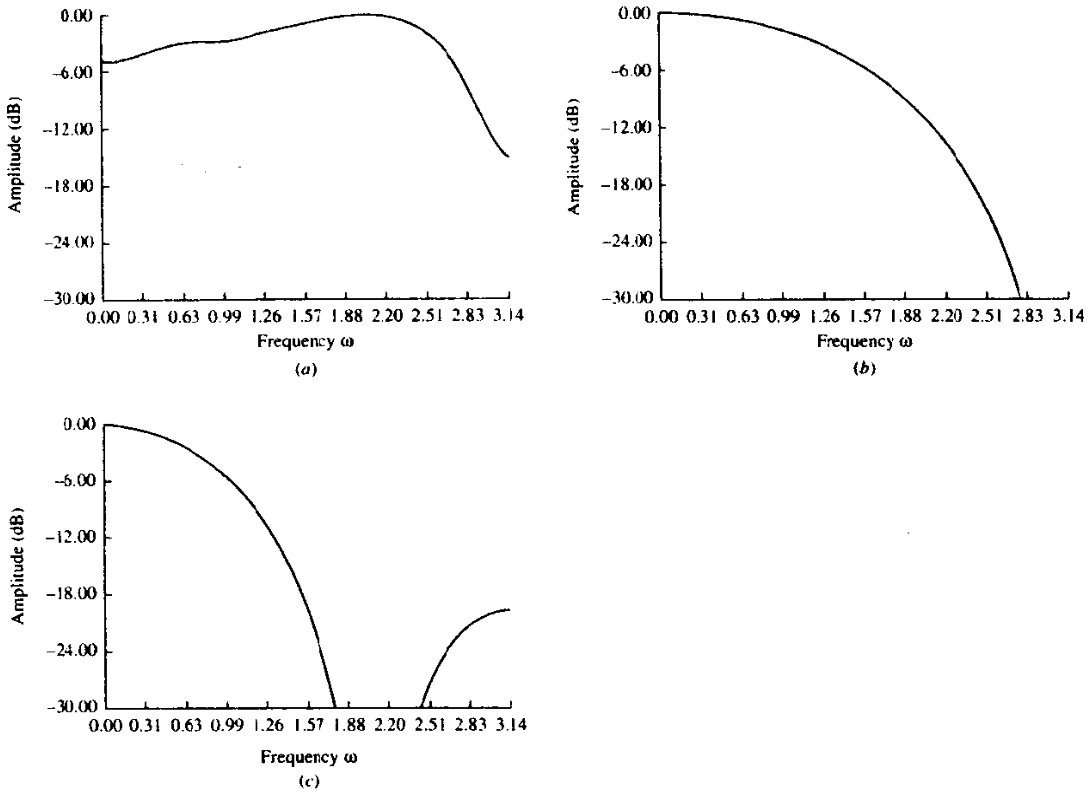


FIGURE 10-2-6 Amplitude spectra for the channels shown in Figs 10-2-5(*a*), (*b*), and (*c*), respectively.

The basic limitation of the linear equalizer to cope with severe ISI has motivated a considerable amount of research into nonlinear equalizers with low computational complexity. The decision-feedback equalizer described in Section 10-3 is shown to be an effective solution to this problem.

10-2-4 Fractionally Spaced Equalizers

In the linear equalizer structures that we have described in the previous section, the equalizer taps are spaced at the reciprocal of the symbol rate, i.e., at the reciprocal of the signaling rate $1/T$. This tap spacing is optimum if the equalizer is preceded by a filter matched to the channel distorted transmitted pulse. When the channel characteristics are unknown, the receiver filter is usually matched to the transmitted signal pulse and the sampling time is optimized for this suboptimum filter. In general, this approach leads to an equalizer performance that is very sensitive to the choice of sampling time.

The limitations of the symbol rate equalizer are most easily evident in the frequency domain. From (9-2-5), the spectrum of the signal at the input to the equalizer may be expressed as

$$Y_T(f) = \frac{1}{T}\sum_n X\left(f - \frac{n}{T}\right)e^{j2\pi(f-n/T)\tau_0} \tag{10-2-66}$$

where $Y_T(f)$ is the folded or aliased spectrum, where the folding frequency is $1/2T$. Note that the received signal spectrum is dependent on the choice of the sampling delay τ_0. The signal spectrum at the output of the equalizer is $C_T(f)Y_T(f)$, where

$$C_T(f) = \sum_{k=-K}^{K} c_k e^{-j2\pi fkT} \tag{10-2-67}$$

It is clear from these relationships that the symbol rate equalizer can only compensate for the frequency response characteristics of the aliased received signal. It cannot compensate for the channel distortion inherent in $X(f)e^{j2\pi f\tau_0}$.

In contrast to the symbol rate equalizer, a *fractionally spaced equalizer* (FSE) is based on sampling the incoming signal at least as fast as the Nyquist rate. For example, if the transmitted signal consists of pulses having a raised cosine spectrum with a roll-off factor β, its spectrum extends to $F_{\max} = (1+\beta)/2T$. This signal can be sampled at the receiver at a rate

$$2F_{\max} = \frac{1+\beta}{T} \tag{10-2-68}$$

and then passed through an equalizer with tap spacing of $T/(1+\beta)$. For example, if $\beta = 1$, we would have a $\frac{1}{2}T$-spaced equalizer. If $\beta = 0.5$, we would have a $\frac{2}{3}T$-spaced equalizer, and so forth. In general, then, a digitally implemented fractionally spaced equalizer has tap spacing of MT/N where M and N are integers and $N > M$. Usually, a $\frac{1}{2}T$-spaced equalizer is used in many applications.

Since the frequency response of the FSE is

$$C_{T'}(f) = \sum_{k=-K}^{K} c_k e^{-j2\pi fkT'} \tag{10-2-69}$$

where $T' = MT/N$, it follows that $C_{T'}(f)$ can equalize the received signal spectrum beyond the Nyquist frequency $f = 1/2T$ to $f = (1+\beta)/T = N/MT$ The equalized spectrum is

$$\begin{aligned} C_{T'}(f)Y_{T'}(f) &= C_{T'}(f)\sum_n X\left(f - \frac{n}{T'}\right)e^{j2\pi(f-n/T')\tau_0} \\ &= C_{T'}(f)\sum_n X\left(f - \frac{nN}{MT}\right)e^{j2\pi(f-nN/MT)\tau_0} \end{aligned} \tag{10-2-70}$$

Since $X(f) = 0$ for $|f| > N/MT$, (10-2-70) may be expressed as

$$C_{T'}(f)Y_{T'}(f) = C_{T'}(f)X(f)e^{j2\pi f\tau_0}, \qquad |f| \le \frac{1}{2T'} \tag{10-2-71}$$

Thus, we observe that the FSE compensates for the channel distortion in the received signal before the aliasing effects due to symbol rate sampling. In other words, $C_{T'}(f)$ can compensate for any arbitrary timing phase.

The FSE output is sampled at the symbol rate $1/T$ and has the spectrum

$$\sum_k C_{T'}\left(f-\frac{k}{T}\right)X\left(f-\frac{k}{T}\right)e^{j2\pi(f-k/T)\tau_0} \tag{10-2-72}$$

In effect, the optimum FSE is equivalent to the optimum linear receiver consisting of the matched filter followed by a symbol rate equalizer.

Let us now consider the adjustment of the tap coefficients in the FSE. The input to the FSE may be expressed as

$$y\left(\frac{kMT}{N}\right)=\sum_n I_n x\left(\frac{kMT}{N}-nT\right)+v\left(\frac{kMT}{N}\right) \tag{10-2-73}$$

In each symbol interval, the FSE produces an output of the form

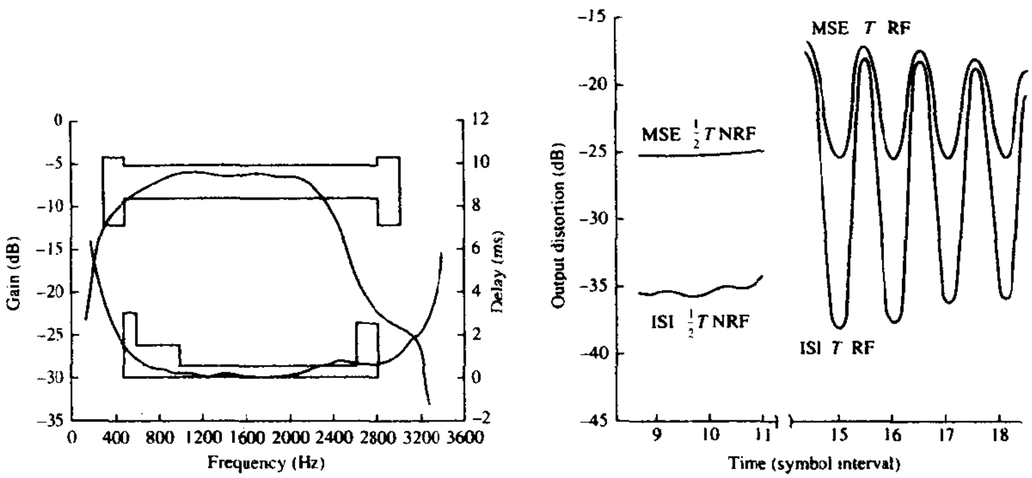
$$\hat{I}_k=\sum_{n=-K}^{K} c_n y\left(kT-\frac{nMT}{N}\right) \tag{10-2-74}$$

where the coefficients of the equalizer are selected to minimize the MSE. This optimization leads to a set of linear equations for the equalizer coefficients that have the solution

$$\mathbf{C}_{\text{opt}}=\mathbf{A}^{-1}\boldsymbol{\alpha} \tag{10-2-75}$$

where $\mathbf{A}$ is the covariance matrix of the input data and $\boldsymbol{\alpha}$ is the vector of cross-correlations. These equations are identical in form to those for the symbol rate equalizer, but there are some subtle differences. One is that $\mathbf{A}$ is Hermitian, but not Toeplitz. In addition, $\mathbf{A}$ exhibits periodicities that are inherent in a cyclostationary process, as shown by Qureshi (1985). As a result of the fractional spacing, some of the eigenvalues of $\mathbf{A}$ are nearly zero. Attempts have been made by Long *et al.* (1988a, b) to exploit this property in the coefficient adjustment.

An analysis of the performance of fractionally spaced equalizers, including their convergence properties, is given in a paper by Ungerboeck (1976). Simulation results demonstrating the effectiveness of the FSE over a symbol rate equalizer have also been given in the papers by Qureshi and Forney (1977) and Gitlin and Weinstein (1981). We cite two examples from these papers. First, Fig. 10-2-7 illustrates the performance of the symbol rate equalizer and a $\frac{1}{2}T$-FSE for a channel with high-end amplitude distortion, whose characteristics are also shown in this figure. The symbol-spaced equalizer was preceded with a filter matched to the transmitted pulse that had a (square-root) raised cosine spectrum with a 20% roll-off ($\beta=0.2$). The FSE did not have any filter preceding it. The symbol rate was 2400 symbols/s and the modulation was QAM. The received SNR was 30 dB. Both equalizers had 31 taps; hence, the $\frac{1}{2}T$-FSE spanned one-half of the time interval of the



(a) Channel with high-end amplitude distortion (HA)

(b) Equalizer performance

FIGURE 10-2-7 T and $\frac{1}{2}T$ equalizer performance as a function of timing phase for 2400 symbols per second. (NRF indicates no receiver filter.) [*From Qureshi and Forney (1977).* © *1977 IEEE.*]

symbol rate equalizer. Nevertheless, the FSE outperformed the symbol rate equalizer when the latter was optimized at the best sampling time. Furthermore, the FSE did not exhibit any sensitivity to timing phase, as illustrated in Fig. 10-2-7.

Similar results were obtained by Gitlin and Weinstein. For a channel with poor envelope delay characteristics, the SNR performance of the symbol rate equalizer and a $\frac{1}{2}T$-FSE are illustrated in Fig. 10-2-8. In this case, both equalizers had the same time span. The T-spaced equalizer had 24 taps while the FSE had 48 taps. The symbol rate was 2400 symbols/s and the data rate was 9600 bits/s with 16-QAM modulation. The signal pulse had a raised cosine spectrum with $\beta = 0.12$. Note again that the FSE outperformed the T-spaced equalizer by several decibels, even when the latter was adjusted for optimum

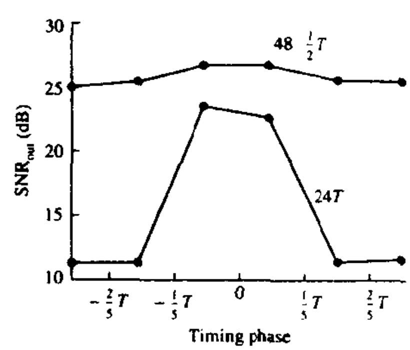


FIGURE 10-2-8 Performance of T and $\frac{1}{2}T$ equalizers as a function of timing phase for 2400 symbols/s 16-QAM on a channel with poor envelope delay. [*From Gitlin and Weinstein (1981). Reprinted with permission from* Bell System Technical Journal. *©1981 AT & T.*]

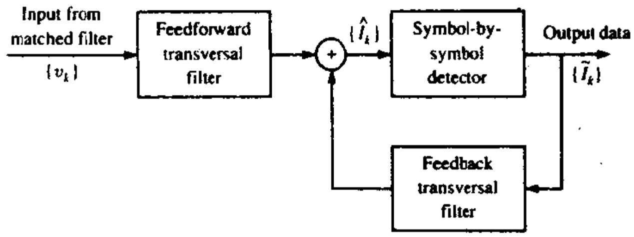


FIGURE 10-3-1 Structure of decision-feedback equalizer.

sampling. The results in these two papers clearly demonstrate the superior performance achieved with a fractionally spaced equalizer.

10-3 DECISION-FEEDBACK EQUALIZATION

The *decision-feedback equalizer* (DFE), depicted in Fig. 10-3-1, consists of two filters, a feedforward filter and a feedback filter. As shown, both have taps spaced at the symbol interval T. The input to the feedforward section is the received signal sequence $\{v_k\}$. In this respect, the feedforward filter is identical to the linear transversal equalizer described in Section 10-2. The feedback filter has as its input the sequence of decisions on previously detected symbols. Functionally, the feedback filter is used to remove that part of the intersymbol interference from the present estimate caused by previously detected symbols.

10-3-1 Coefficient Optimization

From the description given above, it follows that the equalizer output can be expressed as

$$\hat{I}_k = \sum_{j=-K_1}^{0} c_j v_{k-j} + \sum_{j=1}^{K_2} c_j \tilde{I}_{k-j} \tag{10-3-1}$$

where $\hat{I}_k$ is an estimate of the kth information symbol, $\{c_j\}$ are the tap coefficients of the filter, and $\{\tilde{I}_{k-1}, \ldots, \tilde{I}_{k-K_2}\}$ are previously detected symbols. The equalizer is assumed to have $(K_1 + 1)$ taps in its feedforward section and K_2 in its feedback section. It should be observed that this equalizer is nonlinear because the feedback filter contains previously detected symbols $\{\tilde{I}_k\}$.

Both the peak distortion criterior and the MSE criterion result in a mathematically tractable optimization of the equalizer coefficients, as can be concluded from the papers by George *et al.* (1971). Price (1972), Salz (1973), and Proakis (1975). Since the MSE criterion is more prevalent in practice, we focus our attention on it. Based on the assumption that previously detected symbols in the feedback filter are correct, the minimization of MSE

$$J(K_1, K_2) = E\,|I_k - \hat{I}_k|^2 \tag{10-3-2}$$

leads to the following set of linear equations for the coefficients of the feedforward filter:

$$\sum_{j=-K_1}^{0} \psi_{lj} c_j = f^*_{-l}, \qquad l = -K_1, \ldots, -1, 0 \tag{10-3-3}$$

where

$$\psi_{lj} = \sum_{m=0}^{-l} f^*_m f_{m+l-j} + N_0 \delta_{lj}, \qquad l, j = -K_1, \ldots, -1, 0 \tag{10-3-4}$$

The coefficients of the feedback filter of the equalizer are given in terms of the coefficients of the feedforward section by the following expression:

$$c_k = -\sum_{j=-K_1}^{0} c_j f_{k-j}, \qquad k = 1, 2, \ldots, K_2 \tag{10-3-5}$$

The values of the feedback coefficients result in complete elimination of intersymbol interference from previously detected symbols, provided that previous decisions are correct and that $K_2 \geq L$ (see Problem 10-9).

10-3-2 Performance Characteristics of DFE

We now turn our attention to the performance achieved with decision-feedback equalization. The exact evaluation of the performance is complicated to some extent by occasional incorrect decisions made by the detector, which then propagate down the feedback section. In the absence of decision errors, the minimum MSE is given as

$$J_{\min}(K_1) = 1 - \sum_{j=-K_1}^{0} c_j f_{-j} \tag{10-3-6}$$

By going to the limit $(K_1 \to \infty)$ of an infinite number of taps in the feedforward filter, we obtain the smallest achievable MSE, denoted as $J_{\min}$. With some effort $J_{\min}$ can be expressed in terms of the spectral characteristics of the channel and additive noise, as shown by Salz (1973). This more desirable form for $J_{\min}$ is

$$J_{\min} = \exp\left\{\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \ln\left[\frac{N_0}{X(e^{j\omega T}) + N_0}\right] d\omega\right\} \tag{10-3-7}$$

The corresponding output SNR is

$$\begin{aligned}\gamma_\infty &= \frac{1 - J_{\min}}{J_{\min}} \\ &= -1 + \exp\left\{\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \ln\left[\frac{N_0 + X(e^{j\omega T})}{N_0}\right] d\omega\right\}\end{aligned} \tag{10-3-8}$$

We observe again that, in the absence of intersymbol interference, $X(e^{j\omega T}) = 1$ and, hence, $J_{\min} = N_0/(1 + N_0)$. The corresponding output SNR is $\gamma_\infty = 1/N_0$.

Example 10-3-1

It is interesting to compare the value of J_{min} for the decision-feedback equalizer with the value of J_{min} obtained with the linear MSE equalizer. For example, ler us consider the discrete-time equivalent channel consisting of two taps f_0 and f_1. The minimum MSE for this channel is

$$\begin{aligned} J_{min} &= \exp\left\{\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \ln\left[\frac{N_0}{1+N_0+2\,|f_0|\,|f_1|\cos(\omega T+\theta)}\right] d\omega\right\} \\ &= N_0 \exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(1+N_0+2\,|f_0|\,|f_1|\cos\omega)\,d\omega\right] \\ &= \frac{2N_0}{1+N_0+\sqrt{(1+N_0)^2-4\,|f_0 f_1|^2}} \end{aligned} \tag{10-3-9}$$

Note that J_{min} is maximized when $|f_0| = |f_1| = \sqrt{\tfrac{1}{2}}$. Then

$$\begin{aligned} J_{min} &= \frac{2N_0}{1+N_0+\sqrt{(1+N_0)^2-1}} \\ &\approx 2N_0, \qquad N_0 \ll 1 \end{aligned} \tag{10-3-10}$$

The corresponding output SNR is

$$\gamma_\infty \approx \frac{1}{2N_0}, \qquad N_0 \ll 1 \tag{10-3-11}$$

Therefore, there is a 3 dB degradation in output SNR due to the presence of intersymbol interference. In comparison, the performance loss for the linear equalizer is very severe. Its output SNR as given by (10-2-53) is $\gamma_\infty \approx (2/N_0)^{1/2}$ for $N_0 \ll 1$.

Example 10-3-2

Consider the exponentially decaying channel characteristic of the form

$$f_k = (1-a^2)^{1/2} a^k, \qquad k = 0, 1, 2, \ldots \tag{10-3-12}$$

where $a < 1$. The output SNR of the decision-feedback equalizer is

$$\begin{aligned} \gamma_\infty &= -1 + \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left[\frac{1+a^2+(1-a^2)/N_0 - 2a\cos\omega}{1+a^2-2a\cos\omega}\right] d\omega\right\} \\ &= -1 + \frac{1}{2N_0}\{1-a^2+N_0(1+a^2)+\sqrt{[1-a^2+N_0(1+a^2)]^2-4a^2N_0^2}\} \\ &\approx \frac{(1-a^2)[1+N_0(1+a^2)/(1-a^2)]-N_0}{N_0} \\ &\approx \frac{1-a^2}{N_0}, \qquad N_0 \ll 1 \end{aligned} \tag{10-3-13}$$

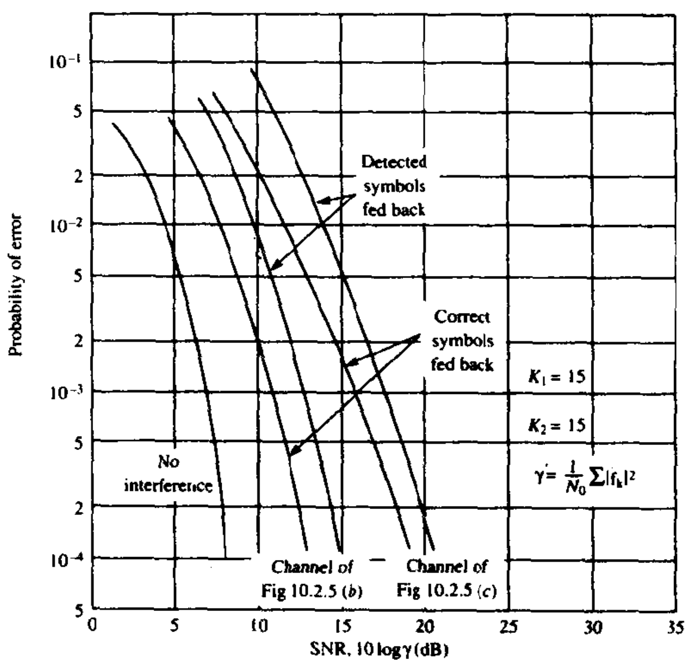
Thus, the loss in SNR is $10 \log_{10}(1 - a^2)$ dB. In comparison, the linear equalizer has a loss of $10 \log_{10}[(1 - a^2)/(1 + a^2)]$ dB.

These results illustrate the superiority of the decision-feedback equalizer over the linear equalizer when the effect of decision errors on performance is neglected. It is apparent that a considerable gain in performance can be achieved relative to the linear equalizer by the inclusion of the decision-feedback section, which eliminates the intersymbol interference from previously detected symbols.

One method of assessing the effect of decision errors on the error rate performance of the decision-feedback equalizer is Monte Carlo simulation on a digital computer. For purposes of illustration, we offer the following results for binary PAM signaling through the equivalent discrete-time channel models shown in Figs 10-2-5(b) and (c).

The results of the simulation are displayed in Fig. 10-3-2. First of all, a comparison of these results with those presented in Fig. 10-2-4 leads us to conclude that the decision-feedback equalizer yields a significant improvement in performance relative to the linear equalizer having the same number of taps. Second, these results indicate that there is still a significant degradation in performance of the decision-feedback equalizer due to the residual intersymbol interference, especially on channels with severe distortion such as the one

FIGURE 10-3-2 *Performance of decision-feedback equalizer with and without error propagation.*

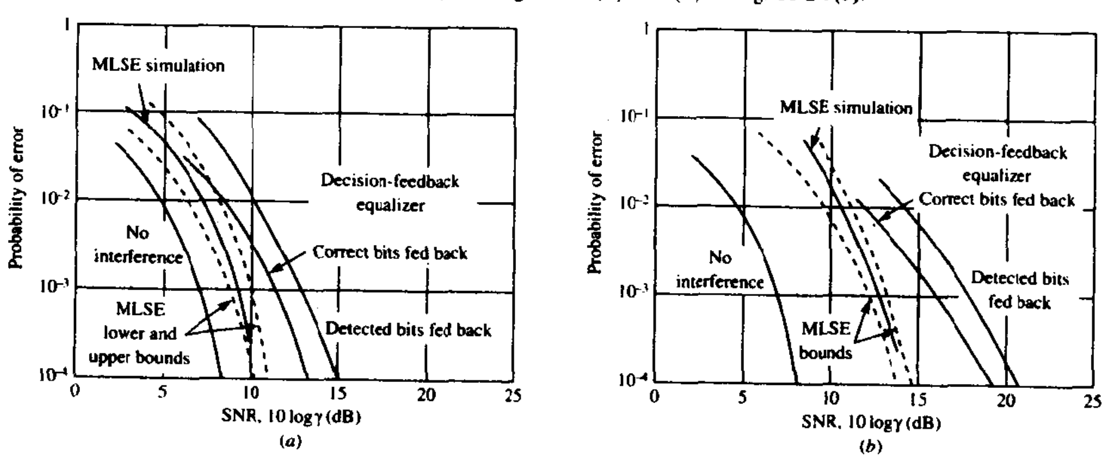


shown in Fig. 10-2-5(c). Finally, the performance loss due to incorrect decisions being fed back is 2 dB, approximately, for the channel responses under consideration. Additional results on the probability of error for a decision-feedback equalizer with error propagation may be found in the papers by Duttweiler *et al.* (1974) and Beaulieu (1992).

The structure of the DFE that is analyzed above employs a T-spaced filter for the feedforward section. The optimality of such a structure is based on the assumption that the analog filter preceding the DFE is matched to the channel-corrupted pulse response and its output is sampled at the optimum time instant. In practice, the channel response is not known a priori, so it is not possible to design an ideal matched filter. In view of this difficulty, it is customary in practical applications to use a fractionally spaced feedforward filter. Of course, the feedback filter tap spacing remains at T. The use of the FSE for the feedforward filter eliminates the system sensitivity to a timing error.

Performance Comparison with MLSE We conclude this subsection on the performance of the DFE by comparing its performance against that of MLSE. For the two-path channel with $f_0 = f_1 = \sqrt{\frac{1}{2}}$, we have shown that MLSE suffers no SNR loss while the decision-feedback equalizer suffers a 3 dB loss. On channels with more distortion, the SNR advantage of MLSE over decision-feedback equalization is even greater. Figure 10-3-3 illustrates a comparison of the error rate performance of these two equalization techniques, obtained via Monte Carlo simulation, for binary PAM and the channel characteristics shown in Figs 10-2-5(b) and (c). The error rate curves for the two methods have different slopes; hence the difference in SNR increases as the error

FIGURE 10-3-3 Comparison of performance between MLSE and decision-feedback equalization for channel characteristics shown (a) in Fig. 10-2-5(b) and (b) in Fig. 10-2-5(c).



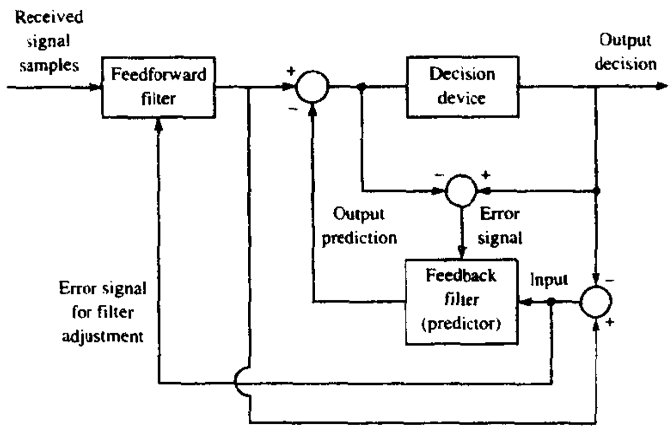


FIGURE 10-3-4 Block diagram of predictive DFE.

probability decreases. As a benchmark, the error rate for the AWGN channel with no intersymbol interference is also shown in Fig. 10-3-3.

10-3-3 Predictive Decision-Feedback Equalizer

Belfiore and Park (1979) proposed another DFE structure that is equivalent to the one shown in Fig. 10-3-1 under the condition that the feedforward filter has an infinite number of taps. This structure consists of a FSE as a feedforward filter and a linear predictor as a feedback filter, as shown in the configuration given in Fig. 10-3-4. Let us briefly consider the performance characteristics of this equalizer.

First of all, the noise at the output of the infinite length feedforward filter has the power spectral density

$$\frac{N_0 X(e^{j\omega T})}{|N_0 + X(e^{j\omega T})|^2}, \qquad |\omega| \le \frac{\pi}{T} \tag{10-3-14}$$

The residual intersymbol interference has the power spectral density

$$\left|1 - \frac{X(e^{j\omega T})}{N_0 + X(e^{j\omega T})}\right|^2 = \frac{N_0^2}{|N_0 + X(e^{j\omega T})|^2}, \qquad |\omega| \le \frac{\pi}{T} \tag{10-3-15}$$

The sum of these two spectra represents the power spectral density of the total noise and intersymbol interference at the output of the feedforward filter. Thus, on adding (10-3-14) and (10-3-15), we obtain

$$E(\omega) = \frac{N_0}{N_0 + X(e^{j\omega T})}, \qquad |\omega| \le \frac{\pi}{T} \tag{10-3-16}$$

As we have observed previously, if $X(e^{j\omega T}) = 1$, the channel is ideal and, hence, it is not possible to reduce the MSE any further. On the other hand, if there is channel distortion, the power in the error sequence at the output of the feedforward filter can be reduced by means of linear prediction based on past values of the error sequence.

If $\mathscr{B}(\omega)$ represents the frequency response of the infinite length feedback predictor, i.e.,

$$\mathscr{B}(\omega) = \sum_{n=1}^{\infty} b_n e^{-j\omega nT} \tag{10-3-17}$$

then the error at the output of the predictor is

$$E(\omega) - E(\omega)\mathscr{B}(\omega) = E(\omega)[1 - \mathscr{B}(\omega)] \tag{10-3-18}$$

The minimization of the mean square value of this error, i.e.,

$$J = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} |1 - \mathscr{B}(\omega)|^2 |E(\omega)|^2 \, d\omega \tag{10-3-19}$$

over the predictor coefficients $\{b_n\}$ yields the optimum predictor in the form

$$\mathscr{B}(\omega) = 1 - \frac{G(\omega)}{g_0} \tag{10-3-20}$$

where $G(\omega)$ is the solution to the spectral factorization

$$G(\omega)G^*(-\omega) = \frac{1}{|E(\omega)|^2} \tag{10-3-21}$$

and

$$G(\omega) = \sum_{n=0}^{\infty} g_n e^{-j\omega nT} \tag{10-3-22}$$

The output of the infinite length linear predictor is a white noise sequence with power spectral density $1/g_0^2$ and the corresponding minimum MSE is given by (10-3-7). Therefore, the MSE performance of the infinite-length predictive DFE is identical to the conventional DFE.

Although these two DFE structures result in equivalent performance if their lengths are infinite, the predictive DFE is suboptimum if the lengths of the two filters are finite. The reason for the optimality of the conventional DFE is relatively simple. The optimization of its tap coefficients in the feedforward and feedback filters is done jointly. Hence, it yields the minimum MSE. On the other hand, the optimizations of the feedforward filter and the feedback predictor in the predictive DFE are done separately. Hence, its MSE is at least as large as that of the conventional DFE. In spite of this suboptimality of the predictive DFE, it is suitable as an equalizer for trellis-coded signals, where the conventional DFE is not as suitable, as described in the next chapter.

10-4 BIBLIOGRAPHICAL NOTES AND REFERENCES

Channel equalization for digital communications was developed by Lucky (1965, 1966), who focused on linear equalizers that were optimized using the peak distortion criterion. The mean square error criterion for optimization of the equalizer coefficients was proposed by Widrow (1966).

Decision-feedback equalization was proposed and analyzed by Austin (1967). Analyses of the performance of the DFE can be found in the papers by Monsen (1971), George *et al.* (1971), Price (1972), Salz (1973), Duttweiler *et al.* (1974), and Altekar and Beaulieu (1993).

The use of the Viterbi algorithm as the optimal maximum-likelihood sequence estimator for symbols corrupted by ISI was proposed and analyzed by Forney (1972) and Omura (1971). Its use for carrier-modulated signals was considered by Ungerboeck (1974) and MacKenchnie (1973).

PROBLEMS

10-1 In a binary PAM system, the input to the detector is

$$y_m = a_m + n_m + i_m$$

where $a_m = \pm 1$ is the desired signal, n_m is a zero-mean Gaussian random variable with variance σ_n^2, and i_m represents the ISI due to channel distortion. The ISI term is a random variable that takes the values $-\frac{1}{2}$, 0, and $\frac{1}{2}$ with probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, respectively. Determine the average probability of error as a function of σ_n^2.

10-2 In a binary PAM system, the clock that specifies the sampling of the correlator output is offset from the optimum sampling time by 10%.

a If the signal pulse used is rectangular, determine the loss in SNR due to the mistiming.

b Determine the amount of ISI introduced by the mistiming and determine its effect on performance.

10-3 The frequency response characteristic of a lowpass channel can be approximated by

$$H(f) = \begin{cases} 1 + \alpha \cos 2\pi f t_0 & (|\alpha| < 1,\ |f| \le W) \\ 0 & (\text{otherwise}) \end{cases}$$

where W is the channel bandwidth. An input signal $s(t)$ whose spectrum is bandlimited to W Hz is passed through the channel.

a Show that

$$y(t) = s(t) + \tfrac{1}{2}\alpha[s(t - t_0) + s(t + t_0)]$$

Thus, the channel produces a pair of echoes.

b Suppose that the received signal $y(t)$ is passed through a filter matched to $s(t)$. Determine the output of the matched filter at $t = kT$, $k = 0, \pm 1, \pm 2, \ldots$, where T is the symbol duration.

c What is the ISI pattern resulting from the channel if $t_0 = T$?

10-4 A wireline channel of length 1000 km is used to transmit data by means of binary

PAM. Regenerative repeaters are spaced 50 km apart along the system. Each segment of the channel has an ideal (constant) frequency response over the frequency band $0 \leq f \leq 1200$ Hz and an attenuation of 1 dB/km. The channel noise is AWGN.

a What is the highest bit rate that can be transmitted without ISI?

b Determine the required $\mathscr{E}_b/N_0$ to achieve a bit error of $P_2 = 10^{-7}$ for each repeater.

c Determine the transmitted power at each repeater to achieve the desired $\mathscr{E}_b/N_0$, where $N_0 = 4.1 \times 10^{-21}$ W/Hz.

10-5 Prove the relationship in (10-1-13) for the autocorrelation of the noise at the output of the matched filter.

10-6 In the case of PAM with correlated noise, the correlation metrics in the Viterbi algorithm may be expressed in general as (Ungerboeck, 1974)

$$CM(\mathbf{I}) = 2\sum_n I_n r_n - \sum_n \sum_m I_n I_m x_{n-m}$$

where $x_n = x(nT)$ is the sampled signal output of the matched filter, $\{I_n\}$ is the data sequence, and $\{r_n\}$ is the received signal sequence at the output of the matched filter. Determine the metric for the duobinary signal.

10-7 Consider the use of a (square-root) raised cosine signal pulse with a roll-off factor of unity for transmission of binary PAM over an ideal bandlimited channel that passes the pulse without distortion. Thus, the transmitted signal is

$$v(t) = \sum_{k=-\infty}^{\infty} I_k g_T(t - kT_b)$$

where the signal interval $T_b = \frac{1}{2}T$. Thus, the symbol rate is double of that for no ISI.

a Determine the ISI values at the output of a matched filter demodulator.

b Sketch the trellis for the maximum-likelihood sequence detector and label the states.

10-8 A binary antipodal signal is transmitted over a nonideal band-limited channel, which introduces ISI over two adjacent symbols. For an isolated transmitted signal pulse $s(t)$, the (noise-free) output of the demodulator is $\sqrt{\mathscr{E}_b}$ at $t = T$, $\sqrt{\mathscr{E}_b}/4$ at $t = 2T$, and zero for $t = kT$, $k > 2$, where $\mathscr{E}_b$ is the signal energy and T is the signaling interval.

a Determine the average probability of error, assuming that the two signals are equally probable and the additive noise is white and gaussian.

b By plotting the error probability obtained in (a) and that for the case of no ISI, determine the relative difference in SNR of the error probability of 10^{-6}.

10-9 Derive the expression in (10-3-5) for the coefficients in the feedback filter of the DFE.

10-10 Binary PAM is used to transmit information over an unequalized linear filter channel. When $a = 1$ is transmitted, the noise-free output of the demodulator is

$$x_m = \begin{cases} 0.3 & (m = 1) \\ 0.9 & (m = 0) \\ 0.3 & (m = -1) \\ 0 & (\text{otherwise}) \end{cases}$$

a Design a three-tap zero-forcing linear equalizer so that the output is

$$q_m = \begin{cases} 1 & (m = 0) \\ 0 & (m = \pm 1) \end{cases}$$

b Determine q_m for $m = \pm 2, \pm 3$, by convolving the impulse response of the equalizer with the channel response.

10-11 The transmission of a signal pulse with a raised cosine spectrum through a channel results in the following (noise-free) sampled output from the demodulator:

$$x_k = \begin{cases} -0.5 & (k = -2) \\ 0.1 & (k = -1) \\ 1 & (k = 0) \\ -0.2 & (k = 1) \\ 0.05 & (k = 2) \\ 0 & (\text{otherwise}) \end{cases}$$

a Determine the tap coefficients of a three-tap linear equalizer based on the zero-forcing criterion.

b For the coefficients determined in (a), determine the output of the equalizer for the case of the isolated pulse. Thus, determine the residual ISI and its span in time.

10-12 A nonideal band-limited channel introduces ISI over three successive symbols. The (noise-free) response of the matched filter demodulator sampled at the sampling time kT is

$$\int_{-\infty}^{\infty} s(t)s(t - kT)\,dt = \begin{cases} \mathscr{E}_b & (k = 0) \\ 0.9\mathscr{E}_b & (k = \pm 1) \\ 0.1\mathscr{E}_b & (k = \pm 2) \\ 0 & (\text{otherwise}) \end{cases}$$

a Determine the tap coefficients of a three-tap linear equalizer that equalizes the channel (received signal) response to an equivalent partial response (duobinary) signal

$$y_k = \begin{cases} \mathscr{E}_b & (k = 0, 1) \\ 0 & (\text{otherwise}) \end{cases}$$

b Suppose that the linear equalizer in (a) is followed by a Viterbi sequence detector for the partial signal. Give an estimate of the error probability if the additive noise is white and gaussian, with power spectral density $\frac{1}{2}N_0$ W/Hz.

***10-13** Determine the tap weight coefficients of a three-tap zero-forcing equalizer if the ISI spans three symbols and is characterized by the values $x(0) = 1, x(-1) = 0.3, x(1) = 0.2$. Also determine the residual ISI at the output of the equalizer for the optimum tap coefficients.

10-14 In line-of-sight microwave radio transmission, the signal arrives at the receiver via two propagation paths: the direct path and a delayed path that occurs due to signal reflection from surrounding terrain. Suppose that the received signal has the form

$$r(t) = s(t) + \alpha s(t - T) + n(t)$$

where $s(t)$ is the transmitted signal, α is the attenuation ($\alpha < 1$) of the secondary path and $n(t)$ is AWGN.

a Determine the output of the demodulator at $t = T$ and $t = 2T$ that employs a filter matched to $s(t)$.

b Determine the probability of error for a symbol-by-symbol detector if the transmitted signal is binary antipodal and the detector ignores the ISI.

c What is the error-rate performance of a simple (one-tap) DFE that estimates α and removes the ISI? Sketch the detector structure that employs a DFE.

10-15 Repeat Problem 10-10 using the MMSE as the criterion for optimizing the tap coefficients. Assume that the noise power spectral density is 0.1 W/Hz.

10-16 In a magnetic recording channel, where the readback pulse resulting from a positive transition in the write current has the form

$$p(t) = \left[1 + \left(\frac{2t}{T_{50}}\right)^2\right]^{-1}$$

a linear equalizer is used to equalize the pulse to a partial response. The parameter T_{50} is defined as the width of the pulse at the 50% amplitude level. The bit rate is $1/T_b$ and the ratio of $T_{50}/T_b = \Delta$ is the normalized density of the recording. Suppose the pulse is equalized to the partial-response values

$$x(nT) = \begin{cases} 1 & (n = -1, 1) \\ 2 & (n = 0) \\ 0 & (\text{otherwise}) \end{cases}$$

where $x(t)$ represents the equalized pulse shape.

a Determine the spectrum $X(f)$ of the band-limited equalized pulse.

b Determine the possible output levels at the detector, assuming that successive transitions can occur at the rate $1/T_b$.

c Determine the error rate performance of the symbol-by-symbol detector for this signal, assuming that the additive noise is zero-mean gaussian with variance σ^2.

10-17 Sketch the trellis for the Viterbi detector of the equalized signal in Problem 10-16 and label all the states. Also, determine the minimum euclidean distance between merging paths.

10-18 Consider the problem of equalizing the discrete-time equivalent channel shown in Fig. P10-18. The information sequence $\{I_n\}$ is binary (± 1) and uncorrelated.

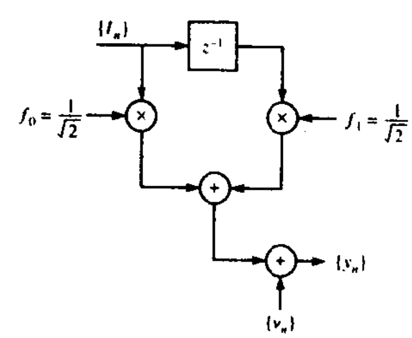


FIGURE P10-18

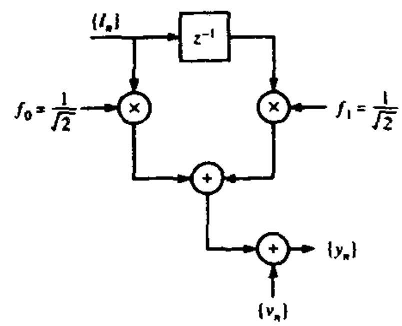


FIGURE P10-21

The additive noise $\{v_n\}$ is white and real-valued, with variance N_0. The received sequence $\{y_n\}$ is processed by a linear three-tap equalizer that is optimized on the basis of the MSE criterion.

a Determine the optimum coefficients of the equalizer as a function of N_0.

b Determine the three eigenvalues λ_1, λ_2, and λ_3 of the covariance matrix Γ and the corresponding (normalized to unit length) eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

c Determine the minimum MSE for the three-tap equalizer as a function of N_0.

d Determine the output SNR for the three-tap equalizer as a function of N_0. How does this compare with the output SNR for the infinite-tap equalizer? For example, evaluate the output SNR for these two equalizers when $N_0 = 0.1$.

10-19 Use the orthogonality principle to derive the equations for the coefficients in a decision-feedback equalizer based on the MSE criterion and given by (10-3-3) and (10-3-5).

10-20 Suppose that the discrete-time model for the intersymbol interference is characterized by the tap coefficients $f_0, f_1, \ldots, f_L$. From the equations for the tap coefficients of a decision-feedback equalizer (DFE), show that only L taps are needed in the feedback filter of the DFE. That is, if $\{c_k\}$ are the coefficients of the feedback filter then $c_k = 0$ for $k \geq L + 1$.

10-21 Consider the channel model shown in Fig. P10-21. $\{v_n\}$ is a real-valued white-noise sequence with zero mean and variance N_0. Suppose the channel is to be equalized by DFE having a two-tap feedforward filter (c_0, c_{-1}) and a one-tap feedback filter (c_1). The $\{c_i\}$ are optimized using the MSE criterion.

a Determine the optimum coefficients and their approximate values for $N_0 \ll 1$.

b Determine the exact value of the minimum MSE and a first-order approximation appropriate to the case $N_0 \ll 1$.

c Determine the exact value of the output SNR for the three-tap equalizer as a function of N_0 and a first-order approximation appropriate to the case $N_0 \ll 1$.

d Compare the results in (b) and (c) with the performance of the infinite-tap DFE.

e Evaluate and compare the exact values of the output SNR for the three-tap and infinite-tap DFE in the special cases where $N_0 = 0.1$ and 0.01. Comment on how well the three-tap equalizer performs relative to the infinite-tap equalizer.

10-22 A pulse and its (raised-cosine) spectral characteristic are shown in Fig. P10-22. This pulse is used for transmitting digital information over a band-limited channel at a rate $1/T$ symbols/s.

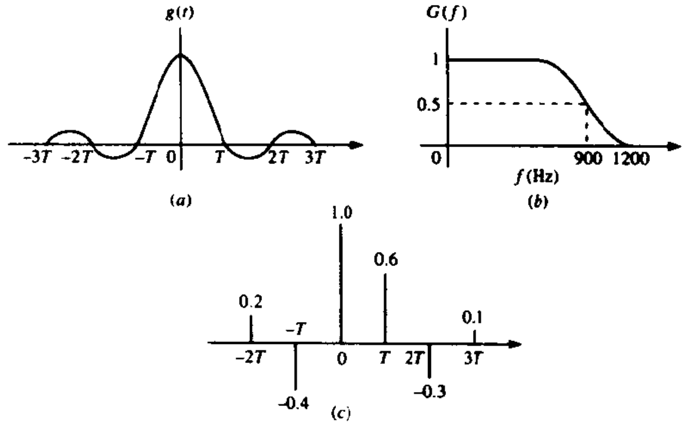


FIGURE P10-22

a What is the roll-off factor β?

b What is the pulse rate?

c The channel distorts the signal pulses. Suppose the sampled values of the filtered received pulse $x(t)$ are as shown in Fig. P10-22(c) It is obvious that there are five interfering signal components. Give the sequence of +1s and −1s that will cause the largest (destructive or constructive) interference and the corresponding value of the interference (the peak distortion).

d What is the probability of occurrence of the worst sequence obtained in (c), assuming that all binary digits are equally probable and independent?

10-23 A time-dispersive channel having an impulse response $h(t)$ is used to transmit four-phase PSK at a rate $R = 1/T$ symbols/s. The equivalent discrete-time channel is shown in Fig. P10-23. The sequence $\{\eta_k\}$ is a white noise sequence having zero mean and variance $\sigma^2 = N_0$.

a What is the sampled autocorrelation function sequence $\{x_k\}$ defined by

$$x_k = \int_{-\infty}^{\infty} h^*(t)h(t + kT)\, dt$$

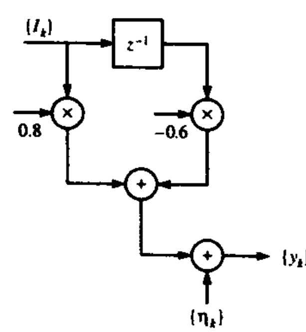


FIGURE P10-23

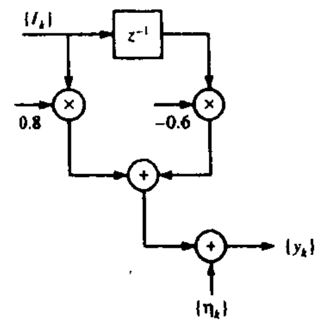


FIGURE P10-24

for this channel?

b The minimum MSE performance of a linear equalizer and a decision-feedback equalizer having an infinite number of taps depends on the *folded spectrum of the channel*

$$\frac{1}{T}\sum_{n=-\infty}^{\infty}\left|H\left(\omega+\frac{2\pi n}{T}\right)\right|^2$$

where $H(\omega)$ is the Fourier transform of $h(t)$. Determine the folded spectrum of the channel given above.

c Use your answer in (b) to express the minimum MSE of a linear equalizer in terms of the folded spectrum of the channel. (You may leave your answer in integral form.)

d Repeat (c) for an infinite-tap decision-feedback equalizer.

10-24 Consider a four-level PAM system with possible transmitted levels, 3, 1, −1, and −3. The channel through which the data are transmitted introduces intersymbol interference over two successive symbols. The equivalent discrete-time channel model is shown in Fig. P10-24. $\{\eta_k\}$ is a sequence of real-valued independent zero-mean gaussian noise variables with variance $\sigma^2 = N_0$. The received sequence is

$$y_1 = 0.8I_1 + n_1$$
$$y_2 = 0.8I_2 - 0.6I_1 + n_2$$
$$y_3 = 0.8I_3 - 0.6I_2 + n_3$$
$$\vdots$$
$$y_k = 0.8I_k - 0.6I_{k-1} + n_k$$

a Sketch the tree structure, showing the possible signal sequences for the received signals y_1, y_2 and y_3.

b Suppose the Viterbi algorithm is used to detect the information sequence. How many probabilities must be computed at each stage of the algorithm?

c How many surviving sequences are there in the Viterbi algorithm for this channel?

d Suppose that the received signals are

$$y_1 = 0.5, \quad y_2 = 2.0, \quad y_3 = -1.0$$

Determine the surviving sequences through stage y_3 and the corresponding metrics.

e Give a tight upper bound for the probability of error for four-level PAM transmitted over this channel.

10-25 A transversal equalizer with K taps has an impulse response

$$e(t) = \sum_{k=0}^{K-1} c_k \delta(t - kT)$$

where T is the delay between adjacent taps, and a transfer function

$$E(z) = \sum_{k=0}^{K-1} c_k z^{-k}$$

The *discrete Fourier transform* (DFT) of the equalizer coefficients $\{c_k\}$ is defined as

$$E_n \equiv E(z)|_{z=e^{j2\pi n/K}} = \sum_{k=0}^{K-1} c_k e^{-j2\pi kn/K}, \qquad n = 0, 1, \ldots, K-1$$

The *inverse DFT* is defined as

$$b_k = \frac{1}{K} \sum_{n=0}^{K-1} E_n e^{j2\pi nk/K}, \qquad k = 0, 1, \ldots, K-1$$

a Show that $b_k = c_k$, by substituting for E_n in the above expression.

b From the relations given above, derive an equivalent filter structure having the z transform

$$E(z) = \underbrace{\frac{1 - z^{-K}}{K}}_{E_1(z)} \sum_{n=0}^{K-1} \underbrace{\frac{E_n}{1 - e^{j2\pi n/K} z^{-1}}}_{E_2(z)}$$

c If $E(z)$ is considered as two separate filters $E_1(z)$ and $E_2(z)$ in cascade, sketch a block diagram for each of the filters, using z^{-1} to denote a unit of delay.

d In the transversal equalizer, the adjustable parameters are the equalizer coefficients $\{c_k\}$. What are the adjustable parameters of the equivalent equalizer in (b), and how are they related to $\{c_k\}$?