

COMMUNICATION

SYSTEMS

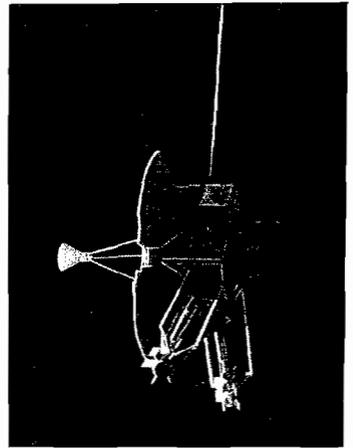
Haykin

4th Ed

COMMUNICATION SYSTEMS

4th Edition

Simon Haykin



Google
Exhibit 1011



COMMUNICATION SYSTEMS

4TH EDITION

COMMUNICATION SYSTEMS

4TH EDITION

Simon Haykin

McMaster University



JOHN WILEY & SONS, INC.

New York ■ Chichester ■ Weinheim ■ Brisbane ■ Singapore ■ Toronto

Editor
Marketing Manager
Associate Production Director
Senior Production Editor
Cover Designer
Illustration Coordinator
Illustration Studio

Bill Zobrist
Katherine Hepburn
Lucille Buonocore
Monique Calello
Madelyn Lesure
Gene Aiello
Wellington Studios

Cover Photo NASA/Photo Researchers, Inc.

This book was set in 10/12 Times Roman by UG / GGS Information Services, Inc. and printed and bound by Hamilton Printing Company. The cover was printed by Phoenix Color Corporation.

This book is printed on acid-free paper. 

The paper in this book was manufactured by a mill whose forest management programs include sustained yield harvesting of its timberlands. Sustained yield harvesting principles ensure that the numbers of trees cut each year does not exceed the amount of new growth.

Copyright © 2001, John Wiley & Sons, Inc. All rights Reserved.

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, except as permitted under Sections 107 or 1089 of the 1976 United States Copyright Act, without either the prior written permission of the Publisher, or authorization through payment of the appropriate per-copy fee to the Copyright Clearance Center, 222 Rosewood Drive, Danvers, MA 01923, (508) 750-8400, fax (508) 750-4470. Requests to the Publisher for permission should be addressed to the Permissions Department, John Wiley & Sons, Inc., 605 Third Avenue, New York, NY 10158-0012, (212) 850-6011, fax (212) 850-6008, E-Mail: PERMREQ@WILEY.COM.

To order books or for customer service call 1-800-CALL-WILEY (225-5945).

Library of Congress Cataloging-in-Publication Data

Haykin, Simon

Communication systems / Simon Haykin.—4th ed.

p. cm.

ISBN 0-471-17869-1 (cloth : alk. paper)

1. Telecommunication. 2. Signal theory (Telecommunication) I. Title.

TK5101 .H37 2000

621.382—dc21

99-042977

Printed in the United States of America

10 9 8 7 6 5 4 3 2

In loving memory of Vera

PREFACE

Electrical engineering education has undergone some radical changes during the past couple of decades and continues to do so. A modern undergraduate program in electrical engineering includes the following two introductory courses:

- ▶ *Signals and Systems*, which provides a balanced and integrated treatment of continuous-time and discrete-time forms of signals and systems. The Fourier transform (in its different forms), Laplace transform, and z -transform are treated in detail. Typically, the course also includes an elementary treatment of communication systems.
- ▶ *Probability and Random Processes*, which develops an intuitive grasp of discrete and continuous random variables and then introduces the notion of a random process and its characteristics.

Typically, these two introductory courses lead to a senior-level course on communication systems.

The fourth edition of this book has been written with this background and primary objective in mind. Simply put, the book provides a modern treatment of communication systems at a level suitable for a one- or two-semester senior undergraduate course. The emphasis is on the statistical underpinnings of communication theory with applications.

The material is presented in a logical manner, and it is illustrated with examples, with the overall aim being that of helping the student develop an intuitive grasp of the theory under discussion. Except for the Background and Preview chapter, each chapter ends with numerous problems designed not only to help the students test their understanding of the material covered in the chapter but also to challenge them to extend this material. Every chapter includes notes and references that provide suggestions for further reading. Sections or subsections that can be bypassed without loss of continuity are identified with a footnote.

A distinctive feature of the book is the inclusion of eight computer experiments using MATLAB. This set of experiments provides the basis of a “Software Laboratory”, with each experiment being designed to extend the material covered in the pertinent chapter. Most important, the experiments exploit the unique capabilities of MATLAB in an instructive manner. The MATLAB codes for all these experiments are available on the Wiley Web site: <http://www.wiley.com/college/haykin/>.

The Background and Preview chapter presents introductory and motivational material, paving the way for detailed treatment of the many facets of communication systems in the subsequent 10 chapters. The material in these chapters is organized as follows:

- ▶ Chapter 1 develops a detailed treatment of *random*, or *stochastic*, processes, with particular emphasis on their partial characterization (i.e., second-order statistics). In effect, the discussion is restricted to wide-sense stationary processes. The correlation

properties and power spectra of random processes are described in detail. Gaussian processes and narrowband noise feature prominently in the study of communication systems, hence their treatment in the latter part of the chapter. This treatment naturally leads to the consideration of the Rayleigh and Rician distributions that arise in a communications environment.

▶ Chapter 2 presents an integrated treatment of *continuous-wave (CW) modulation* (i.e., *analog communications*) and their different types, as outlined here:

(i) *Amplitude modulation*, which itself can assume one of the following forms (depending on how the spectral characteristics of the modulated wave are specified):

- ▶ Full amplitude modulation
- ▶ Double sideband-suppressed carrier modulation
- ▶ Quadrature amplitude modulation
- ▶ Single sideband modulation
- ▶ Vestigial sideband modulation

(ii) *Angle modulation*, which itself can assume one of two interrelated forms:

- ▶ Phase modulation
- ▶ Frequency modulation

The time-domain and spectral characteristics of these modulated waves, methods for their generation and detection, and the effects of channel noise on their performances are discussed.

▶ Chapter 3 covers *pulse modulation* and discusses the processes of sampling, quantization, and coding that are fundamental to the digital transmission of analog signals. This chapter may be viewed as the transition from analog to digital communications. Specifically, the following types of pulse modulation are discussed:

(i) *Analog pulse modulation*, where only time is represented in discrete form; it embodies the following special forms:

- ▶ Pulse amplitude modulation
- ▶ Pulse width (duration) modulation
- ▶ Pulse position modulation

The characteristics of pulse amplitude modulation are discussed in detail, as it is basic to all forms of pulse modulation, be they of the analog or digital type.

(ii) *Digital pulse modulation*, in which both time and signal amplitude are represented in discrete form; it embodies the following special forms:

- ▶ Pulse-code modulation
- ▶ Delta modulation
- ▶ Differential pulse-code modulation

In delta modulation, the sampling rate is increased far in excess of that used in pulse-code modulation so as to simplify implementation of the system. In contrast, in differential pulse-code modulation, the sampling rate is reduced through the use of a predictor that exploits the correlation properties of the information-bearing signal.

(iii) MPEG/audio coding standard, which includes a psychoacoustic model as a key element in the design of the encoder.

▶ Chapter 4 covers *baseband pulse transmission*, which deals with the transmission of pulse-amplitude modulated signals in their baseband form. Two important issues are discussed: the effects of channel noise and limited channel bandwidth on the performance of a digital communication system. Assuming that the channel noise is additive

and white, this effect is minimized by using a *matched filter*, which is basic to the design of communication receivers. As for limited channel bandwidth, it manifests itself in the form of a phenomenon known as *intersymbol interference*. To combat the degrading effects of this signal-dependent interference, we may use either a *pulse-shaping filter* or *correlative encoder/decoder*; both of these approaches are discussed. The chapter includes a discussion of digital subscriber lines for direct communication between a subscriber and an Internet service provider. This is followed by a derivation of the optimum linear receiver for combatting the combined effects of channel noise and intersymbol interference, which, in turn, leads to an introductory treatment of *adaptive equalization*.

- ▶ Chapter 5 discusses *signal-space analysis* for an additive white Gaussian noise channel. In particular, the foundations for the geometric representation of signals with finite energy are established. The correlation receiver is derived, and its equivalence with the matched filter receiver is demonstrated. The chapter finishes with a discussion of the probability of error and its approximate calculation.
- ▶ Chapter 6 discusses *passband data transmission*, where a sinusoidal carrier wave is employed to facilitate the transmission of the digitally modulated wave over a band-pass channel. This chapter builds on the geometric interpretation of signals presented in Chapter 5. In particular, the effect of channel noise on the performance of digital communication systems is evaluated, using the following modulation techniques:
 - (i) *Phase-shift keying*, which is the digital counterpart to phase modulation with the phase of the carrier wave taking on one of a prescribed set of discrete values.
 - (ii) Hybrid amplitude/phase modulation schemes including *quadrature-amplitude modulation* (QAM), and *carrierless amplitude/phase modulation* (CAP).
 - (iii) *Frequency-shift keying*, which is the digital counterpart of frequency modulation with the frequency of the carrier wave taking on one of a prescribed set of discrete values.
 - (iv) Generic multichannel modulation, followed by *discrete multitone*, the use of which has been standardized in asymmetric digital subscriber lines.

In a digital communication system, timing is everything, which means that the receiver must be *synchronized* to the transmitter. In this context, we speak of the receiver being coherent or noncoherent. In a *coherent receiver*, provisions are made for the recovery of both the carrier phase and symbol timing. In a *noncoherent receiver* the carrier phase is ignored and provision is only made for symbol timing. Such a strategy is dictated by the fact that the carrier phase may be random, making phase recovery a costly proposition. Synchronization techniques are discussed in the latter part of the chapter, with particular emphasis on discrete-time signal processing.

- ▶ Chapter 7 introduces *spread-spectrum modulation*. Unlike traditional forms of modulation discussed in earlier chapters, channel bandwidth is purposely sacrificed in spread-spectrum modulation for the sake of security or protection against interfering signals. The direct-sequence and frequency-hop forms of spread-spectrum modulation are discussed.
- ▶ Chapter 8 deals with *multiuser radio communications*, where a multitude of users have access to a common radio channel. This type of communication channel is well represented in satellite and wireless communication systems, both of which are discussed. The chapter includes a presentation of link budget analysis, emphasizing the related antenna and propagation concepts, and noise calculations.
- ▶ Chapter 9 develops the *fundamental limits in information theory*, which are embodied in Shannon's theorems for data compaction, data compression, and data trans-

mission. These theorems provide upper bounds on the performance of information sources and communication channels. Two concepts, basic to formulation of the theorems, are (1) the *entropy* of a source (whose definition is analogous to that of entropy in thermodynamics), and (2) channel capacity.

- ▶ Chapter 10 deals with *error-control coding*, which encompasses techniques for the encoding and decoding of digital data streams for their reliable transmission over noisy channels. Four types of error-control coding are discussed:
 - (i) *Linear block codes*, which are completely described by sets of linearly independent code words, each of which consists of message bits and parity-check bits. The parity-check bits are included for the purpose of error control.
 - (ii) *Cyclic codes*, which form a subclass of linear block codes.
 - (iii) *Convolutional codes*, which involve operating on the message sequence continuously in a serial manner.
 - (iv) *Turbo codes*, which provide a novel method of constructing good codes that approach Shannon's channel capacity in a physically realizable manner. Methods for the generation of these codes and their decoding are discussed.

The book also includes supplementary material in the form of six appendices as follows:

- ▶ Appendix 1 reviews probability theory.
- ▶ Appendix 2, on the representation of signals and systems, reviews the Fourier transform and its properties, the various definitions of bandwidth, the Hilbert transform, and the low-pass equivalents of narrowband signals and systems.
- ▶ Appendix 3 presents an introductory treatment of the Bessel function and its modified form. Bessel functions arise in the study of frequency modulation, noncoherent detection of signals in noise, and symbol timing synchronization.
- ▶ Appendix 4 introduces the confluent hypergeometric function, the need for which arises in the envelope detection of amplitude-modulated signals in noise.
- ▶ Appendix 5 provides an introduction to cryptography, which is basic to secure communications.
- ▶ Appendix 6 includes 12 useful tables of various kinds.

As mentioned previously, the primary purpose of this book is to provide a modern treatment of communication systems suitable for use in a one- or two-semester undergraduate course at the senior level. The make-up of the material for the course is naturally determined by the background of the students and the interests of the teachers involved. The material covered in the book is both broad and deep enough to satisfy a variety of backgrounds and interests, thereby providing considerable flexibility in the choice of course material. As an aid to the teacher of the course, a detailed solutions manual for all the problems in the book is available from the publisher.

Acknowledgments

I wish to express my deep gratitude to Dr. Gregory J. Pottie (University of California, Los Angeles), Dr. Santosh Venkatesh (University of Pennsylvania), Dr. Stephen G. Wilson (University of Virginia), Dr. Gordon Stüber (Georgia Institute of Technology), Dr. Venugopal Veeraralli (Cornell University), and Dr. Granville E. Ott (University of Texas at Austin)

for critical reviews of an earlier version of the manuscript and for making numerous suggestions that have helped me shape the book into its present form. The treatment of the effect of noise on envelope detection presented in Chapter 2 is based on course notes made available to me by Dr. Santosh Venkatesh, for which I am grateful. I am grateful to Dr. Gordon Stüber for giving permission to reproduce Figure 6.32.

I am indebted to Dr. Michael Moher (Communications Research Centre, Ottawa) for reading five chapters of an earlier version of the manuscript and for making many constructive comments on turbo codes. I am equally indebted to Dr. Brendan Frey (University of Waterloo, Ontario) for his invaluable help in refining the material on turbo codes, comments on low-density parity-check codes, for providing the software to plot Fig. 9.18, and giving me the permission to reproduce Figures 10.27 and 10.33. I am grateful to Dr. David Conn (McMaster University, Ontario) for his critical reading of the Background and Preview Chapter and for making suggestions on how to improve the presentation of the material therein.

I also wish to thank Dr. Jean-Jacque Werner (Lucent Technologies, Holmdel), Dr. James Mazo (Lucent Technologies, Murray Hill), Dr. Andrew Viterbi (Qualcom, San Diego), Dr. Radford Neal (University of Toronto, Ontario), Dr. Yitzhak (Irwin) Kalet (Technion, Israel), Dr. Walter Chen (Motorola), Dr. John Cioffi (Stanford University), Dr. Jon Mark (University of Waterloo, Ontario), and Dr. Robert Dony (University of Guelph, Ontario); I thank them all for their helpful comments on selected sections in the book. Corrections and suggestions for improvements to the book made by Dr. Donald Wunsch II (University of Missouri) are also appreciated.

I am grateful to my graduate student Mathini Sellathurai (McMaster University) for performing the computer experiments in the book, and Hugh Pasika (McMaster University) for many useful comments on the Background and Preview Chapter and for doing the computations on some graphical plots in the book. Proofreading of the page proofs by Mathini Sellathurai and Alpesh Patel is much appreciated.

I am particularly grateful to my editor at Wiley, Bill Zobrist, for his strong support and help throughout the writing of the book. I am indebted to Monique Calello, Senior Production Editor at Wiley, for her tireless effort in overseeing the production of the book in its various stages. I thank Katherine Hepburn for advertising and marketing the book. I thank Karen Tongish for her careful copyediting of the manuscript, Katrina Avery for her careful proofreading of the page proofs, and Kristen Maus for composing the index of the book.

Last but by no means least, as always, I am grateful to my Technical Coordinator, Lola Brooks, for her tireless effort in typing the manuscript of the book. I also wish to record my gratitude to Brigitte Maier, Assistant Librarian, and Regina Bendig, Reference Librarian, at McMaster University, for helping me on numerous occasions in tracing references for the bibliography.

*Simon Haykin
Ancaster, Ontario
January, 2000*

CONTENTS

BACKGROUND AND PREVIEW

1

1. The Communication Process 1
2. Primary Communication Resources 3
3. Sources of Information 3
4. Communication Networks 10
5. Communication Channels 15
6. Modulation Process 19
7. Analog and Digital Types of Communication 21
8. Shannon's Information Capacity Theorem 23
9. A Digital Communication Problem 24
10. Historical Notes 26
Notes and References 29

CHAPTER 1 *Random Processes*

31

- 1.1 Introduction 31
- 1.2 Mathematical Definition of a Random Process 32
- 1.3 Stationary Processes 33
- 1.4 Mean, Correlation, and Covariance Functions 35
- 1.5 Ergodic Processes 41
- 1.6 Transmission of a Random Process Through a Linear Time-Invariant Filter 42
- 1.7 Power Spectral Density 44
- 1.8 Gaussian Process 54
- 1.9 Noise 58
- 1.10 Narrowband Noise 64
- 1.11 Representation of Narrowband Noise in Terms of In-phase and Quadrature Components 64
- 1.12 Representation of Narrowband Noise in Terms of Envelope and Phase Components 67
- 1.13 Sine Wave Plus Narrowband Noise 69
- 1.14 Computer Experiments: Flat-Fading Channel 71

1.15	Summary and Discussion	75
	<i>Notes and References</i>	77
	<i>Problems</i>	78

CHAPTER 2 *Continuous-Wave Modulation*

88

2.1	Introduction	88
2.2	Amplitude Modulation	90
2.3	Linear Modulation Schemes	93
2.4	Frequency Translation	103
2.5	Frequency-Division Multiplexing	105
2.6	Angle Modulation	107
2.7	Frequency Modulation	109
2.8	Nonlinear Effects in FM Systems	126
2.9	Superheterodyne Receiver	128
2.10	Noise in CW Modulation Systems	130
2.11	Noise in Linear Receivers using Coherent Detection	132
2.12	Noise in AM Receivers using Envelope Detection	135
2.13	Noise in FM Receivers	142
2.14	Computer Experiments: Phase-locked Loop	157
2.15	Summary and Discussion	162
	<i>Notes and References</i>	165
	<i>Problems</i>	166

CHAPTER 3 *Pulse Modulation*

183

3.1	Introduction	183
3.2	Sampling Process	184
3.3	Pulse-Amplitude Modulation	188
3.4	Other Forms of Pulse Modulation	191
3.5	Bandwidth-Noise Trade-off	193
3.6	Quantization Process	193
3.7	Pulse-Code Modulation	201
3.8	Noise Considerations in PCM Systems	209
3.9	Time-Division Multiplexing	211
3.10	Digital Multiplexers	214
3.11	Virtues, Limitations, and Modifications of PCM	217
3.12	Delta Modulation	218
3.13	Linear Prediction	223
3.14	Differential Pulse-Code Modulation	227
3.15	Adaptive Differential Pulse-Code Modulation	229

- 3.16 Computer Experiment: Adaptive Delta Modulation 232
- 3.17 MPEG Audio Coding Standard 234
- 3.18 Summary and Discussion 236
 - Notes and References* 238
 - Problems* 239

CHAPTER 4 Baseband Pulse Transmission

247

- 4.1 Introduction 247
- 4.2 Matched Filter 248
- 4.3 Error Rate Due to Noise 253
- 4.4 Intersymbol Interference 259
- 4.5 Nyquist's Criterion for Distortionless Baseband Binary Transmission 261
- 4.6 Correlative-Level Coding 267
- 4.7 Baseband M -ary PAM Transmission 275
- 4.8 Digital Subscriber Lines 277
- 4.9 Optimum Linear Receiver 282
- 4.10 Adaptive Equalization 287
- 4.11 Computer Experiments: Eye Patterns 293
- 4.12 Summary and Discussion 296
 - Notes and References* 297
 - Problems* 300

CHAPTER 5 Signal-Space Analysis

309

- 5.1 Introduction 309
- 5.2 Geometric Representation of Signals 311
- 5.3 Conversion of the Continuous AWGN Channel into a Vector Channel 318
- 5.4 Likelihood Functions 322
- 5.5 Coherent Detection of Signals in Noise: Maximum Likelihood Decoding 322
- 5.6 Correlation Receiver 326
- 5.7 Probability of Error 328
- 5.8 Summary and Discussion 337
 - Notes and References* 337
 - Problems* 338

CHAPTER 6 Passband Digital Transmission

344

- 6.1 Introduction 344
- 6.2 Passband Transmission Model 348
- 6.3 Coherent Phase-Shift Keying 349

6.4	Hybrid Amplitude/Phase Modulation Schemes	368
6.5	Coherent Frequency-Shift Keying	380
6.6	Detection of Signals with Unknown Phase	403
6.7	Noncoherent Orthogonal Modulation	407
6.8	Noncoherent Binary Frequency-Shift Keying	413
6.9	Differential Phase-Shift Keying	414
6.10	Comparison of Digital Modulation Schemes Using a Single Carrier	417
6.11	Voiceband Modems	420
6.12	Multichannel Modulation	431
6.13	Discrete Multitone	440
6.14	Synchronization	448
6.15	Computer Experiments: Carrier Recovery and Symbol Timing	458
6.16	Summary and Discussion	464
	<i>Notes and References</i>	465
	<i>Problems</i>	468

CHAPTER 7 Spread-Spectrum Modulation **479**

7.1	Introduction	479
7.2	Pseudo-Noise Sequences	480
7.3	A Notion of Spread Spectrum	488
7.4	Direct-Sequence Spread Spectrum with Coherent Binary Phase-Shift Keying	490
7.5	Signal-Space Dimensionality and Processing Gain	493
7.6	Probability of Error	497
7.7	Frequency-Hop Spread Spectrum	499
7.8	Computer Experiments: Maximal-Length and Gold Codes	505
7.9	Summary and Discussion	508
	<i>Notes and References</i>	509
	<i>Problems</i>	509

CHAPTER 8 Multiuser Radio Communications **512**

8.1	Introduction	512
8.2	Multiple-Access Techniques	513
8.3	Satellite Communications	514
8.4	Radio Link Analysis	517
8.5	Wireless Communications	529
8.6	Statistical Characterization of Multipath Channels	535
8.7	Binary Signaling over a Rayleigh Fading Channel	542
8.8	TDMA and CDMA Wireless Communication Systems	547
8.9	Source Coding of Speech for Wireless Communications	550

- 8.10 Adaptive Antenna Arrays for Wireless Communications 553
- 8.11 Summary and Discussion 559
 - Notes and References* 560
 - Problems* 562

CHAPTER 9 Fundamental Limits in Information Theory

567

- 9.1 Introduction 567
- 9.2 Uncertainty, Information, and Entropy 568
- 9.3 Source-Coding Theorem 574
- 9.4 Data Compaction 575
- 9.5 Discrete Memoryless Channels 581
- 9.6 Mutual Information 584
- 9.7 Channel Capacity 587
- 9.8 Channel-Coding Theorem 589
- 9.9 Differential Entropy and Mutual Information for Continuous Ensembles 593
- 9.10 Information Capacity Theorem 597
- 9.11 Implications of the Information Capacity Theorem 601
- 9.12 Information Capacity of Colored Noise Channel 607
- 9.13 Rate Distortion Theory 611
- 9.14 Data Compression 614
- 9.15 Summary and Discussion 616
 - Notes and References* 617
 - Problems* 618

CHAPTER 10 Error-Control Coding

626

- 10.1 Introduction 626
- 10.2 Discrete-Memoryless Channels 629
- 10.3 Linear Block Codes 632
- 10.4 Cyclic Codes 641
- 10.5 Convolutional Codes 654
- 10.6 Maximum Likelihood Decoding of Convolutional Codes 660
- 10.7 Trellis-Coded Modulation 668
- 10.8 Turbo Codes 674
- 10.9 Computer Experiment: Turbo Decoding 682
- 10.10 Low-Density Parity-Check Codes 683
- 10.11 Irregular Codes 691
- 10.12 Summary and Discussion 693
 - Notes and References* 694
 - Problems* 696

APPENDIX 1	Probability Theory	703	
APPENDIX 2	Representation of Signals and Systems	715	
APPENDIX 3	Bessel Functions	735	
APPENDIX 4	Confluent Hypergeometric Functions	740	
APPENDIX 5	Cryptography	742	
APPENDIX 6	Tables	761	
GLOSSARY			771
BIBLIOGRAPHY			777
INDEX			792

BACKGROUND AND PREVIEW

The background and preview material presented herein sets the stage for a statistical treatment of communication systems in subsequent chapters. In particular, we describe the following:

- ▶ *The communication process.*
- ▶ *Primary communication resources, namely, transmitted power and channel bandwidth.*
- ▶ *Sources of information.*
- ▶ *The two primary types of switching: circuit switching and packet switching.*
- ▶ *Communication channels for the transportation of information-bearing signals from the transmitter to the receiver.*
- ▶ *The modulation process, which is basic to communication systems.*
- ▶ *Analog and digital types of communication systems.*
- ▶ *Shannon's information capacity theorem.*
- ▶ *A digital communications problem.*

The chapter concludes with some historical notes, as a source of motivation for the reader.

The Communication Process

Today, *communication* enters our daily lives in so many different ways that it is very easy to overlook the multitude of its facets. The telephones at our hands, the radios and televisions in our living rooms, the computer terminals with access to the Internet in our offices and homes, and our newspapers are all capable of providing rapid communications from every corner of the globe. Communication provides the senses for ships on the high seas, aircraft in flight, and rockets and satellites in space. Communication through a wireless telephone keeps a car driver in touch with the office or home miles away. Communication keeps a weather forecaster informed of conditions measured by a multitude of sensors. Indeed, the list of applications involving the use of communication in one way or another is almost endless.

In the most fundamental sense, communication involves implicitly the transmission of *information* from one point to another through a succession of processes, as described here:

1. The generation of a *message signal*: voice, music, picture, or computer data.
2. The description of that message signal with a certain measure of precision, by a set of *symbols*: electrical, aural, or visual.
3. The *encoding* of these symbols in a form that is suitable for transmission over a physical medium of interest.
4. The *transmission* of the encoded symbols to the desired destination.
5. The *decoding* and *reproduction* of the original symbols.
6. The *re-creation* of the original message signal, with a definable degradation in quality; the degradation is caused by imperfections in the system.

There are, of course, many other forms of communication that do not directly involve the human mind in real time. For example, in *computer communications* involving communication between two or more computers, human decisions may enter only in setting up the programs or commands for the computer, or in monitoring the results.

Irrespective of the form of communication process being considered, there are three basic elements to every communication system, namely, *transmitter*, *channel*, and *receiver*, as depicted in Figure 1. The transmitter is located at one point in space, the receiver is located at some other point separate from the transmitter, and the channel is the physical medium that connects them. The purpose of the transmitter is to convert the *message signal* produced by the *source of information* into a form suitable for transmission over the channel. However, as the transmitted signal propagates along the channel, it is distorted due to channel imperfections. Moreover, noise and interfering signals (originating from other sources) are added to the channel output, with the result that the *received signal* is a corrupted version of the *transmitted signal*. The receiver has the task of operating on the received signal so as to reconstruct a recognizable form of the original message signal for a user.

There are two basic modes of communication:

1. *Broadcasting*, which involves the use of a single powerful transmitter and numerous receivers that are relatively inexpensive to build. Here information-bearing signals flow only in one direction.
2. *Point-to-point communication*, in which the communication process takes place over a link between a single transmitter and a receiver. In this case, there is usually a bidirectional flow of information-bearing signals, which requires the use of a transmitter and receiver at each end of the link.

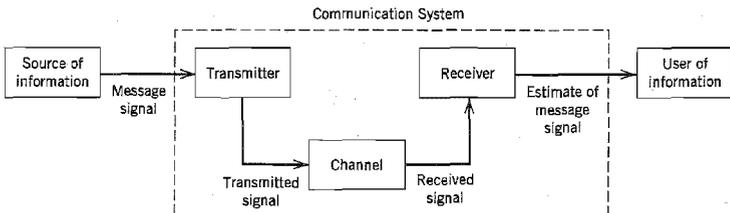


FIGURE 1 Elements of a communication system.

The broadcasting mode of communication is exemplified by radio and television, and the ubiquitous telephone provides the means for one form of point-to-point communication. Another example of point-to-point communication is the link between an Earth station and a robot navigating the surface of a distant planet.

All these different communication systems as well as others not mentioned here share a common feature: The underlying communication process in each and every one of them is *statistical* in nature. Indeed, it is for this important reason that much of this book is devoted to the statistical underpinnings of communication systems. In so doing, we develop an exposition of the fundamental issues involved in the study of different communication methodologies and thereby provide a natural forum for their comparative evaluations.

Primary Communication Resources

In a communication system, two primary resources are employed: *transmitted power* and *channel bandwidth*. The transmitted power is the average power of the transmitted signal. The channel bandwidth is defined as the band of frequencies allocated for the transmission of the message signal. A general system design objective is to use these two resources as efficiently as possible. In most communication channels, one resource may be considered more important than the other. We may therefore classify communication channels as *power limited* or *band limited*. For example, the telephone circuit is a typical band-limited channel, whereas a space communication link or satellite channel is typically power limited.

When the spectrum of a message signal extends down to zero or low frequencies, we define the bandwidth of the signal as that upper frequency above which the spectral content of the signal is negligible and therefore unnecessary for transmitting information. For example, the average voice spectrum extends well beyond 10 kHz, though most of the average power is concentrated in the range of 100 to 600 Hz, and a band from 300 to 3100 Hz gives good articulation. Accordingly, we find that telephone circuits that respond well to this latter range of frequencies give quite satisfactory commercial telephone service.

Another important point that we have to keep in mind is the unavoidable presence of noise in a communication system. *Noise* refers to unwanted waves that tend to disturb the transmission and processing of message signals in a communication system. The sources of noise may be internal or external to the system.

A quantitative way to account for the effect of noise is to introduce *signal-to-noise ratio* (SNR) as a system parameter. For example, we may define the SNR at the receiver input as *the ratio of the average signal power to the average noise power*, both being measured at the same point. The customary practice is to express the SNR in *decibels* (dBs), defined as 10 times the logarithm (to base 10) of the power ratio. For example, signal-to-noise ratios of 10, 100, and 1,000 correspond to 10, 20, and 30 dBs, respectively.

Sources of Information

The telecommunications environment is dominated by four important sources of information: *speech*, *music*, *pictures*, and *computer data*. A source of information may be characterized in terms of the signal that carries the information. A *signal* is defined as a single-valued function of *time* that plays the role of the independent variable; at every instant of time, the function has a unique value. The signal can be *one-dimensional*, as in the case of speech, music, or computer data; *two-dimensional*, as in the case of pictures;

three-dimensional, as in the case of video data; and *four-dimensional*, as in the case of volume data over time. In the sequel, we elaborate on different sources of information.

- (i) *Speech* is the primary method of human communication. Specifically, the speech communication process involves the transfer of information from a speaker to a listener, which takes place in three successive stages:
 - ▶ *Production*. An intended message in the speaker's mind is represented by a *speech signal* that consists of sounds (i.e., pressure waves) generated inside the vocal tract and whose arrangement is governed by the rules of language.
 - ▶ *Propagation*. The sound waves propagate through the air at a speed of 300 m/s, reaching the listener's ears.
 - ▶ *Perception*. The incoming sounds are deciphered by the listener into a received message, thereby completing the chain of events that culminate in the transfer of information from the speaker to the listener.

The speech-production process may be viewed as a form of filtering, in which a *sound source excites a vocal tract filter*. The vocal tract consists of a tube of nonuniform cross-sectional area, beginning at the *glottis* (i.e., the opening between the vocal cords) and ending at the *lip*. As the sound propagates along the vocal tract, the spectrum (i.e., frequency content) is shaped by the frequency selectivity of the vocal tract; this effect is somewhat similar to the resonance phenomenon observed in organ pipes. The important point to note here is that the power spectrum (i.e., the distribution of long-term average power versus frequency) of speech approaches zero for zero frequency and reaches a peak in the neighborhood of a few hundred hertz. To put matters into proper perspective, however, we have to keep in mind that the hearing mechanism is very sensitive to frequency. Moreover, the type of communication system being considered has an important bearing on the band of frequencies considered to be "essential" for the communication process. For example, as mentioned previously, a bandwidth of 300 to 3100 Hz is considered adequate for commercial telephonic communication.

- (ii) The second source of information, *music*, originates from instruments such as the piano, violin, and flute. The note made by a musical instrument may last for a short time interval as in the pressing of a key on a piano, or it may be sustained for a long time interval as in the example of a flute player holding a prolonged note. Typically, music has two structures: a *melodic* structure consisting of a time sequence of sounds, and a *harmonic* structure consisting of a set of simultaneous sounds. Like a speech signal, a musical signal is bipolar. However, a musical signal differs from a speech signal in that its spectrum occupies a much wider band of frequencies that may extend up to about 15 kHz. Accordingly, musical signals demand a much wider channel bandwidth than speech signals for their transmission.
- (iii) The third source of information, *pictures*, relies on the human visual system for its perception. The picture can be *dynamic*, as in television, or *static*, as in facsimile. Taking the case of television first, the pictures in motion are converted into electrical signals to facilitate their transport from the transmitter to the receiver. To do so, each complete picture is *sequentially scanned*. The scanning process is carried out in a *TV camera*. In a black-and-white TV, the camera contains optics designed to focus an image on a *photocathode* consisting of a large number of photosensitive elements. The charge pattern so generated on the photosensitive surface is scanned by an *electron beam*, thereby producing an output current that varies *temporally* with the way in which the brightness of the original picture varies *spatially* from one point to another. The resulting output current is called a *video signal*. The type of scanning

used in television is a form of spatial sampling called *raster scanning*, which converts a two-dimensional image intensity into a one-dimensional waveform; it is somewhat analogous to the manner in which we read a printed paper in that the scanning is performed from left to right on a line-by-line basis. In North American analog television, a picture is divided into 525 lines, which constitute a *frame*. Each frame is decomposed into two *interlaced fields*, each of which consists of 262.5 lines. For convenience of presentation, we will refer to the two fields as I and II. The scanning procedure is illustrated in Figure 2. The lines of field I are depicted as solid lines, and those of field II are depicted as dashed lines. The *start* and *end* of each field are also included in the figure. Field I is scanned first. The scanning spot of the TV camera moves with constant velocity across each line of the field from left to right, and the image intensity at the center of the spot is measured; the scanning spot itself is partly responsible for local spatial averaging of the image. When the end of a particular line is reached, the scanning spot quickly flies back (in a horizontal direction) to the start of the next line down in the field. This flyback is called the *horizontal retrace*. The scanning process described here is continued until the whole field has been accounted for. When this condition is reached, the scanning spot moves quickly (in a vertical direction) from the end of field I to the start of field II. This second flyback is called the *vertical retrace*. Field II is treated in the same fashion as field I. The time taken for each field to be scanned is $1/60$ s. Correspondingly, the time taken for a frame or a complete picture to be scanned is $1/30$ s. With 525 lines in a frame, the *line-scanning frequency* equals 15.75 kHz. Thus, by flashing 30 still pictures per second on the display tube of the TV receiver, the human eye perceives them to be moving pictures. This effect is due to a phenomenon known as the *persistence of vision*. During the horizontal- and vertical-retrace intervals, the picture tube is made inoperative by means of *blanking pulses* that are generated at the transmitter. Moreover, synchronization between the various scanning operations at both transmitter and receiver is accomplished by means of special pulses that are transmitted during the blanking periods; thus, the synchronizing pulses do not show on the reproduced picture. The reproduction quality of a TV picture is limited by two basic factors:

1. The number of lines available in a raster scan, which limits resolution of the picture in the vertical direction.
2. The channel bandwidth available for transmitting the video signal, which limits resolution of the picture in the horizontal direction.

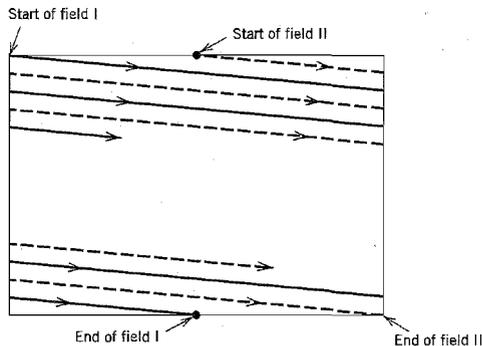


FIGURE 2 Interlaced raster scan.

For each direction, *resolution* is expressed in terms of the maximum number of lines alternating between black and white that can be resolved in the TV image along the pertinent direction by a human observer. In the NTSC (National Television System Committee) system, which is the North American standard, the parameter values used result in a *video bandwidth* of 4.2 MHz, which extends down to zero frequency. This bandwidth is orders of magnitude larger than that of a speech signal. Note also that whereas a speech signal is bipolar, a video (television) signal is inherently positive (i.e., unipolar).

In color TV, the perception of color is based on the three types of color receptors (cones) in the human eye: red, green, and blue, whose wavelengths are 570 nm, 535 nm, and 445 nm, respectively. These three colors are referred to as *primary colors* because any other color found in nature can be approximated by an additive mixture of them. This physical reality is indeed the basis for the transmission of color in commercial TV broadcasting. The three primary colors are represented by the video signals $m_R(t)$, $m_G(t)$, and $m_B(t)$, respectively. To conserve bandwidth and produce a picture that can be viewed on a conventional black-and-white (monochrome) television receiver, the transmission of these three primary colors is accomplished by observing that they can be uniquely represented by any three signals that are independent linear combinations of $m_R(t)$, $m_G(t)$, and $m_B(t)$. The three signals are as follows:

- ▶ A *luminance signal*, $m_L(t)$, which produces a black-and-white version of the color picture when it is received on a conventional monochrome television receiver.
- ▶ A pair of signals, $m_I(t)$ and $m_Q(t)$, called the *chrominance signals*, which indicate the way the color of the picture departs from shades of gray.

The luminance signal $m_L(t)$ is assigned the entire 4.2 MHz bandwidth. Owing to certain properties of human vision, tests show that if the nominal bandwidths of the chrominance signals $m_I(t)$ and $m_Q(t)$ are 1.6 MHz and 0.6 MHz, respectively, satisfactory color reproduction is possible.

Turning next to a *facsimile (fax) machine*, the purpose of this machine is to transmit still pictures over a communication channel (most notably, a telephone channel). Such a machine provides a highly popular facility for the transmission of handwritten or printed text from one point to another; transmitting text by facsimile is treated simply like transmitting a picture. The basic principle employed for signal generation in a facsimile machine is to scan an original document (picture) and use an image sensor to convert the light to an electrical signal.

- (iv) Finally, *personal computers (PCs)* have become an integral part of our daily lives. We use them for electronic mail, exchange of software, and sharing of resources. The text transmitted by a PC is usually encoded using the *American Standard Code for Information Interchange (ASCII)*, which is the first code developed specifically for computer communications. Each character in ASCII is represented by seven *data bits* constituting a unique binary pattern made up of 0s and 1s; bit is acronym for binary digit. Thus a total of $2^7 = 128$ different characters can be represented in ASCII. The characters are various lowercase and uppercase letters, numbers, special control symbols, and punctuation symbols commonly used such as @, \$, and %. Some of the special “control” symbols, such as BS (backspace) and CR (carriage return), are used to control the printing of characters on a page. Other symbols, such as ENQ (enquiry) and ETB (end of transmission block), are used for communication purposes. (A complete listing of ASCII characters is given in Table A6.1.) The seven data bits are ordered starting with the most significant bit b_7 down to the least significant bit b_1 ,

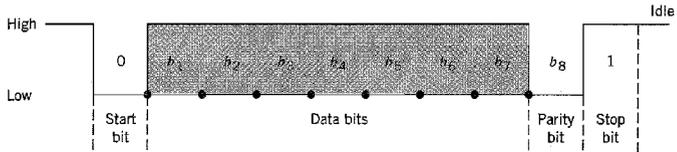


FIGURE 3 The bit format for sending asynchronous serial data used in the RS-232 standard.

as illustrated in Figure 3. At the end of the data bits, an extra bit b_8 is appended for the purpose of *error detection*. This error-detection bit is called a *parity bit*. A sequence of eight bits is referred to as a *byte*, or an *octet*. The parity bit is set in such a way that the total number of 1s in each byte is odd for odd parity and even for even parity. Suppose, for example, the communicators agree to use even parity; then the parity bit will be a 0 when the number of 1s in the data bit is even and a 1 when it is odd. Hence, if a single bit in a byte is received in error and thereby violates the even parity rule, it can be detected and then corrected through retransmission. Personal computers are often connected via their RS (recommended standard)-232 ports. When ASCII data (in fact, all character data) are transmitted through these ports, a start bit, set to 0, and one or more stop bits, set to 1, as shown in Figure 3, are added to provide character framing. When the transmission is idle, a long series of 1s is sent so as to keep the circuit connection alive. In Figure 3, symbols 0 and 1 are designated as “low” and “high,” respectively. They are also sometimes referred to as “space” and “mark,” respectively; the latter terminology comes from the days of telegraphy. The text prepared on a PC is usually stored and then transmitted over a communication channel (e.g., a telephone channel) with a single character being sent at a time. This form of data transmission is called *asynchronous transmission*, as opposed to *synchronous transmission* in which a whole sequence of encoded characters is sent over the channel in one long transmission. Encoded characters produced by a mixture of asynchronous and synchronous terminals are combined by means of *data multiplexers*. The multiplexed stream of data so formed is then applied to a device called a *modem* (modulator-demodulator) for the purpose of transmission over the channel.

In summary, computer-generated data and television signals are both *wide-band signals*, in that their power content occupies a wide range of frequencies. Another important characteristic of data communication between personal computers is *burstiness*, which means that information is usually transmitted from one terminal to another in bursts with silent periods between bursts. Indeed, data traffic involving computers in one form or another tends to be of a bursty nature. This is to be contrasted with traffic in a digital transmission network due to voice or interactive video, which, relatively speaking, is *continuous*.

Another way in which we use the computer is to *download* compressed forms of text, audio, and video data from a service provider at a remote location. *Data compression* provides a practical means for the efficient storage and transmission of these kinds of data. A data compression system consists of an *encoder* and a *decoder*, where the compression of an incoming data stream and its reconstruction are performed, respectively. Basically, there are two forms of data compression:

1. *Lossless compression* operates by removing the *redundant* information contained in the data of interest. The compression is said to be lossless because it is com-

pletely reversible in that the original data can be reconstructed exactly. Lossless compression is also referred to as *data compaction*.

2. *Lossy compression* involves the loss of information in a controlled manner; the compression may therefore not be completely reversible. Lossy compression is, however, capable of achieving a compression ratio higher than that attainable with lossless methods.

For digital text, lossless compression is required. In this context, we mention the *Lempel-Ziv algorithm*, which is intrinsically adaptive and capable of encoding groups of source symbols that occur frequently. It achieves a compression of approximately 55 percent on ordinary English text, which, loosely speaking, corresponds to the compression that would be achieved by encoding pairs of letters. The Lempel-Ziv algorithm is a form of *entropic coding*, or *source coding*, which is discussed in Chapter 9.

In many other applications, lossy compression is usually the preferred approach as its use can substantially reduce the data size without significantly altering the perceptual quality of an image or audio signal. For such applications, this form of data compression is acceptable, and in high-throughput data-transmission applications such as the Internet, it is a necessity. But in some other applications such as a clinical setting, the quality of a medical image (e.g., digital x-ray radiograph) must not be degraded on reconstruction.

For digital audio and video applications involving storage or transmission to be viable in today's marketplace, we need *standard* compression algorithms that enable the interoperability of equipment produced by different manufacturers. In this context, we mention three prominent standard compression algorithms that cater to different needs:

- ▶ The *JPEG image coding standard*² is designed to compress full-color or grayscale images of natural, real-world scenes by exploiting known limitations of the human visual system; JPEG stands for Joint Photographic Experts Group. At the input to the encoder, *picture elements*, or *pixels*, are grouped into 8×8 blocks, which are applied to a relative of the Fourier transform known as the *discrete cosine transform* (DCT)³. The DCT decomposes each block of pixels into a set of 64 coefficients that closely satisfy two related objectives:

1. The coefficients should be as *uncorrelated* as possible.
2. The energy of the input signal should be packed into the *smallest number of coefficients possible*.

The next operation in the encoder is that of *quantization*, where each of the 64 DCT coefficients is *rounded off*. In JPEG, quantization is performed in conjunction with a *quantization table* supplied by the user as an input to the encoder. Each element of the table is an integer from 1 to 255 that specifies the step size of the DCT coefficients, which, in turn, permits the representation of each quantized DCT coefficient by an 8-bit code word. Basically, the purpose of quantization is to discard information that is not perceptually discernible. Quantization is a *many-to-one mapping* and therefore the principal source of lossiness in the encoder. The final operation in the encoder is that of *Huffman coding*, which is a form of entropic (source) coding also discussed in Chapter 9. Huffman coding achieves additional data compression in a lossless manner by encoding the quantized DCT coefficients in accordance with their statistical characteristics. At the decoder, data reconstruction is performed through a sequence of operations that are the inverse

of those in the encoder, namely, Huffman decoding, dequantization in accordance with the quantization table, and finally the inverse DCT.

- ▶ The *MPEG-1/video coding standard*⁴ is designed primarily to compress video signals at 30 frames per second (fps) into bit streams running at the rate of 1.5 megabits per second (Mb/s); MPEG stands for Motion Photographic Experts Group. The MPEG-1 video coding standard achieves this design goal by exploiting four basic forms of redundancy inherently present in video data:

1. Interframe (temporal) redundancy.
2. Interpixel redundancy within a frame.
3. Psychovisual redundancy.
4. Entropic coding redundancy.

It is the exploitation of interframe redundancy that distinguishes MPEG-1 from JPEG. In principle, neighboring frames in typical video sequences are highly *correlated*. The meaning of this high correlation is that, in an average sense, a video signal does not change rapidly from one frame to the next, and as a result, the difference between adjacent frames has a variance (i.e., average power) that is much smaller than the variance of the video signal itself. Accordingly, the interframe redundancy can be significantly reduced to produce a more efficiently compressed video signal. This reduction is achieved through the use of *prediction* to estimate each frame from its neighbors; the resulting prediction error is transmitted for motion estimation and compensation. The prediction is nonlinear by virtue of the nature of the problem. As with JPEG, the interpixel redundancy is reduced through the combined use of the DCT, quantization, and lossless entropic coding. The net result is that full-motion video becomes a 1.5 Mb/s stream of computer data that can be stored on compact discs or integrated with texts and graphics. Most important, the full-motion video and associated audio can be delivered over existing computer and telecommunication networks, which, in turn, makes it possible to fulfill the need for *video-on-demand* on the Internet.

- ▶ The *MPEG-1/audio coding standard*⁵ is based on *perceptual coding*, which is a waveform-preserving process; that is, the amplitude-time waveform of the decoded audio signal closely approximates that of the original audio signal. In basic terms, the encoding process encompasses four distinct operations:

1. *Time-frequency mapping*, whereby the input audio signal is decomposed into multiple subbands.
2. *Psychoacoustic modeling*, which simultaneously operates on the input audio signal to compute certain thresholds using known rules from the psychoacoustic behavior of the human auditory system.
3. *Quantization and coding*, which, in conjunction with the psychoacoustic model, works on the output of the time-frequency mapper so as to maintain the noise resulting from quantization process at an inaudible level.
4. *Frame-packing*, which is used to format the quantized audio samples into a decodable bit stream.

The psychoacoustic model builds on a perceptual phenomenon known as *auditory masking*. Specifically, the human ear does *not* perceive quantization noise in a given frequency band if the average noise power lies below the *masking threshold* (i.e., the threshold of just noticeable distortion). For a given frequency band of interest, the masking threshold varies with frequency across that band. The *min-*

imum masking threshold is the one that is employed in the psychoacoustic model on a band-by-band basis. For example, the net result of using the MPEG-1 standard on the two audio channels of a stereo program is that each digitized audio signal, coming in at the rate of 768 kilobits per second (kb/s), is compressed to a rate as low as 16 kb/s. (The incoming data rate of 768 kb/s corresponds to a sampling rate of 48 kHz, with each sample being represented by a 16-bit code word.) Thus the MPEG-1/audio coding standard is suitable for the storage of audio signals in inexpensive media or their transmission over channels with limited bandwidth, while at the same time maintaining perceptual quality.

Communication Networks⁶

A *communication network* (or simply *network*), illustrated in Figure 4, consists of an interconnection of a number of *routers* made up of intelligent processors (e.g., microprocessors). The primary purpose of these processors is to route data through the network, hence the name. Each router has one or more *hosts* attached to it; hosts are devices that communicate with one another. The network is designed to serve as a shared resource for moving data exchanged between hosts in an efficient manner and to provide a framework to support new applications and services.

The telephone network is an example of a communication network in which *circuit switching* is used to provide a dedicated communication path, or *circuit*, between two hosts. The circuit consists of a connected sequence of links from source to destination. For example, the links may consist of time slots for which a common channel is available for access by a multitude of users. The circuit, once in place, remains uninterrupted for the duration of transmission. Circuit switching is usually controlled by a centralized hierarchical control mechanism with knowledge of the network's organization. To establish a circuit-switched connection, an available path through the network is seized and then dedicated to the exclusive use of the two hosts wishing to communicate. In particular, a call-request signal must propagate all the way to the destination and be acknowledged before transmission can begin. Then, the network is effectively transparent to the users. This means that during the connection time, the bandwidth and resources allocated to the circuit are essentially "owned" by the two hosts until the circuit is disconnected. The circuit

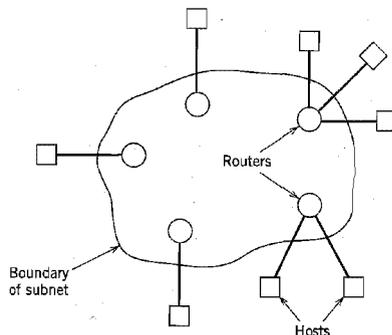


FIGURE 4 Communication network.

thus represents an efficient use of resources only to the extent that the allocated bandwidth is properly used. Although the telephone network is used to transmit data, voice constitutes the bulk of the network's traffic. Indeed, circuit switching is well suited to the transmission of voice signals, since voice gives rise to a stream traffic and voice conversations tend to be of long duration (about 2 minutes on the average) compared to the time required for setting up the circuit (about 0.1 to 0.5 seconds). Moreover, in most voice conversations, there is information flow for a relatively large percentage of the connection time, which makes circuit switching all the more suitable for voice conversations.

In circuit switching, a communication link is shared between the different sessions using that link on a *fixed* allocation basis. In *packet switching*, on the other hand, the sharing is done on a *demand* basis, so it has an advantage over circuit switching in that when a link has traffic to send, the link may be more fully utilized.

The network principle of packet switching is "store and forward." Specifically, in a *packet-switched network*, any message larger than a specified size is subdivided prior to transmission into segments not exceeding the specified size. The segments are commonly referred to as *packets*. The original message is reassembled at the destination on a packet-by-packet basis. The network may be viewed as a distributed pool of *network resources* (i.e., channel bandwidth, buffers, and switching processors) whose capacity is *shared dynamically* by a community of competing hosts wishing to communicate. In contrast, in a circuit-switched network, resources are dedicated to a pair of hosts for the entire period they are in session. Accordingly, packet switching is far better suited to a computer-communication environment in which bursts of data are exchanged between hosts on an occasional basis. The use of packet switching, however, requires that careful *control* be exercised on user demands; otherwise, the network may be seriously abused.

The design of a *data network* (i.e., a network in which the hosts are all made up of computers and terminals) may proceed in an orderly way by looking at the network in terms of a *layered architecture*, regarded as a hierarchy of nested layers. *Layer* refers to a process or device inside a computer system, designed to perform a specific function. Naturally, the designers of a layer will be intimately familiar with its internal details and operation. At the system level, however, a user views the layer merely as a "black box" that is described in terms of inputs, outputs, and the functional relation between outputs and inputs. In a layered architecture, each layer regards the next lower layer as one or more black boxes with some given functional specification to be used by the given higher layer. Thus, the highly complex communication problem in data networks is resolved as a manageable set of well-defined interlocking functions. It is this line of reasoning that has led to the development of the *open systems interconnection* (OSI)⁷ *reference model* by a subcommittee of the International Organization for Standardization. The term *open* refers to the ability of any two systems conforming to the reference model and its associated standards to interconnect.

In the OSI reference model, the communications and related-connection functions are organized as a series of layers, or *levels*, with well-defined *interfaces*, and with each layer built on its predecessor. In particular, each layer performs a related subset of primitive functions, and it relies on the next lower layer to perform additional primitive functions. Moreover, each layer offers certain services to the next higher layer and shields the latter from the implementation details of those services. Between each pair of layers, there is an *interface*. It is the interface that defines the services offered by the lower layer to the upper layer.

The OSI model is composed of seven layers, as illustrated in Figure 5; this figure also includes a description of the functions of the individual layers of the model. Layer k on system A , say, communicates with layer k on some other system B in accordance with a

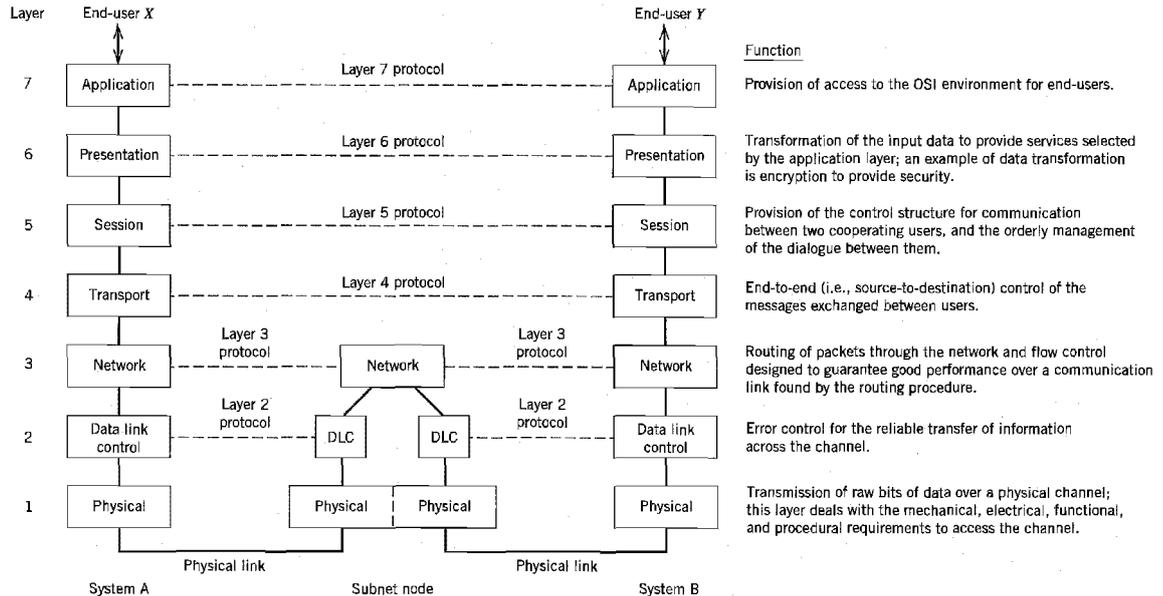


FIGURE 5 OSI model; the acronym DLC in the middle of the figure stands for *data link control*.

set of rules and conventions, collectively constituting the layer k protocol, where $k = 1, 2, \dots, 7$. (The term *protocol* has been borrowed from common usage, describing conventional social behavior between human beings.) The entities that comprise the corresponding layers on different systems are referred to as *peer processes*. In other words, communication is achieved by having the peer processes in two different systems communicate via a protocol, with the protocol itself being defined by a set of rules of procedure. Physical communication between peer processes exists only at layer 1. On the other hand, layers 2 through 7 are in *virtual communication* with their distant peers. However, each of these six layers can exchange data and control information with its neighboring layers (below and above) through layer-to-layer interfaces. In Figure 5, physical communication is shown by solid lines and virtual communication by dashed lines.

Our primary interest in this book is in the physical layer of the OSI model.

■ INTERNET

Any discussion of computer networks naturally leads to the *Internet*. In the Internet paradigm, the underlying network technology is decoupled from the applications at hand by adopting an abstract definition of network service. In more specific terms, we may say the following:

- ▷ The applications are carried out independently of the technology employed to construct the network.
- ▷ By the same token, the network technology is capable of evolving without affecting the applications.

The Internet architecture, depicted in Figure 6, has three functional blocks: hosts, subnets, and routers. The hosts constitute nodes of the network, where data originate or where they are delivered. The routers constitute intermediate nodes that are used to cross subnet boundaries. Within a subnet, all the hosts belonging to that subnet exchange data directly; see, for example, subnets 1 and 3 in Figure 6.

Like other computer networks, the Internet has a layered set of protocols. In particular, the exchange of data between the hosts and routers is accomplished by means of the *Internet protocol* (IP), as illustrated in Figure 7. The IP is a universal protocol that resides in the network layer (i.e., layer 3 of the OSI reference model). It is simple, defining an addressing plan with a built-in capability to transport data in the form of packets from node to node. In crossing a subnetwork boundary, the routers make the decisions as to how the packets addressed for a specified destination should be routed. This is done on the basis of routing tables that are developed through the use of custom protocols for

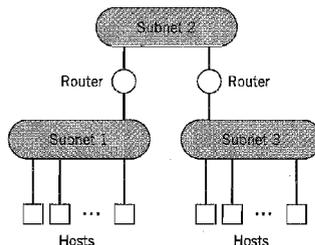


FIGURE 6 An interconnected network of subnets.

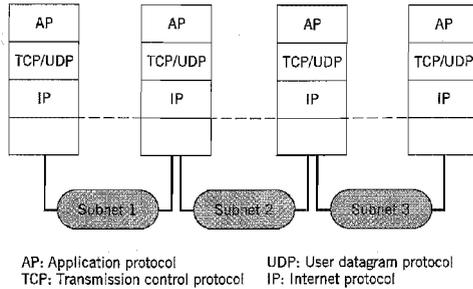


FIGURE 7 Illustrating the network architecture of the Internet.

exchanging pertinent information with other routers. The net result of using the layered set of protocols is the delivery of *best effort service*. That is, the Internet offers to deliver each packet of data, but there are no guarantees on the transit time experienced in delivery or even whether the packets will be delivered to the intended recipient.

■ BROADBAND NETWORKS

With the ever-increasing demand for new services (e.g., video on demand, multimedia communications) and the availability of key enabling technologies (e.g., optical fibers, digital switches), the telephone network is evolving into an all-purpose broadband network known as the *broadband integrated services digital network* (B-ISDN). The underlying technology that makes B-ISDN possible is a user-network interface protocol called the *asynchronous transfer mode* (ATM). ATM is a high-bandwidth, low-delay, packet-like technique used for switching and multiplexing; it is independent of the physical means of transport. The low-delay feature of the technique is needed to support real-time services such as voice. The high-bandwidth feature is required to handle video on demand. Simply put, ATM is both a technology that is hidden from the users and a connection-oriented service that is visible to the users.

As the name implies, ATM is not synchronous (i.e., tied to a master clock). It allows for the transport of digital information in the form of small, fixed-size packets called *cells*. The key feature of ATM to note here is that the connection-oriented service preserves call sequencing, which means that no reassembly of cells is needed prior to presenting the traffic stream to the destination host. The deployment of a cell-switching technology in B-ISDN is a gigantic break with the traditional use of circuit switching in the telephone network.

The primary purpose of ATM is to allocate *network resources* (i.e., bandwidth, buffers, and processing horsepower) efficiently so as to guarantee the expected *quality of service* (QoS) for each connection. QoS is measured in terms of three parameters:

- ▶ *Cell loss ratio*, defined as the ratio of the number of cells lost in transport across the network to the total number of cells pumped into the network.
- ▶ *Cell delay*, defined as the time taken for a cell of a particular connection to transit across the network.
- ▶ *Cell delay variation*, defined as the dispersion or jitter about the mean cell delay.

Quality of service offered in B-ISDN is to be contrasted with *best effort service* offered by the Internet.

TABLE 1 *Hierarchy of SONET data rates*

<i>Level^a</i>	<i>Data Rate (Mb/s)</i>
OC-1	51.84
OC-3	155.52
OC-9	466.56
OC-12	622.08
OC-18	933.12
OC-24	1,244.16
OC-36	1,866.24
OC-48	2,488.32

^aOC stands for optical carrier level.

After their generation, the ATM cells are structured for transport across the network. The cells in B-ISDN are placed on an optical transmission system called the *synchronous optical network* (SONET);⁸ optical fibers are discussed in the next section. SONET uses time-division multiplexing, whereby the entire bandwidth of an optical fiber is devoted to different incoming data streams on a time-shared basis, hence the need for a synchronous operation. SONET is controlled by a master clock with an accuracy of about 1 part in 10⁹. Thus bits of data are sent on a SONET line at extremely precise intervals, controlled by the master clock. Nevertheless, SONET permits the irregular time arrivals of ATM cells.

The basic SONET frame is a block of 810 bytes put out every 125 μ s for an overall data rate of 51.84 Mb/s. Having 8000 frames every second exactly matches the sampling rate of 8 kHz, which is the standard sampling rate for the digital transmission of voice signals across the telephone network. The basic data rates of 51.84 Mb/s are synchronously byte-interleaved to generate a hierarchy of data rates, as summarized in Table 1.

Communication Channels

The transmission of information across a communication network is accomplished in the physical layer by means of a *communication channel*. Depending on the mode of transmission used, we may distinguish two basic groups of communication channels: channels based on *guided propagation* and those based on *free propagation*. The first group includes telephone channels, coaxial cables, and optical fibers. The second group includes wireless broadcast channels, mobile radio channels, and satellite channels. These six channels are described in what follows.

- (i) As mentioned earlier, a typical telephone network uses circuit switching to establish an end-to-end communication link on a temporary basis. The primary purpose of the network is to ensure that the telephone transmission between a speaker at one end of the link and a listener at the other end is an acceptable substitute for face-to-face conversation. In this form of communication, the message source is the sound produced by the speaker's voice, and the ultimate destination is the listener's ear. The telephone channel, however, supports only the transmission of electrical signals. Accordingly, appropriate transducers are used at the transmitting and receiving ends of the system. Specifically, a *microphone* is placed near the speaker's

mouth to convert sound waves into an electrical signal, and the electrical signal is converted back into acoustic form by means of a *moving-coil receiver* placed near the listener's ear. Present-day designs of these two transducers have been perfected so as to respond well to frequencies ranging from 20 to 8000 Hz; moreover, a pair of them can be compactly packaged inside a single telephone set that is easy to speak into or listen from. The telephone channel is a *bandwidth-limited channel*. The restriction on bandwidth arises from the requirement of sharing the channel among a multitude of users at any one time. A practical solution to the telephonic communication problem must therefore minimize the channel bandwidth requirement, subject to a satisfactory transmission of human voice. To meet this requirement, the transducers and channel specifications must conform to standards based on subjective tests that are performed on the intelligibility, or articulation, of telephone signals by representative male and female speakers. A speech signal (male or female) is essentially limited to a band from 300 to 3100 Hz in the sense that frequencies outside this band do not contribute much to articulation efficiency. This frequency band may therefore be viewed as a rough guideline for the passband of a telephone channel that provides a satisfactory service, as illustrated in Figure 8 for a typical toll connection. Figure 8a shows the insertion loss of the channel plotted versus frequency; *insertion loss* (in dB) is defined as $10 \log_{10}(P_0/P_L)$, where P_L is the power delivered to a load from a source via the channel and P_0 is the power delivered to the same load when it is connected directly to the source. Figure 8b shows the corresponding plot of the envelope (group) delay (in milliseconds) versus frequency; *envelope delay* is defined as the negative of the derivative of the phase response with respect to the angular frequency $\omega = 2\pi f$. The plots of Figure 8 clearly illustrate the *dispersive* nature of the telephone channel.

The telephone channel is built using twisted pairs for signal transmission. A *twisted pair* consists of two solid copper conductors, each of which is encased in a polyvinylchloride (PVC) sheath. Typically, each pair has a twist rate of 2 to 12 twists per foot, and a characteristic impedance of 90 to 110 ohms. Twisted pairs are usually made up into cables, with each cable consisting of many pairs in close proximity to

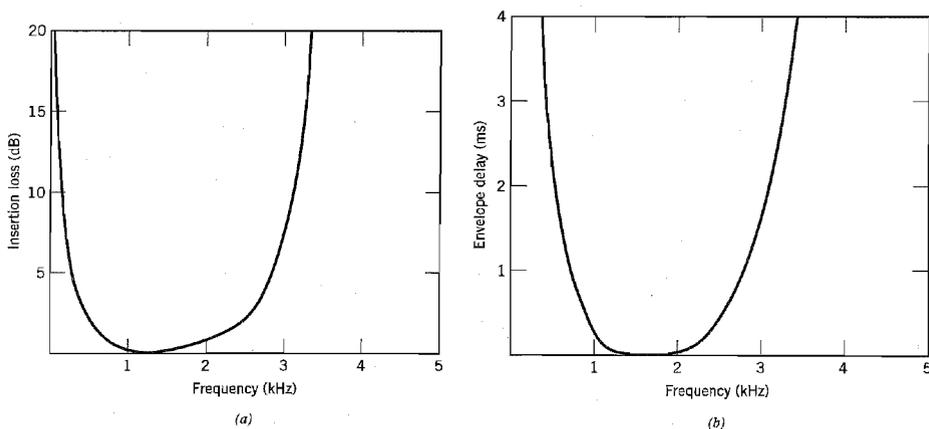


FIGURE 8 Characteristics of typical telephone connection: (a) Insertion loss. (b) Envelope delay. (Adapted from Bellamy, 1991.)

each other. Twisted pairs are naturally susceptible to *electromagnetic interference* (EMI), the effects of which are mitigated through twisting the wires.

- (ii) A *coaxial cable* consists of an inner conductor and an outer conductor, separated by a dielectric insulating material. The inner conductor is made of a copper wire encased inside the dielectric material. As for the outer conductor, it is made of copper, tinned copper, or copper-coated steel. Typically, a coaxial cable has a characteristic impedance of 50 or 75 ohms. Compared to a twisted-pair cable, a coaxial cable offers a greater degree of immunity to EMI. Moreover, because of their much higher bandwidth, coaxial cables can support the transmission of digital data at much higher bit rates than twisted pairs. Rates up to 20 Mb/s are feasible using coaxial cables, with 10 Mb/s being the standard.

Whereas the use of a twisted pair has been confined mainly to point-to-point service, a coaxial cable can operate as a multiple-access medium by using high-impedance taps. A common application of coaxial cables is as the transmission medium for local area networks in an office environment.

Another common application of coaxial cables is in *cable-television systems*, also known as *community-antenna television (CATV) systems*. In this application coaxial cables are used to distribute television, audio, and data signals from the head end to the subscribers. The *head end* is the central originating unit of the CATV system, where all signals are carried and processed.

- (iii) An *optical fiber* is a dielectric wave guide that transports light signals from one place to another just as a twisted-wire pair or a coaxial cable transports electrical signals. It consists of a central *core* within which the propagating electromagnetic field is confined and which is surrounded by a *cladding* layer, which is itself surrounded by a thin protective *jacket*.⁹ The core and cladding are both made of pure silica glass, whereas the jacket is made of plastic. Optical fibers have unique characteristics that make them highly attractive as a transmission medium. In particular, they offer the following unique characteristics:

- ▶ *Enormous potential bandwidth*, resulting from the use of optical carrier frequencies around 2×10^{14} Hz; with such a high carrier frequency and a bandwidth roughly equal to 10 percent of the carrier frequency, the theoretical bandwidth of a lightwave system is around 2×10^{13} Hz, which is very large indeed.
- ▶ *Low transmission losses*, as low as 0.1 dB/km.
- ▶ *Immunity to electromagnetic interference*, which is an inherent characteristic of an optical fiber viewed as a dielectric waveguide.
- ▶ *Small size and weight*, characterized by a diameter no greater than that of a human hair.
- ▶ *Ruggedness and flexibility*, exemplified by very high tensile strengths and the possibility of being bent or twisted without damage.

Last, but by no means least, optical fibers offer the potential for low-cost line communications since they are fabricated from sand, which, unlike the copper used in metallic conductors, is not a scarce resource. The unique properties of optical fibers have fuelled phenomenal advances in lightwave systems technology, which have, in turn, revolutionized long-distance communications and continue to do so.

- (iv) *Wireless broadcast channels* support the transmission of radio and television signals. The information-bearing signal, representing speech, music, or pictures, is modulated onto a carrier frequency that identifies the transmitting station; modulation is described in the next section. The transmission originates from an *antenna* that acts as the transition or matching unit between the source of the modulated signal and

electromagnetic waves in free space. The objective in designing the antenna is to excite the waves in the required direction or directions, as efficiently as possible. Typically, the transmitting antenna is mounted on a tower to provide an unobstructed view of the surrounding area, as far afield as possible. By virtue of the phenomenon of *diffraction*, which is a fundamental property of wave motion, radio waves are bent around the earth's surface. Propagation beyond the line of sight is thereby made possible, albeit with somewhat greater loss than is incurred in free space.

At the receiving end, an antenna is used to pick up the radiated waves, establishing a communication link to the transmitter. Most radio receivers are of the *superheterodyne* type. This technique consists of down-converting the received signal to some convenient frequency band, called the *intermediate frequency (IF) band*, and then recovering the original information-bearing signal by means of an appropriate detector.

- (v) A *mobile radio channel* extends the capability of the public telecommunications network by introducing *mobility* into the network by virtue of its ability to broadcast. The term *mobile radio* is usually meant to encompass terrestrial situations where a radio transmitter or receiver is capable of being moved, regardless of whether it actually moves or not. The major propagation effects encountered in the use of a mobile radio in built-up areas are due to the fact that the antenna of the mobile unit may lie well below the surrounding buildings. Simply put, there is no “line-of-sight” path for communication; rather, radio propagation takes place mainly by way of scattering from the surfaces of the surrounding buildings and by diffraction over and/or around them. The end result is that energy reaches the receiving antenna via more than one path. In a mobile radio environment, we thus speak of a *multipath phenomenon* in that the various incoming radio waves reach their destination from different directions and with different time delays. Indeed, there may be a multitude of propagation paths with different electrical lengths, and their contributions to the received signal could combine in a variety of ways. Consequently, the received signal strength varies with location in a very complicated fashion, and so a mobile radio channel may be viewed as a *linear time-varying channel* that is statistical in nature.
- (vi) Finally, a *satellite channel* adds another invaluable dimension to the public telecommunications network by providing *broad-area coverage* in both a continental and an intercontinental sense. Moreover, access to remote areas not covered by conventional cable or fiber communications is also a distinct feature of satellites. In almost all satellite communication systems, the satellites are placed in *geostationary orbit*. For the orbit to be geostationary, it has to satisfy two requirements. First, the orbit is *geosynchronous*, which requires the satellite to be at an altitude of 22,300 miles; a geosynchronous satellite orbits the Earth in 24 hours (i.e., the satellite is synchronous with the Earth's rotation). Second, the satellite is placed in orbit directly above the equator on an eastward heading (i.e., it has zero inclination). Viewed from Earth, a satellite in geostationary orbit appears to be stationary in the sky. Consequently, an Earth station does *not* have to track the satellite; rather, it merely has to point its antenna along a fixed direction, pointing toward the satellite. By so doing, the system design is simplified considerably. Communications satellites in geostationary orbit offer the following unique system capabilities:
 - ▶ Broad-area coverage.
 - ▶ Reliable transmission links.
 - ▶ Wide transmission bandwidths.

In terms of services, satellites can provide fixed point-to-point links extending over long distances and into remote areas, communication to mobile platforms (e.g., aircraft, ships), or broadcast capabilities. Indeed, communications satellites play a key role in the notion of the whole world being viewed as a “global village.” In a typical satellite communication system, a message signal is transmitted from an Earth station via an *uplink* to a satellite, amplified in a *transponder* (i.e., electronic circuitry) on board the satellite, and then retransmitted from the satellite via a *downlink* to another Earth station. With the satellite positioned in geostationary orbit, it is always visible to all the Earth stations located inside the satellite antenna’s coverage zones on the Earth’s surface. In effect, the satellite acts as a powerful *repeater* in the sky. The most popular frequency band for satellite communications is 6 GHz for the uplink and 4 GHz for the downlink. The use of this frequency band offers the following attributes:

- ▶ Relatively inexpensive microwave equipment.
- ▶ Low attenuation due to rainfall; rainfall is a primary atmospheric cause of signal loss.
- ▶ Insignificant sky background noise; the sky background noise (due to random noise emissions from galactic, solar, and terrestrial sources) reaches its lowest level between 1 and 10 GHz.

In the 6/4-GHz band, a typical satellite is assigned a 500 MHz bandwidth that is divided among 12 transponders on board the satellite. Each transponder, using approximately 36 MHz of the satellite bandwidth, corresponds to a specific radio channel. A single transponder can carry at least one color television signal, 1200 voice circuits, or digital data at a rate of 50 Mb/s.

To summarize, a communication channel is central to the operation of a communication system. Its properties determine both the information-carrying capacity of the system and the quality of service offered by the system. We may classify communication channels in different ways:

- ▶ A channel may be *linear* or *nonlinear*; a wireless radio channel is linear, whereas a satellite channel is usually (but not always) nonlinear.
- ▶ A channel may be *time invariant* or *time varying*; an optical fiber is time invariant, whereas a mobile radio channel is typically time varying.
- ▶ A channel may be *bandwidth limited* or *power limited* (i.e., limited in the available transmitted power); a telephone channel is bandwidth limited, whereas an optical fiber link and a satellite channel are both power limited.

Now that we have some understanding of sources of information and communication channels, we may return to the block diagram of a communication system shown in Figure 1.

Modulation Process

The purpose of a communication system is to deliver a message signal from an information source in recognizable form to a user destination, with the source and the user being physically separated from each other. To do this, the transmitter modifies the message signal into a form suitable for transmission over the channel. This modification is achieved by means of a process known as *modulation*, which involves varying some parameter of a *carrier wave* in accordance with the message signal. The receiver re-creates the original

message signal from a degraded version of the transmitted signal after propagation through the channel. This re-creation is accomplished by using a process known as *demodulation*, which is the reverse of the modulation process used in the transmitter. However, owing to the unavoidable presence of noise and distortion in the received signal, we find that the receiver cannot re-create the original message signal exactly. The resulting degradation in overall system performance is influenced by the type of modulation scheme used. Specifically, we find that some modulation schemes are less sensitive to the effects of noise and distortion than others.

We may classify the modulation process into *continuous-wave modulation* and *pulse modulation*. In continuous-wave (CW) modulation, a sinusoidal wave is used as the carrier. When the amplitude of the carrier is varied in accordance with the message signal, we have *amplitude modulation* (AM), and when the angle of the carrier is varied, we have *angle modulation*. The latter form of CW modulation may be further subdivided into *frequency modulation* (FM) and *phase modulation* (PM), in which the instantaneous frequency and phase of the carrier, respectively, are varied in accordance with the message signal.

In pulse modulation, on the other hand, the carrier consists of a periodic sequence of rectangular pulses. Pulse modulation can itself be of an analog or digital type. In *analog pulse modulation*, the amplitude, duration, or position of a pulse is varied in accordance with sample values of the message signal. In such a case, we speak of *pulse-amplitude modulation* (PAM), *pulse-duration modulation* (PDM), and *pulse-position modulation* (PPM).

The standard digital form of pulse modulation is known as *pulse-code modulation* (PCM) that has no CW counterpart. PCM starts out essentially as PAM, but with an important modification: The amplitude of each modulated pulse (i.e., *sample* of the original message signal) is *quantized* or *rounded off* to the nearest value in a prescribed set of *discrete* amplitude levels and then *coded* into a corresponding sequence of binary symbols. The binary symbols 0 and 1 are themselves represented by pulse signals that are suitably shaped for transmission over the channel. In any event, as a result of the quantization process, some information is always lost and the original message signal cannot therefore be reconstructed exactly. However, provided that the number of quantizing (discrete amplitude) levels is large enough, the distortion produced by the quantization process is not discernible to the human ear in the case of a speech signal or the human eye in the case of a two-dimensional image. Among all the different modulation schemes, pulse-code modulation has emerged as the preferred method of modulation for the transmission of analog message signals for the following reasons:

- ▶ *Robustness* in noisy environments by *regenerating* the transmitted signal at regular intervals.
- ▶ *Flexible* operation.
- ▶ *Integration* of diverse sources of information into a common format.
- ▶ *Security* of information in its transmission from source to destination.

In introducing the idea of modulation, we stressed its importance as a process that ensures the transmission of a message signal over a prescribed channel. There is another important benefit, namely, *multiplexing*, that results from the use of modulation. Multiplexing is the process of combining several message signals for their simultaneous transmission over the same channel. Three commonly used methods of multiplexing are as follows:

- ▶ *Frequency-division multiplexing* (FDM), in which CW modulation is used to translate each message signal to reside in a specific frequency slot inside the passband of

the channel by assigning it a distinct carrier frequency; at the receiver, a bank of filters is used to separate the different modulated signals and prepare them individually for demodulation.

- ▶ *Time-division multiplexing* (TDM), in which pulse modulation is used to position samples of the different message signals in nonoverlapping time slots.
- ▶ *Code-division multiplexing* (CDM), in which each message signal is identified by a distinctive code.

In FDM the message signals overlap with each other at the channel input; hence the system may suffer from *crosstalk* (i.e., interaction between message signals) if the channel is non-linear. In TDM the message signals use the full passband of the channel, but on a time-shared basis. In CDM the message signals are permitted to overlap in both time and frequency across the channel.

Mention should also be made of *wavelength-division multiplexing* (WDM), which is special to optical fibers. In WDM, wavelength is used as a new degree of freedom by concurrently operating distinct portions of the wavelength spectrum (i.e., distinct *colors*) that are accessible within the optical fiber. However, recognizing the reciprocal relationship that exists between the wavelength and frequency of an electromagnetic wave, we may say that WDM is a form of FDM.

Analog and Digital Types of Communication

Typically, in the design of a communication system the information source, communication channel, and information sink (end user) are all specified. The challenge is to design the transmitter and the receiver with the following guidelines in mind:

- ▶ Encode/modulate the message signal generated by the source of information, transmit it over the channel, and produce an “estimate” of it at the receiver output that satisfies the requirements of the end user.
- ▶ Do all of this at an affordable cost.

We have the option of using a digital or analog communication system.

Consider first the case of a *digital communication* system represented by the block diagram of Figure 9, the rationale for which is rooted in information theory. The functional blocks of the transmitter and the receiver, starting from the far end of the channel, are paired as follows:

- ▶ *Source encoder-decoder.*
- ▶ *Channel encoder-decoder.*
- ▶ *Modulator-demodulator.*

The source encoder removes redundant information from the message signal and is responsible for the efficient use of the channel. The resulting sequence of symbols is called the *source code word*. The data stream is processed next by the channel encoder, which produces a new sequence of symbols called the *channel code word*. The channel code word is longer than the source code word by virtue of the *controlled redundancy* built into its construction. Finally, the modulator represents each symbol of the channel code word by a corresponding analog symbol, appropriately selected from a finite set of possible analog symbols. The sequence of analog symbols produced by the modulator is called a *waveform*, which is suitable for transmission over the channel. At the receiver, the channel output (received signal) is processed in reverse order to that in the transmitter, thereby recon-

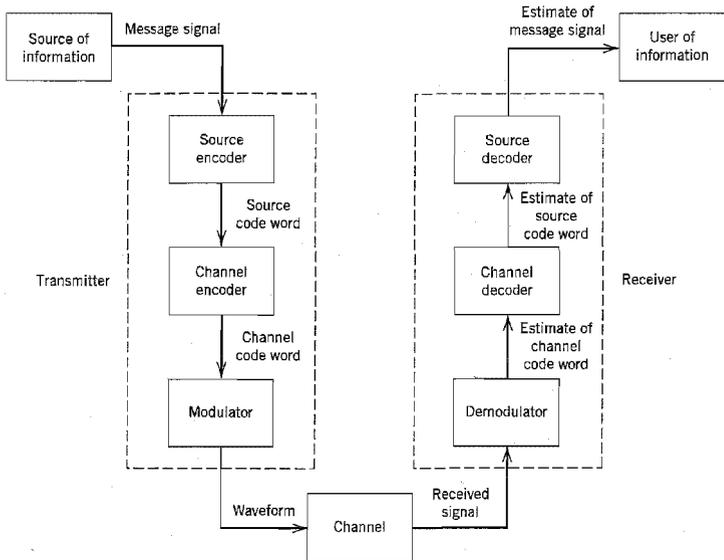


FIGURE 9 Block diagram of digital communication system.

structing a recognizable version of the original message signal. The reconstructed message signal is finally delivered to the user of information at the destination. From this description it is apparent that the design of a digital communication system is rather complex in conceptual terms but easy to build. Moreover, the system is *robust*, offering greater tolerance of physical effects (e.g., temperature variations, aging, mechanical vibrations) than its analog counterpart.

In contrast, the design of an *analog communication system* is simple in conceptual terms but difficult to build because of stringent requirements on linearity and system adjustment. For example, voice communication requires nonlinear distortion products at least 40 dB below the wanted message signal. In signal-processing terms, the transmitter consists of a modulator and the receiver consists of a demodulator, the details of which are determined by the type of CW modulation used.

The conceptual simplicity of analog communications is due to the fact that analog modulation techniques, exemplified by their wide use in radio and television, make relatively superficial changes to the message signal in order to prepare it for transmission over the channel. More specifically, there is no significant effort made by the system designer to tailor the waveform of the transmitted signal to suit the channel at any deeper level. On the other hand, digital communication theory endeavors to find a finite set of waveforms that are closely matched to the characteristics of the channel and which are therefore more tolerant of channel impairments. In so doing, reliable communication is established over the channel. In the selection of good waveforms for digital communication over a noisy channel, the design is influenced solely by the channel characteristics. However, once the appropriate set of waveforms for transmission over the channel has been selected, the source information can be encoded into the channel waveforms, and the efficient trans-

mission of information from the source to the user is thereby ensured. In summary, the use of digital communications provides the capability for information transmission that is both *efficient* and *reliable*.

From this discussion, it is apparent that the use of digital communications requires a considerable amount of electronic circuitry, but nowadays electronics are inexpensive, due to the ever-increasing availability of very-large-scale integrated (VLSI) circuits in the form of silicon chips. Thus although cost considerations used to be a factor in selecting analog communications over digital communications in the past, that is no longer the case.

Despite the trend toward the ever-increasing use of digital communications, a strong case can be made for the study of analog communications for two important reasons:

1. As long as we hear and see analog communications around us via radio and television, we need to understand how these communications systems work. Moreover, the study of analog modulation motivates other digital modulation schemes.
2. Analog devices and circuits have a natural affinity for operating at very high speeds and they consume very little power compared to their digital counterparts. Accordingly, the implementation of very high-speed or very low-power communication systems dictates the use of an analog approach.

Shannon's Information Capacity Theorem

The goal of a communication system designer is to configure a system that transports a message signal from a source of interest across a noisy channel to a user at the other end of the channel with the following objective:

The message signal is delivered to the user both efficiently and reliably, subject to certain design constraints: allowable transmit power, available channel bandwidth, and affordable cost of building the system.

In the case of a digital communication system, reliability is commonly expressed in terms of *bit error rate* (BER) or *probability of bit error* measured at the receiver output. Clearly, the smaller the BER, the more reliable the communication system is. A question that comes to mind in this context is whether it is possible to design a communication system that operates with zero BER even through the channel is noisy. In an *ideal* setting, the answer to this question is an emphatic *yes*. The answer is embodied in one of Shannon's celebrated theorems,¹⁰ which is called the information capacity theorem.

Let B denote the channel bandwidth, and let SNR denote the received signal-to-noise ratio. The *information capacity theorem* states that ideally these two parameters are related as

$$C = B \log_2(1 + \text{SNR}) \text{ b/s} \quad (1)$$

where C is the information capacity of the channel. The *information capacity* is defined as the maximum rate at which information can be transmitted across the channel without error; it is measured in *bits per second* (b/s). For a prescribed channel bandwidth B and received SNR, the information capacity theorem tells us that a message signal can be transmitted through the system without error even when the channel is noisy, provided that the *actual signaling rate* R in bits per second, at which data are transmitted through the channel, is less than the information capacity C .

Unfortunately, Shannon's information capacity theorem does not tell us how to design the system. Nevertheless, from a design point of view, the theorem is very valuable for the following reasons:

1. The information capacity theorem provides a *bound* on what rate of data transmission is theoretically attainable for prescribed values of channel bandwidth B and received SNR. On this basis, we may use the ratio

$$\eta = \frac{R}{C}$$

as a measure of the *efficiency* of the digital communication system under study. The closer η is to unity, the more efficient the system is.

2. Equation (1) provides a basis for the trade-off between channel bandwidth B and received SNR. In particular, for a prescribed signaling rate R , we may reduce the required SNR by increasing the channel bandwidth B , hence the motivation for using a wideband modulated scheme (e.g., pulse-code modulation) for improved noise performance.
3. Equation (1) provides an idealized framework for comparing the noise performance of one modulation scheme against another.

A Digital Communication Problem

When we speak of a digital communication system having a low bit error rate, say, the implication is that only a small fraction in a long stream of binary symbols is decoded in error by the receiver. The issue of the receiver determining whether a binary symbol sent over a noisy channel is decoded in error or not is of fundamental importance to the design of digital communication systems. It is therefore appropriate briefly to discuss this basic issue so as to motivate the study of communication systems.

Suppose we have a random binary signal, $m(t)$, consisting of symbols 1 and 0 that are equally likely. Symbol 1 is represented by a constant level $+1$, and symbol 0 is represented by a constant level -1 , each of which lasts for a duration T . Such a signal may represent the output of a digital computer or the digitized version of a speech signal. To facilitate the transmission of this signal over a communication channel, we employ a simple modulation scheme known as *phase-shift keying*. Specifically, the information bearing signal $m(t)$ is multiplied by a sinusoidal carrier wave $A_c \cos(2\pi f_c t)$, where A_c is the carrier amplitude, f_c is the carrier frequency, and t is time. Figure 10a shows a block diagram of the transmitter, the output of which is defined by

$$s(t) = \begin{cases} A_c \cos(2\pi f_c t) & \text{for symbol 1} \\ -A_c \cos(2\pi f_c t) & \text{for symbol 0} \end{cases} \quad (2)$$

where $0 \leq t \leq T$. The carrier frequency f_c is a multiple of $1/T$.

The channel is assumed to be distortionless but noisy, as depicted in Figure 10b. The received signal $x(t)$ is thus defined by

$$x(t) = s(t) + w(t) \quad (3)$$

where $w(t)$ is the additive channel noise.

The receiver consists of a correlator followed by a decision-making device, as depicted in Figure 10c. The *correlator* multiplies the received signal $x(t)$ by a locally generated

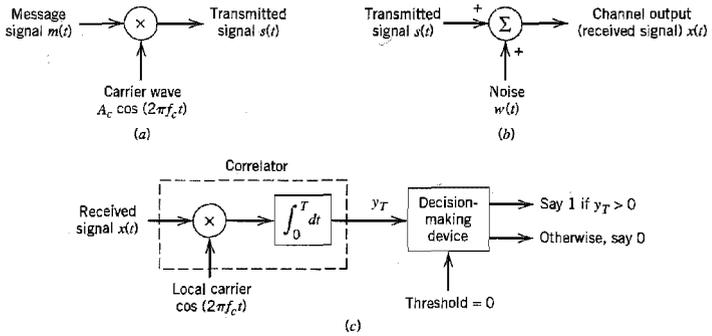


FIGURE 10 Elements of a digital communication system. (a) Block diagram of transmitter. (b) Block diagram of channel. (c) Block diagram of receiver.

carrier $\cos(2\pi f_c t)$ and then integrates the product over the symbol interval $0 \leq t \leq T$, producing the output

$$y_T = \int_0^T x(t) \cos(2\pi f_c t) dt \tag{4}$$

Substituting Equations (2) and (3) into (4) and invoking the assumption that the carrier frequency f_c is a multiple of $1/T$, we obtain (after the simplification of terms)

$$y_T = \begin{cases} +\frac{A_c}{2} + w_T & \text{for symbol 1} \\ -\frac{A_c}{2} + w_T & \text{for symbol 0} \end{cases} \tag{5}$$

where w_T is the contribution of the correlator output due to the channel noise $w(t)$. To reconstruct the original binary signal $m(t)$, the correlator output y_T is compared against a threshold of zero volts by the *decision-making device*, the operation of which is based on the following rule:

If the correlator output y_T is greater than zero, the receiver outputs symbol 1; otherwise, it outputs symbol 0.

With this background, we may now discuss/raise some basic issues. First, from Fourier analysis we find that the *time-bandwidth product* of a pulse signal is constant. This means that the bandwidth of a rectangular pulse of duration T is inversely proportional to T . The transmitted signal in Figure 10a consists of the product of this rectangular signal and the sinusoidal carrier $A_c \cos(2\pi f_c t)$. The multiplication of a signal by a sinusoid has the effect of shifting the Fourier transform of the signal to the right by f_c and to the left by an equal amount, except for the scaling factor of $1/2$. It follows therefore that the bandwidth of the transmitted signal $m(t)$, and therefore the required channel bandwidth, is inversely proportional to the reciprocal of the symbol duration T . For the problem at hand, the reciprocal of T is also the signaling rate of the system in b/s.

There are, however, some other issues that require theoretical considerations:

1. What is the justification for the receiver structure of Figure 10c?
2. The noise contribution w_T is the value of a random variable W produced by sampling a certain realization $w(t)$ of the channel noise at time $t = T$ in accordance with Equations (3) and (4). How do we relate the statistics of the random variable W to the statistical characteristics of the channel noise?
3. The receiver of Figure 10c makes occasional errors due to the random nature of the correlator output. That is, the receiver decides in favor of symbol 0 given that symbol 1 was actually transmitted, and vice versa. What is the probability of decision errors?

Moreover, there are some important practical issues that need attention:

1. Channel bandwidth is a highly valuable resource. How do we choose a modulation scheme that conserves bandwidth in a cost-effective manner?
2. The binary signal $m(t)$ may include redundant symbols introduced into it through the use of channel encoding so as to provide protection against channel noise. How do we design the channel encoder in the transmitter and the channel decoder in the receiver so as to come very close to Shannon's information capacity theorem in a physically realizable manner?
3. The locally generated carrier in the receiver of Figure 10c is physically separate from the carrier source used for modulation in the transmitter. How do we *synchronize* the receiver to the transmitter with respect to both the carrier phase and symbol timing so as to justify the use of Equation (4) as the basis of decision-making in the reconstruction of the original binary signal?

The theoretical and practical issues raised here in the context of the simple digital communication system of Figure 10 are addressed in the following chapters of the book.

Historical Notes¹¹

A preview of communications would be incomplete without a history of the subject. In this final section of this introductory chapter we present some historical notes on communications; each paragraph focuses on some important and related events. It is hoped that this material will provide a sense of inspiration and motivation for the reader.

In 1837, the telegraph was perfected by Samuel Morse, a painter. With the words "What hath God wrought," transmitted by Morse's electric telegraph between Washington, D.C., and Baltimore, Maryland, in 1844, a completely revolutionary means of real-time, long-distance communications was triggered. The telegraph is the forerunner of digital communications in that the *Morse code* is a *variable-length* ternary code using an alphabet of four symbols: a dot, a dash, a letter space, and a word space; short sequences represent frequent letters, whereas long sequences represent infrequent letters. This type of signaling is ideal for manual keying. Subsequently, Emile Baudot developed a *fixed-length* binary code for telegraphy in 1875. In Baudot's telegraphic code, well-suited for use with teletypewriters, each code word consists of five equal-length code elements, and each element is assigned one of two possible states: a *mark* or a *space* (i.e., symbol 1 or 0 in today's terminology).

In 1864, James Clerk Maxwell formulated the electromagnetic theory of light and predicted the existence of radio waves; the underlying set of equations bears his name. The

existence of radio waves was established experimentally by Heinrich Hertz in 1887. In 1894, Oliver Lodge demonstrated wireless communication over a relatively short distance (150 yards). Then, on December 12, 1901, Guglielmo Marconi received a radio signal at Signal Hill in Newfoundland; the radio signal had originated in Cornwall, England, 1700 miles away across the Atlantic. The way was thereby opened toward a tremendous broadening of the scope of communications. In 1906, Reginald Fessenden, a self-educated academic, made history by conducting the first radio broadcast.

In 1875, the *telephone* was invented by Alexander Graham Bell, a teacher of the deaf. The telephone made real-time transmission of speech by electrical encoding and replication of sound a practical reality. The first version of the telephone was crude and weak, enabling people to talk over short distances only. When telephone service was only a few years old, interest developed in automating it. Notably, in 1897, A. B. Strowger, an undertaker from Kansas City, Missouri, devised the automatic *step-by-step switch* that bears his name; of all the electromechanical switches devised over the years, the Strowger switch was the most popular and widely used.

In 1904, John Ambrose Fleming invented the *vacuum-tube diode*, which paved the way for the invention of the *vacuum-tube triode* by Lee de Forest in 1906. The discovery of the triode was instrumental in the development of transcontinental telephony in 1913 and signaled the dawn of wireless voice communications. Indeed, until the invention and perfection of the transistor, the triode was the supreme device for the design of electronic amplifiers.

In 1918, Edwin H. Armstrong invented the *superheterodyne radio* receiver; to this day, almost all radio receivers are of this type. In 1933, Armstrong demonstrated another revolutionary concept, namely, a modulation scheme that he called *frequency modulation* (FM); Armstrong's paper making the case for FM radio was published in 1936.

The first all-electronic *television* system was demonstrated by Philo T. Farnsworth in 1928, and then by Vladimir K. Zworykin in 1929. By 1939, the British Broadcasting Corporation (BBC) was broadcasting television on a commercial basis.

In 1928, Harry Nyquist published a classic paper on the theory of signal transmission in telegraphy. In particular, Nyquist developed criteria for the correct reception of telegraph signals transmitted over dispersive channels in the absence of noise. Much of Nyquist's early work was applied later to the transmission of digital data over dispersive channels.

In 1937, Alec Reeves invented *pulse-code modulation* (PCM) for the digital encoding of speech signals. The technique was developed during World War II to enable the encryption of speech signals; indeed, a full-scale, 24-channel system was used in the field by the United States military at the end of the war. However, PCM had to await the discovery of the transistor and the subsequent development of large-scale integration of circuits for its commercial exploitation.

In 1943, D. O. North devised the *matched filter* for the optimum detection of a known signal in additive white noise. A similar result was obtained in 1946 independently by J. H. Van Vleck and D. Middleton, who coined the term *matched filter*.

In 1947, the geometric representation of signals was developed by V. A. Kotelnikov in a doctoral dissertation presented before the Academic Council of the Molotov Energy Institute in Moscow. This method was subsequently brought to full fruition by John M. Wozencraft and Irwin M. Jacobs in a landmark textbook published in 1965.

In 1948, the theoretical foundations of digital communications were laid by Claude Shannon in a paper entitled "A Mathematical Theory of Communication." Shannon's paper was received with immediate and enthusiastic acclaim. It was perhaps this response

that emboldened Shannon to amend the title of his paper to “The Mathematical Theory of Communication” when it was reprinted a year later in a book co-authored with Warren Weaver. It is noteworthy that prior to the publication of Shannon’s 1948 classic paper, it was believed that increasing the rate of information transmission over a channel would increase the probability of error; the communication theory community was taken by surprise when Shannon proved that this was not true, provided that the transmission rate was below the channel capacity. Shannon’s 1948 paper was followed by some significant advances in coding theory, which include the following:

- ▶ Development of the first nontrivial *error-correcting codes* by M. J. E. Golay in 1949 and Richard W. Hamming in 1950.
- ▶ Development of *turbo codes* by C. Berrou, A. Glavieux, and P. Thitimajshima in 1993; turbo codes provide near-optimum error-correcting coding and decoding performance in the Shannon sense.

The *transistor* was invented in 1948 by Walter H. Brattain, John Bardeen, and William Shockley at Bell Laboratories. The first silicon integrated circuit (IC) was produced by Robert Noyce in 1958. These landmark innovations in solid-state devices and integrated circuits led to the development of *very-large-scale integrated* (VLSI) circuits and single-chip *microprocessors*, and with them the nature of signal processing and the telecommunications industry changed forever.

The invention of the transistor in 1948 spurred the application of electronics to switching and digital communications. The motivation was to improve reliability, increase capacity, and reduce cost. The first call through a stored-program system was placed in March 1958 at Bell Laboratories, and the first commercial telephone service with digital switching began in Morris, Illinois, in June 1960. The first *T-1 carrier system* transmission was installed in 1962 by Bell Laboratories.

During the period 1943 to 1946, the first electronic digital computer, called the ENIAC, was built at the Moore School of Electrical Engineering of the University of Pennsylvania under the technical direction of J. Presper Eckert, Jr., and John W. Mauchly. However, John von Neumann’s contributions were among the earliest and most fundamental to the theory, design, and application of digital computers, which go back to the first draft of a report written in 1945. Computers and terminals started communicating with each other over long distances in the early 1950s. The links used were initially voice-grade telephone channels operating at low speeds (300 to 1200 b/s). Various factors have contributed to a dramatic increase in data transmission rates; notable among them are the idea of *adaptive equalization*, pioneered by Robert Lucky in 1965, and efficient modulation techniques, pioneered by G. Ungerboeck in 1982. Another idea widely employed in computer communications is that of *automatic repeat-request* (ARQ). The ARQ method was originally devised by H. C. A. van Duuren during World War II and published in 1946. It was used to improve radio-telephony for telex transmission over long distances.

From 1950 to 1970, various studies were made on *computer networks*. However, the most significant of them in terms of impact on computer communications was the Advanced Research Project Agency Network (ARPANET), first put into service in 1971. The development of ARPANET was sponsored by the Advanced Research Projects Agency of the U.S. Department of Defense. The pioneering work in *packet switching* was done on ARPANET. In 1985, ARPANET was renamed the *Internet*. The turning point in the evolution of the Internet occurred in 1990 when Tim Berners-Lee proposed a hypermedia software interface to the Internet, which he named the *World Wide Web*.¹² Thereupon, in

the space of only about two years, the Web went from nonexistence to worldwide popularity, culminating in its commercialization in 1994. How do we explain the explosive growth of the Internet? We may answer this question by offering these reasons:¹³

- ▶ Before the Web exploded into existence, the ingredients for its creation were already in place. In particular, thanks to VLSI, personal computers (PCs) had already become ubiquitous in homes throughout the world, and they were increasingly equipped with modems for interconnectivity to the outside world.
- ▶ For about two decades, the Internet had grown steadily (albeit within a confined community of users), reaching a critical threshold of user-value based electronic mail and file transfer.
- ▶ Standards for document description and transfer, hypertext markup language (HTML), and hypertext transfer protocol (HTTP) had been adopted.

Thus, everything needed for creating the Web was already in place except for two critical ingredients: a simple user interface and a brilliant service concept.

In 1955, John R. Pierce proposed the use of satellites for communications. This proposal was preceded, however, by an earlier paper by Arthur C. Clark that was published in 1945, also proposing the idea of using an *Earth-orbiting* satellite as a relay point for communication between two Earth stations. In 1957, the Soviet Union launched Sputnik I, which transmitted telemetry signals for 21 days. This was followed shortly by the launching of Explorer I by the United States in 1958, which transmitted telemetry signals for about five months. A major experimental step in communications satellite technology was taken with the launching of Telstar I from Cape Canaveral on July 10, 1962. The Telstar satellite was built by Bell Laboratories, which had acquired considerable knowledge from pioneering work by Pierce. The satellite was capable of relaying TV programs across the Atlantic; this was made possible only through the user of maser receivers and large antennas.

The use of optical means (e.g., smoke and fire signals) for the transmission of information dates back to prehistoric times. However, no major breakthrough in optical communications was made until 1966, when K. C. Kao and G. A. Hockham of Standard Telephone Laboratories, U.K., proposed the use of a clad glass fiber as a dielectric waveguide. The *laser* (an acronym for light amplification by stimulated emission of radiation) had been invented and developed in 1959 and 1960. Kao and Hockham pointed out that (1) the attenuation in an optical fiber was due to impurities in the glass, and (2) the intrinsic loss, determined by Rayleigh scattering, is very low. Indeed, they predicted that a loss of 20 dB/km should be attainable. This remarkable prediction, made at a time when the power loss in a glass fiber was about 1000 dB/km, was to be demonstrated later. Nowadays, transmission losses as low as 0.1 dB/km are achievable.

The spectacular advances in microelectronics, digital computers, and lightwave systems that we have witnessed to date, and that will continue into the future, are all responsible for dramatic changes in the telecommunications environment; many of these changes are already in place, and more changes will evolve as time goes on.

NOTES AND REFERENCES

1. For essays on an early account of communications and other related disciplines (e.g., electronics, computers, radar, radio astronomy, satellites), see Overhage (1962); in particular, see the chapter on "Communications" by L. V. Berkner, pp. 35–50.

2. The JPEG image coding standard is discussed in the papers by Wallace (1991); see also the article by T. A. Ramstad in the handbook edited by Madiseti and Williams (1998).
3. The discrete cosine transform (DCT) and its inverse for a block of 8×8 source image samples are respectively defined by

$$F(u, v) = \frac{1}{4} C(u)C(v) \left[\sum_{x=0}^7 \sum_{y=0}^7 f(x, y) \cos\left(\frac{(2x+1)u\pi}{16}\right) \cos\left(\frac{(2y+1)v\pi}{16}\right) \right]$$

$$f(x, y) = \frac{1}{4} \left[\sum_{u=0}^7 \sum_{v=0}^7 C(u)C(v)F(u, v) \cos\left(\frac{(2x+1)u\pi}{16}\right) \cos\left(\frac{(2y+1)v\pi}{16}\right) \right]$$

where

$$C(u), C(v) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for } u = 0 \text{ and } v = 0 \\ 1 & \text{otherwise} \end{cases}$$

For a full treatment of the DCT, see Rao and Yip (1990).

4. The MPEG-1 video coding standard is discussed in the paper by Gall (1991); see also the article by A. M. Tekalp in the handbook edited by Madiseti and Williams (1998), which discusses the follow-up versions of the MPEG video coding standard.
5. The MPEG-1 audio coding standard is discussed in the papers by Brandenburg and Stoll (1994) and Pan (1993); see also the article by P. Noll in the handbook edited by Madiseti and Williams (1998), which also discusses the follow-up versions of the MPEG audio coding standard. In particular, the widespread use of the more current standard, MPEG-3 audio, is resulting in a level of piracy that may dwarf the earlier problems of "bootleg" cassette tapes.
6. For a detailed discussion of communication networks, see Tanenbaum (1996).
7. The OSI reference model was developed by a subcommittee of the International Organization for Standardization (ISO) in 1977. For a discussion of the principles involved in arriving at the seven layers of the OSI model and a description of the layers themselves, see Tanenbaum (1996).
8. SONET was originally proposed by Telcordia Technologies Inc. (then known as Bellcore) and standardized by the American National Standards Institute (ANSI). Later, CCITT approved a SONET standard and issued a set of parallel recommendations called *synchronous digital hierarchy* (SDH). The differences between SONET and SDH are of a minor nature.
9. For a thorough and precise analysis of the propagation of light waves in an optical fiber, we need to treat it as a dielectric waveguide and use Maxwell's equations to carry out the analysis; such an analysis is highly mathematical in nature. For a readable account of the analysis, see Chapter 3 of Green, Jr. (1993).
10. For a semitechnical overview of Shannon's theorems on information theory presented in a highly readable fashion, see the book entitled *Silicon Dreams* by Lucky (1989).
11. For a readable account of the history of communications, see Lebow (1995).
12. For a historical account of the development of the Internet, see Leiner et al. (1997).
13. For an insightful essay on new telecommunications services and how society reacts to their development, see Lucky (1997). This paper points to *Metcalf's law*, according to which it seems as if any new telecommunications service must take a long time for it to build to universal acceptance. Lucky cites the World Wide Web as a startling counterexample to Metcalf's law and gives the reasons why.

RANDOM PROCESSES

This chapter presents an introductory treatment of stationary random processes with emphasis on second-order statistics. In particular, it discusses the following issues:

- ▶ *The notion of a random process.*
- ▶ *The requirement that has to be satisfied for a random process to be stationary.*
- ▶ *The partial description of a random process in terms of its mean, correlation, and covariance functions.*
- ▶ *The conditions that have to be satisfied for a stationary random process to be ergodic, a property that enables us to substitute time averages for ensemble averages.*
- ▶ *What happens to a stationary random process when it is transmitted through a linear time-invariant filter?*
- ▶ *The frequency-domain description of a random process in terms of power spectral density.*
- ▶ *The characteristics of an important type of random process known as a Gaussian process.*
- ▶ *Sources of noise and their narrowband form.*
- ▶ *Rayleigh and Rician distributions, which represent two special probability distributions that arise in the study of communication systems.*

1.1 Introduction

The idea of a *mathematical model* used to describe a physical phenomenon is well established in the physical sciences and engineering. In this context, we may distinguish two classes of mathematical models: deterministic and stochastic. A model is said to be *deterministic* if there is no uncertainty about its time-dependent behavior at any instant of time. However, in many real-world problems the use of a deterministic model is inappropriate because the physical phenomenon of interest involves too many unknown factors. Nevertheless, it may be possible to consider a model described in probabilistic terms in that we speak of the *probability* of a future value lying between two specified limits. In such a case, the model is said to be *stochastic* or *random*. A brief review of probability theory is presented in Appendix 1.

Consider, for example, a radio communication system. The received signal in such a system usually consists of an *information-bearing signal* component, a random *interference* component, and *channel noise*. The information-bearing signal component may represent, for example, a voice signal that, typically, consists of randomly spaced bursts of energy of random duration. The interference component may represent spurious electromagnetic waves produced by other communication systems operating in the vicinity of the

radio receiver. A major source of channel noise is *thermal noise*, which is caused by the random motion of the electrons in conductors and devices at the front end of the receiver. We thus find that the received signal is random in nature. Although it is not possible to predict the exact value of the signal in advance, it is possible to describe the signal in terms of statistical parameters such as average power and power spectral density, as discussed in this chapter.

1.2 Mathematical Definition of a Random Process

In light of these introductory remarks, it is apparent that random processes have two properties. First, they are functions of time. Second, they are random in the sense that before conducting an experiment, it is not possible to exactly define the waveforms that will be observed in the future.

In describing a random experiment it is convenient to think in terms of a sample space. Specifically, each outcome of the experiment is associated with a *sample point*. The totality of sample points corresponding to the aggregate of all possible outcomes of the experiment is called the *sample space*. Each sample point of the sample space is a function of time. The sample space or ensemble composed of functions of time is called a *random* or *stochastic process*.¹ As an integral part of this notion, we assume the existence of a probability distribution defined over an appropriate class of sets in the sample space, so that we may speak with confidence of the probability of various events.

Consider, then, a random experiment specified by the outcomes s from some *sample space* S , by the events defined on the sample space S , and by the probabilities of these

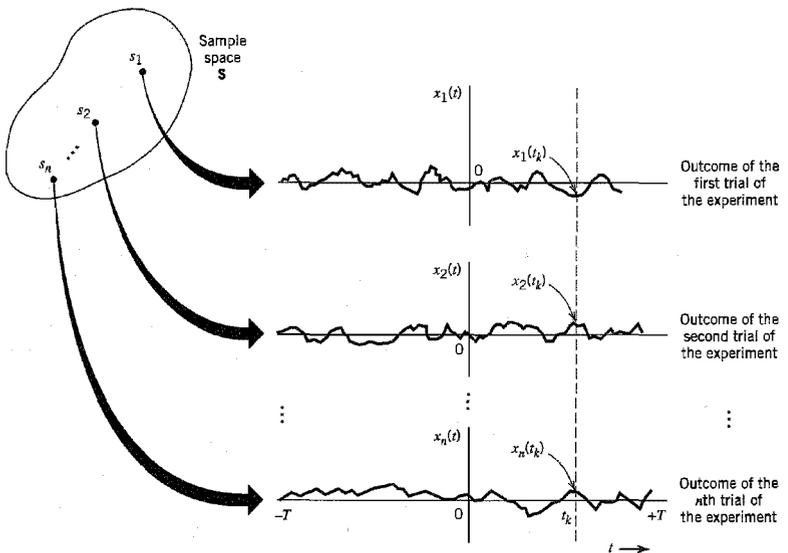


FIGURE 1.1 An ensemble of sample functions.

events. Suppose that we assign to each sample point s a function of time in accordance with the rule:

$$X(t, s), \quad -T \leq t \leq T \quad (1.1)$$

where $2T$ is the *total observation interval*. For a fixed sample point s_j , the graph of the function $X(t, s_j)$ versus time t is called a *realization* or *sample function* of the random process. To simplify the notation, we denote this sample function as

$$x_j(t) = X(t, s_j) \quad (1.2)$$

Figure 1.1 illustrates a set of sample functions $\{x_j(t) | j = 1, 2, \dots, n\}$. From this figure, we note that for a fixed time t_k inside the observation interval, the set of numbers

$$\{x_1(t_k), x_2(t_k), \dots, x_n(t_k)\} = \{X(t_k, s_1), X(t_k, s_2), \dots, X(t_k, s_n)\}$$

constitutes a *random variable*. Thus we have an indexed ensemble (family) of random variables $\{X(t, s)\}$, which is called a *random process*. To simplify the notation, the customary practice is to suppress the s and simply use $X(t)$ to denote a random process. We may now formally define a random process $X(t)$ as *an ensemble of time functions together with a probability rule that assigns a probability to any meaningful event associated with an observation of one of the sample functions of the random process*. Moreover, we may distinguish between a random variable and a random process as follows:

- ▶ For a random variable, the outcome of a random experiment is mapped into a number.
- ▶ For a random process, the outcome of a random experiment is mapped into a waveform that is a function of time.

1.3 Stationary Processes

In dealing with random processes encountered in the real world, we often find that the statistical characterization of a process is independent of the time at which observation of the process is initiated. That is, if such a process is divided into a number of time intervals, the various sections of the process exhibit essentially the same statistical properties. Such a process is said to be *stationary*. Otherwise, it is said to be *nonstationary*. Generally speaking, a stationary process arises from a stable physical phenomenon that has evolved into a steady-state mode of behavior, whereas a nonstationary process arises from an unstable phenomenon.

To be more precise, consider a random process $X(t)$ that is initiated at $t = -\infty$. Let $X(t_1), X(t_2), \dots, X(t_k)$ denote the random variables obtained by observing the random process $X(t)$ at times t_1, t_2, \dots, t_k , respectively. The joint distribution function of this set of random variables is $F_{X(t_1), \dots, X(t_k)}(x_1, \dots, x_k)$. Suppose next we shift all the observation times by a fixed amount τ , thereby obtaining a new set of random variables $X(t_1 + \tau), X(t_2 + \tau), \dots, X(t_k + \tau)$. The joint distribution function of this latter set of random variables is $F_{X(t_1 + \tau), \dots, X(t_k + \tau)}(x_1, \dots, x_k)$. The random process $X(t)$ is said to be *stationary in the strict sense* or *strictly stationary* if the following condition holds:

$$F_{X(t_1 + \tau), \dots, X(t_k + \tau)}(x_1, \dots, x_k) = F_{X(t_1), \dots, X(t_k)}(x_1, \dots, x_k) \quad (1.3)$$

for all time shifts τ , all k , and all possible choices of observation times t_1, \dots, t_k . In other words, a *random process* $X(t)$, initiated at time $t = -\infty$, is *strictly stationary* if the joint distribution of any set of random variables obtained by observing the random process $X(t)$ is invariant with respect to the location of the origin $t = 0$. Note that the finite-dimensional

distributions in Equation (1.3) depend on the relative time separation between random variables but not on their absolute time. That is, the random process has the same probabilistic behavior through all time.

Similarly, we may say that two random processes $X(t)$ and $Y(t)$ are *jointly strictly stationary* if the joint finite-dimensional distributions of the two sets of random variables $X(t_1), \dots, X(t_k)$ and $Y(t'_1), \dots, Y(t'_j)$ are invariant with respect to the origin $t = 0$ for all k and j and all choices of observation times t_1, \dots, t_k and t'_1, \dots, t'_j .

Returning to Equation (1.3), we may distinguish two situations of special interest:

1. For $k = 1$, we have

$$F_{X(t)}(\mathbf{x}) = F_{X(t+\tau)}(\mathbf{x}) = F_X(\mathbf{x}) \quad \text{for all } t \text{ and } \tau \quad (1.4)$$

That is, *the first-order distribution function of a stationary random process is independent of time.*

2. For $k = 2$ and $\tau = -t_1$, we have

$$F_{X(t_1), X(t_2)}(\mathbf{x}_1, \mathbf{x}_2) = F_{X(0), X(t_2-t_1)}(\mathbf{x}_1, \mathbf{x}_2) \quad \text{for all } t_1 \text{ and } t_2 \quad (1.5)$$

That is, *the second-order distribution function of a stationary random process depends only on the time difference between the observation times and not on the particular times at which the random process is observed.*

These two properties have profound implications for the statistical parameterization of a stationary random process; this issue is discussed in Section 1.4.

▼ EXAMPLE 1.1

Consider Figure 1.2, depicting three spatial windows located at times t_1, t_2, t_3 . We wish to evaluate the probability of obtaining a sample function $x(t)$ of a random process $X(t)$ that passes through this set of windows, that is, the probability of the joint event

$$A = \{a_i < X(t_i) \leq b_i\}, \quad i = 1, 2, 3$$

In terms of the joint distribution function, this probability equals

$$P(A) = F_{X(t_1), X(t_2), X(t_3)}(b_1, b_2, b_3) - F_{X(t_1), X(t_2), X(t_3)}(a_1, a_2, a_3)$$

Suppose now the random process $X(t)$ is known to be strictly stationary. An implication of strict stationarity is that the probability of the set of sample functions of this process passing through the windows of Figure 1.3a is equal to the probability of the set of sample functions passing through the corresponding time-shifted windows of Figure 1.3b. Note, however, that it is not necessary that these two sets consist of the same sample functions. ◀

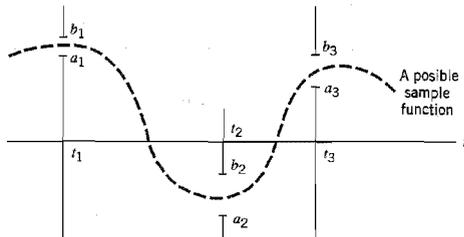


FIGURE 1.2 Illustrating the probability of a joint event.

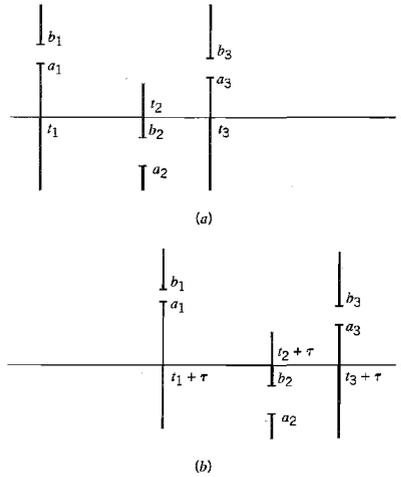


FIGURE 1.3 Illustrating the concept of stationarity in Example 1.1.

1.4 Mean, Correlation, and Covariance Functions

Consider a strictly stationary random process $X(t)$. We define the *mean* of the process $X(t)$ as the expectation of the random variable obtained by observing the process at some time t , as shown by

$$\begin{aligned}\mu_X(t) &= E[X(t)] \\ &= \int_{-\infty}^{\infty} x f_{X(t)}(x) dx\end{aligned}\quad (1.6)$$

where $f_{X(t)}(x)$ is the first-order probability density function of the process. From Equation (1.4) we deduce that for a strictly stationary random process, $f_{X(t)}(x)$ is independent of time t . Consequently, *the mean of a strictly stationary process is a constant*, as shown by

$$\mu_X(t) = \mu_X \quad \text{for all } t \quad (1.7)$$

We define the *autocorrelation function* of the process $X(t)$ as the expectation of the product of two random variables, $X(t_1)$ and $X(t_2)$, obtained by observing the process $X(t)$ at times t_1 and t_2 , respectively. Specifically, we write

$$\begin{aligned}R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1), X(t_2)}(x_1, x_2) dx_1 dx_2\end{aligned}\quad (1.8)$$

where $f_{X(t_1), X(t_2)}(x_1, x_2)$ is the second-order probability density function of the process. From Equation (1.5), we deduce that for a strictly stationary random process, $f_{X(t_1), X(t_2)}(x_1, x_2)$ depends only on the difference between the observation times t_1 and t_2 .

This, in turn, implies that *the autocorrelation function of a strictly stationary process depends only on the time difference $t_2 - t_1$, as shown by*

$$R_X(t_1, t_2) = R_X(t_2 - t_1) \quad \text{for all } t_1 \text{ and } t_2 \quad (1.9)$$

Similarly, the *autocovariance function* of a strictly stationary process $X(t)$ is written as

$$\begin{aligned} C_X(t_1, t_2) &= E[(X(t_1) - \mu_X)(X(t_2) - \mu_X)] \\ &= R_X(t_2 - t_1) - \mu_X^2 \end{aligned} \quad (1.10)$$

Equation (1.10) shows that, like the autocorrelation function, the autocovariance function of a strictly stationary process $X(t)$ depends only on the time difference $t_2 - t_1$. This equation also shows that if we know the mean and autocorrelation function of the process, we can uniquely determine the autocovariance function. The mean and autocorrelation function are therefore sufficient to describe the first two moments of the process.

However, two important points should be carefully noted:

1. The mean and autocorrelation function only provide a *partial description* of the distribution of a random process $X(t)$.
2. The conditions of Equations (1.7) and (1.9), involving the mean and autocorrelation function, respectively, are *not* sufficient to guarantee that the random process $X(t)$ is strictly stationary.

Nevertheless, practical considerations often dictate that we simply limit ourselves to a partial description of the process given by the mean and autocorrelation function. The class of random processes that satisfy Equations (1.7) and (1.9) has been given various names, such as *second-order stationary*, *wide-sense stationary*, or *weakly stationary* processes. Henceforth, we shall simply refer to them as *stationary processes*.²

A stationary process is not necessarily strictly stationary because Equations (1.7) and (1.9) obviously do not imply the invariance of the joint (k -dimensional) distribution of Equation (1.3) with respect to the time shift τ for all k . On the other hand, a strictly stationary process does not necessarily satisfy Equations (1.7) and (1.9) as the first- and second-order moments may not exist. Clearly, however, the class of strictly stationary processes with finite second-order moments forms a subclass of the class of all stationary processes.

■ PROPERTIES OF THE AUTOCORRELATION FUNCTION

For convenience of notation, we redefine the autocorrelation function of a stationary process $X(t)$ as

$$R_X(\tau) = E[X(t + \tau)X(t)] \quad \text{for all } t \quad (1.11)$$

This autocorrelation function has several important properties:

1. The *mean-square value* of the process may be obtained from $R_X(\tau)$ simply by putting $\tau = 0$ in Equation (1.11), as shown by

$$R_X(0) = E[X^2(t)] \quad (1.12)$$

2. The autocorrelation function $R_X(\tau)$ is an even function of τ , that is,

$$R_X(\tau) = R_X(-\tau) \quad (1.13)$$

This property follows directly from the defining equation (1.11). Accordingly, we may also define the autocorrelation function $R_X(\tau)$ as

$$R_X(\tau) = E[X(t)X(t - \tau)]$$

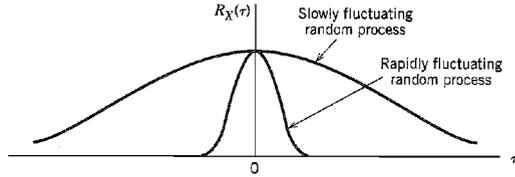


FIGURE 1.4 Illustrating the autocorrelation functions of slowly and rapidly fluctuating random processes.

3. The autocorrelation function $R_X(\tau)$ has its maximum magnitude at $\tau = 0$, that is,

$$|R_X(\tau)| \leq R_X(0) \quad (1.14)$$

To prove this property, consider the nonnegative quantity

$$E[(X(t + \tau) \pm X(t))^2] \geq 0$$

Expanding terms and taking their individual expectations, we readily find that

$$E[X^2(t + \tau)] \pm 2E[X(t + \tau)X(t)] + E[X^2(t)] \geq 0$$

which, in light of Equations (1.11) and (1.12), reduces to

$$2R_X(0) \pm 2R_X(\tau) \geq 0$$

Equivalently, we may write

$$-R_X(0) \leq R_X(\tau) \leq R_X(0)$$

from which Equation (1.14) follows directly.

The physical significance of the autocorrelation function $R_X(\tau)$ is that it provides a means of describing the interdependence of two random variables obtained by observing a random process $X(t)$ at times τ seconds apart. It is therefore apparent that the more rapidly the random process $X(t)$ changes with time, the more rapidly will the autocorrelation function $R_X(\tau)$ decrease from its maximum $R_X(0)$ as τ increases, as illustrated in Figure 1.4. This decrease may be characterized by a *decorrelation time* τ_0 , such that for $\tau > \tau_0$, the magnitude of the autocorrelation function $R_X(\tau)$ remains below some prescribed value. We may thus define the decorrelation time τ_0 of a stationary process $X(t)$ of zero mean as the time taken for the magnitude of the autocorrelation function $R_X(\tau)$ to decrease to 1 percent, say, of its maximum value $R_X(0)$.

► EXAMPLE 1.2 Sinusoidal Wave with Random Phase

Consider a sinusoidal signal with random phase, defined by

$$X(t) = A \cos(2\pi f_c t + \Theta) \quad (1.15)$$

where A and f_c are constants and Θ is a random variable that is *uniformly distributed* over the interval $[-\pi, \pi]$, that is,

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \theta \leq \pi \\ 0, & \text{elsewhere} \end{cases} \quad (1.16)$$

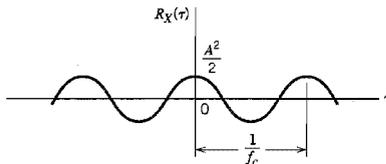


FIGURE 1.5 Autocorrelation function of a sine wave with random phase.

This means that the random variable Θ is equally likely to have any value θ in the interval $[-\pi, \pi]$. Each value of Θ corresponds to a sample in the sample space of the random process $X(t)$.

The process $X(t)$ defined by Equations (1.15) and (1.16) may represent a locally generated carrier in the receiver of a communication system, which is used in demodulation of the received signal. In particular, the random variable Θ denotes the phase difference between this locally generated carrier and the sinusoidal carrier wave used to modulate the message signal in the transmitter.

The autocorrelation function of $X(t)$ is

$$\begin{aligned} R_X(\tau) &= E[X(t + \tau)X(t)] \\ &= E[A^2 \cos(2\pi f_c t + 2\pi f_c \tau + \Theta) \cos(2\pi f_c t + \Theta)] \\ &= \frac{A^2}{2} E[\cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta)] + \frac{A^2}{2} E[\cos(2\pi f_c \tau)] \\ &= \frac{A^2}{2} \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta) d\theta + \frac{A^2}{2} \cos(2\pi f_c \tau) \end{aligned}$$

The first term integrates to zero, and so we get

$$R_X(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau) \quad (1.17)$$

which is plotted in Figure 1.5. We see therefore that the autocorrelation function of a sinusoidal wave with random phase is another sinusoid at the same frequency in the “ τ domain” rather than the original time domain. \blacktriangleleft

► EXAMPLE 1.3 Random Binary Wave

Figure 1.6 shows the sample function $x(t)$ of a process $X(t)$ consisting of a random sequence of binary symbols 1 and 0. The following assumptions are made:

1. The symbols 1 and 0 are represented by pulses of amplitude $+A$ and $-A$ volts, respectively, and duration T seconds.
2. The pulses are not synchronized, so the starting time t_d of the first complete pulse for positive time is equally likely to lie anywhere between zero and T seconds. That is, t_d is the sample value of a uniformly distributed random variable T_d , with its probability density function defined by

$$f_{T_d}(t_d) = \begin{cases} \frac{1}{T}, & 0 \leq t_d \leq T \\ 0, & \text{elsewhere} \end{cases}$$

3. During any time interval $(n - 1)T < t - t_d < nT$, where n is an integer, the presence of a 1 or a 0 is determined by tossing a fair coin; specifically, if the outcome is heads,

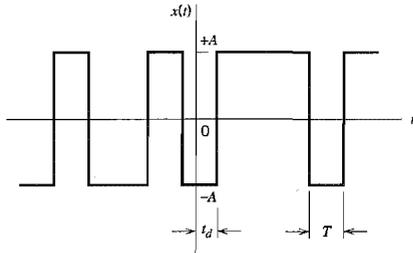


FIGURE 1.6 Sample function of random binary wave.

we have a 1 and if the outcome is tails, we have a 0. These two symbols are thus equally likely, and the presence of a 1 or 0 in any one interval is independent of all other intervals.

Since the amplitude levels $-A$ and $+A$ occur with equal probability, it follows immediately that $E[X(t)] = 0$ for all t , and the mean of the process is therefore zero.

To find the autocorrelation function $R_X(t_k, t_i)$, we have to evaluate $E[X(t_k)X(t_i)]$, where $X(t_k)$ and $X(t_i)$ are random variables obtained by observing the random process $X(t)$ at times t_k and t_i , respectively.

Consider first the case when $|t_k - t_i| > T$. Under this condition the random variables $X(t_k)$ and $X(t_i)$ occur in different pulse intervals and are therefore independent. We thus have

$$E[X(t_k)X(t_i)] = E[X(t_k)]E[X(t_i)] = 0, \quad |t_k - t_i| > T$$

Consider next the case when $|t_k - t_i| < T$, with $t_k = 0$ and $t_i < t_k$. In such a situation we observe from Figure 1.6 that the random variables $X(t_k)$ and $X(t_i)$ occur in the same pulse interval if and only if the delay t_d satisfies the condition $t_d < T - |t_k - t_i|$. We thus obtain the *conditional expectation*:

$$E[X(t_k)X(t_i)|t_d] = \begin{cases} A^2, & t_d < T - |t_k - t_i| \\ 0, & \text{elsewhere} \end{cases}$$

Averaging this result over all possible values of t_d , we get

$$\begin{aligned} E[X(t_k)X(t_i)] &= \int_0^{T-|t_k-t_i|} A^2 f_{T_d}(t_d) dt_d \\ &= \int_0^{T-|t_k-t_i|} \frac{A^2}{T} dt_d \\ &= A^2 \left(1 - \frac{|t_k - t_i|}{T} \right), \quad |t_k - t_i| < T \end{aligned}$$

By similar reasoning for any other value of t_k , we conclude that the autocorrelation function of a random binary wave, represented by the sample function shown in Figure 1.6, is only a function of the time difference $\tau = t_k - t_i$, as shown by

$$R_X(\tau) = \begin{cases} A^2 \left(1 - \frac{|\tau|}{T} \right), & |\tau| < T \\ 0, & |\tau| \geq T \end{cases} \quad (1.18)$$

This result is plotted in Figure 1.7. ◀

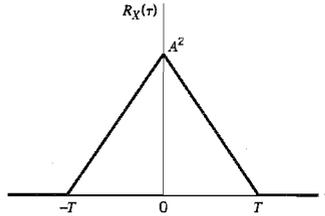


FIGURE 1.7 Autocorrelation function of random binary wave.

■ CROSS-CORRELATION FUNCTIONS

Consider next the more general case of two random processes $X(t)$ and $Y(t)$ with autocorrelation functions $R_X(t, u)$ and $R_Y(t, u)$, respectively. The two *cross-correlation functions* of $X(t)$ and $Y(t)$ are defined by

$$R_{XY}(t, u) = E[X(t)Y(u)] \quad (1.19)$$

and

$$R_{YX}(t, u) = E[Y(t)X(u)] \quad (1.20)$$

where t and u denote two values of time at which the processes are observed. In this case, the correlation properties of the two random processes $X(t)$ and $Y(t)$ may be displayed conveniently in matrix form as follows:

$$\mathbf{R}(t, u) = \begin{bmatrix} R_X(t, u) & R_{XY}(t, u) \\ R_{YX}(t, u) & R_Y(t, u) \end{bmatrix}$$

which is called the *correlation matrix* of the random processes $X(t)$ and $Y(t)$. If the random processes $X(t)$ and $Y(t)$ are each stationary and, in addition, they are jointly stationary, then the correlation matrix can be written as

$$\mathbf{R}(\tau) = \begin{bmatrix} R_X(\tau) & R_{XY}(\tau) \\ R_{YX}(\tau) & R_Y(\tau) \end{bmatrix} \quad (1.21)$$

where $\tau = t - u$.

The cross-correlation function is not generally an even function of τ as was true for the autocorrelation function, nor does it have a maximum at the origin. However, it does obey a certain symmetry relationship as follows (see Problem 1.9):

$$R_{XY}(\tau) = R_{YX}(-\tau) \quad (1.22)$$

► EXAMPLE 1.4 Quadrature-Modulated Processes

Consider a pair of *quadrature-modulated processes* $X_1(t)$ and $X_2(t)$ that are related to a stationary process $X(t)$ as follows:

$$X_1(t) = X(t) \cos(2\pi f_c t + \Theta)$$

$$X_2(t) = X(t) \sin(2\pi f_c t + \Theta)$$

where f_c is a carrier frequency, and the random variable Θ is uniformly distributed over the interval $[0, 2\pi]$. Moreover, Θ is independent of $X(t)$. One cross-correlation function of $X_1(t)$ and $X_2(t)$ is given by

$$\begin{aligned} R_{12}(\tau) &= E[X_1(t)X_2(t-\tau)] \\ &= E[X(t)X(t-\tau) \cos(2\pi f_c t + \Theta) \sin(2\pi f_c t - 2\pi f_c \tau + \Theta)] \\ &= E[X(t)X(t-\tau)]E[\cos(2\pi f_c t + \Theta) \sin(2\pi f_c t - 2\pi f_c \tau + \Theta)] \quad (1.23) \\ &= \frac{1}{2}R_X(\tau)E[\sin(4\pi f_c t - 2\pi f_c \tau + 2\Theta) - \sin(2\pi f_c \tau)] \\ &= -\frac{1}{2}R_X(\tau) \sin(2\pi f_c \tau) \end{aligned}$$

where, in the last line, we have made use of the uniform distribution of the random variable Θ representing phase. Note that at $\tau = 0$, the factor $\sin(2\pi f_c \tau)$ is zero and therefore

$$\begin{aligned} R_{12}(0) &= E[X_1(t)X_2(t)] \\ &= 0 \end{aligned}$$

This shows that the random variables obtained by simultaneously observing the quadrature-modulated processes $X_1(t)$ and $X_2(t)$ at some fixed value of time t are orthogonal to each other. \blacktriangleleft

1.5 Ergodic Processes

The *expectations* or *ensemble averages* of a random process $X(t)$ are averages “across the process.” For example, the mean of a random process $X(t)$ at some fixed time t_k is the expectation of the random variable $X(t_k)$ that describes *all possible values* of the sample functions of the process observed at time $t = t_k$. Naturally, we may also define *long-term sample averages*, or *time averages* that are averages “along the process.” We are therefore interested in relating ensemble averages to time averages, for time averages represent a practical means available to us for the *estimation* of ensemble averages of a random process. The key question, of course, is: When can we substitute time averages for ensemble averages? To explore this issue, consider the sample function $x(t)$ of a stationary process $X(t)$, with the observation interval defined as $-T \leq t \leq T$. The DC *value* of $x(t)$ is defined by the time average

$$\mu_x(T) = \frac{1}{2T} \int_{-T}^T x(t) dt \quad (1.24)$$

Clearly, the time average $\mu_x(T)$ is a random variable, as its value depends on the observation interval and which particular sample function of the random process $X(t)$ is picked for use in Equation (1.24). Since the process $X(t)$ is assumed to be stationary, the mean of the time average $\mu_x(T)$ is given by (after interchanging the operation of expectation and integration):

$$\begin{aligned} E[\mu_x(T)] &= \frac{1}{2T} \int_{-T}^T E[x(t)] dt \\ &= \frac{1}{2T} \int_{-T}^T \mu_X dt \quad (1.25) \\ &= \mu_X \end{aligned}$$

where μ_X is the mean of the process $X(t)$. Accordingly, the time average $\mu_x(T)$ represents an *unbiased estimate* of the ensemble-averaged mean μ_X . We say that the process $X(t)$ is *ergodic in the mean* if two conditions are satisfied:

- ▶ The time average $\mu_x(T)$ approaches the ensemble average μ_X in the limit as the observation interval T approaches infinity; that is,

$$\lim_{T \rightarrow \infty} \mu_x(T) = \mu_X$$

- ▶ The variance of $\mu_x(T)$, treated as a random variable, approaches zero in the limit as the observation interval T approaches infinity; that is,

$$\lim_{T \rightarrow \infty} \text{var}[\mu_x(T)] = 0$$

The other time average of particular interest is the autocorrelation function $R_x(\tau, T)$ defined in terms of the sample function $x(t)$ observed over the interval $-T \leq t \leq T$. Following Equation (1.24), we may formally define the *time-averaged autocorrelation function* of a sample function $x(t)$ as follows:

$$R_x(\tau, T) = \frac{1}{2T} \int_{-T}^T x(t + \tau)x(t) dt \quad (1.26)$$

This second time-average should also be viewed as a random variable with a mean and variance of its own. In a manner similar to ergodicity of the mean, we say that the process $x(t)$ is *ergodic in the autocorrelation function* if the following two limiting conditions are satisfied:

$$\begin{aligned} \lim_{T \rightarrow \infty} R_x(\tau, T) &= R_X(\tau) \\ \lim_{T \rightarrow \infty} \text{var}[R_x(\tau, T)] &= 0 \end{aligned}$$

We could, of course, go on in a similar way to define ergodicity in the most general sense by considering higher-order statistics of the process $X(t)$. In practice, however, ergodicity in the mean and ergodicity in the autocorrelation function, as described here, are often (but not always) considered to be adequate. Note also that the use of Equations (1.24) and (1.26) to compute the time averages $\mu_x(T)$ and $R_x(\tau, T)$ requires that the process $X(t)$ be stationary. In other words, for a random process to be ergodic, it has to be stationary; however, the converse is not necessarily true.

1.6 Transmission of a Random Process Through a Linear Time-Invariant Filter

Suppose that a random process $X(t)$ is applied as input to a *linear time-invariant filter of impulse response $h(t)$* , producing a new random process $Y(t)$ at the filter output, as in Figure 1.8. In general, it is difficult to describe the probability distribution of the output random process $Y(t)$, even when the probability distribution of the input random process $X(t)$ is completely specified for $-\infty < t < \infty$.

In this section, we determine the time-domain form of the input–output relations of the filter for defining the mean and autocorrelation functions of the output random process $Y(t)$ in terms of those of the input $X(t)$, assuming that $X(t)$ is a stationary process.

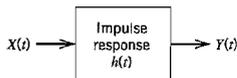


FIGURE 1.8 Transmission of a random process through a linear time-invariant filter.

The transmission of a process through a linear time-invariant filter is governed by the *convolution integral*; for a review of this operation, see Appendix 2. For the problem at hand, we may thus express the output random process $Y(t)$ in terms of the input random process $X(t)$ as

$$Y(t) = \int_{-\infty}^{\infty} b(\tau_1)X(t - \tau_1) d\tau_1$$

where τ_1 is the integration variable. Hence, the mean of $Y(t)$ is

$$\begin{aligned} \mu_Y(t) &= E[Y(t)] \\ &= E\left[\int_{-\infty}^{\infty} b(\tau_1)X(t - \tau_1) d\tau_1\right] \end{aligned} \quad (1.27)$$

Provided that the expectation $E[X(t)]$ is finite for all t and the system is stable, we may interchange the order of expectation and integration in Equation (1.27) and so write

$$\begin{aligned} \mu_Y(t) &= \int_{-\infty}^{\infty} b(\tau_1)E[X(t - \tau_1)] d\tau_1 \\ &= \int_{-\infty}^{\infty} b(\tau_1)\mu_X(t - \tau_1) d\tau_1 \end{aligned} \quad (1.28)$$

When the input random process $X(t)$ is stationary, the mean $\mu_X(t)$ is a constant μ_X , so that we may simplify Equation (1.28) as follows:

$$\begin{aligned} \mu_Y &= \mu_X \int_{-\infty}^{\infty} b(\tau_1) d\tau_1 \\ &= \mu_X H(0) \end{aligned} \quad (1.29)$$

where $H(0)$ is the zero-frequency (DC) response of the system. Equation (1.29) states that the mean of the random process $Y(t)$ produced at the output of a linear time-invariant system in response to $X(t)$ acting as the input process is equal to the mean of $X(t)$ multiplied by the DC response of the system, which is intuitively satisfying.

Consider next the autocorrelation function of the output random process $Y(t)$. By definition, we have

$$R_Y(t, u) = E[Y(t)Y(u)]$$

where t and u denote two values of the time at which the output process is observed. We may therefore use the convolution integral to write

$$R_Y(t, u) = E\left[\int_{-\infty}^{\infty} b(\tau_1)X(t - \tau_1) d\tau_1 \int_{-\infty}^{\infty} b(\tau_2)X(u - \tau_2) d\tau_2\right] \quad (1.30)$$

Here again, provided that the mean-square value $E[X^2(t)]$ is finite for all t and the system is stable, we may interchange the order of the expectation and the integrations with respect to τ_1 and τ_2 in Equation (1.30), obtaining

$$\begin{aligned} R_Y(t, u) &= \int_{-\infty}^{\infty} d\tau_1 b(\tau_1) \int_{-\infty}^{\infty} d\tau_2 b(\tau_2) E[X(t - \tau_1)X(u - \tau_2)] \\ &= \int_{-\infty}^{\infty} d\tau_1 b(\tau_1) \int_{-\infty}^{\infty} d\tau_2 b(\tau_2) R_X(t - \tau_1, u - \tau_2) \end{aligned} \quad (1.31)$$

When the input $X(t)$ is a stationary process, the autocorrelation function of $X(t)$ is only a function of the difference between the observation times $t - \tau_1$ and $u - \tau_2$. Thus, putting $\tau = t - u$ in Equation (1.31), we may write

$$R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b(\tau_1)b(\tau_2)R_X(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2 \quad (1.32)$$

On combining this result with that involving the mean μ_Y , we see that if the input to a stable linear time-invariant filter is a stationary process, then the output of the filter is also a stationary process.

Since $R_Y(0) = E[Y^2(t)]$, it follows that the mean-square value of the output random process $Y(t)$ is obtained by putting $\tau = 0$ in Equation (1.32). We thus get the result

$$E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b(\tau_1)b(\tau_2)R_X(\tau_2 - \tau_1) d\tau_1 d\tau_2 \quad (1.33)$$

which is a constant.

1.7 Power Spectral Density

Thus far we have considered the characterization of stationary processes in linear systems in the time domain. We turn next to the characterization of random processes in linear systems by using frequency-domain ideas. In particular, we wish to derive the frequency-domain equivalent to the result of Equation (1.33) defining the mean-square value of the filter output.

By definition, the impulse response of a linear time-invariant filter is equal to the inverse Fourier transform of the frequency response of the system; a review of the Fourier transform is presented in Appendix 2. Using $H(f)$ to denote the *frequency response* of the system, we may thus write

$$b(\tau_1) = \int_{-\infty}^{\infty} H(f) \exp(j2\pi f\tau_1) df \quad (1.34)$$

Substituting this expression for $b(\tau_1)$ into Equation (1.33), we get

$$\begin{aligned} E[Y^2(t)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} H(f) \exp(j2\pi f\tau_1) df \right] b(\tau_2)R_X(\tau_2 - \tau_1) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} df H(f) \int_{-\infty}^{\infty} d\tau_2 b(\tau_2) \int_{-\infty}^{\infty} R_X(\tau_2 - \tau_1) \exp(j2\pi f\tau_1) d\tau_1 \end{aligned} \quad (1.35)$$

In the last integral on the right-hand side of Equation (1.35), define a new variable

$$\tau = \tau_2 - \tau_1$$

Then we may rewrite Equation (1.35) in the form

$$E[Y^2(t)] = \int_{-\infty}^{\infty} df H(f) \int_{-\infty}^{\infty} d\tau_2 h(\tau_2) \exp(j2\pi f\tau_2) \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau \quad (1.36)$$

However, the middle integral on the right-hand side in Equation (1.36) is simply $H^*(f)$, the complex conjugate of the frequency response of the filter, and so we may simplify this equation as

$$E[Y^2(t)] = \int_{-\infty}^{\infty} df |H(f)|^2 \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau \quad (1.37)$$

where $|H(f)|$ is the *magnitude response* of the filter. We may further simplify Equation (1.37) by recognizing that the last integral is simply the Fourier transform of the auto-correlation function $R_X(\tau)$ of the input random process $X(t)$. This prompts us to introduce the definition of a new parameter

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau \quad (1.38)$$

The function $S_X(f)$ is called the *power spectral density*, or *power spectrum*, of the stationary process $X(t)$. Thus substituting Equation (1.38) into (1.37), we obtain the desired relation:

$$E[Y^2(t)] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df \quad (1.39)$$

Equation (1.39) states that *the mean-square value of the output of a stable linear time-invariant filter in response to a stationary process is equal to the integral over all frequencies of the power spectral density of the input process multiplied by the squared magnitude response of the filter*. This is the desired frequency-domain equivalent to the time-domain relation of Equation (1.33).

To investigate the physical significance of the power spectral density, suppose that the random process $X(t)$ is passed through an ideal narrowband filter with a magnitude response centered about the frequency f_c , as shown in Figure 1.9; that is,

$$|H(f)| = \begin{cases} 1, & |f \pm f_c| < \frac{1}{2}\Delta f \\ 0, & |f \pm f_c| > \frac{1}{2}\Delta f \end{cases} \quad (1.40)$$

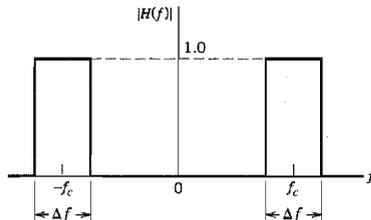


FIGURE 1.9 Magnitude response of ideal narrowband filter.

where Δf is the bandwidth of the filter. Then from Equation (1.39) we find that if the filter bandwidth Δf is sufficiently small compared to the midband frequency f_c and $S_X(f)$ is a continuous function, the mean-square value of the filter output is approximately

$$E[Y^2(t)] \approx (2\Delta f)S_X(f_c) \quad (1.41)$$

The filter, however, passes only those frequency components of the input random process $X(t)$ that lie inside a narrow frequency band of width Δf centered about the frequency $\pm f_c$. Thus $S_X(f_c)$ represents the frequency density of the average power in the random process $X(t)$, evaluated at the frequency $f = f_c$. The dimensions of the power spectral density are therefore in watts per Hertz (W/Hz).

■ PROPERTIES OF THE POWER SPECTRAL DENSITY

The power spectral density $S_X(f)$ and the autocorrelation function $R_X(\tau)$ of a stationary process $X(t)$ form a Fourier-transform pair with τ and f as the variables of interest, as shown by the pair of relations

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau \quad (1.42)$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \exp(j2\pi f\tau) df \quad (1.43)$$

Equations (1.42) and (1.43) are basic relations in the theory of spectral analysis of random processes, and together they constitute what are usually called the *Einstein-Wiener-Khintchine relations*.³

The Einstein-Wiener-Khintchine relations show that if either the autocorrelation function or power spectral density of a random process is known, the other can be found exactly. But these functions display different aspects of the correlation information about the process. It is commonly accepted that for practical purposes, the power spectral density is the more useful "parameter."

We now wish to use this pair of relations to derive some general properties of the power spectral density of a stationary process.

Property 1

The zero-frequency value of the power spectral density of a stationary process equals the total area under the graph of the autocorrelation function; that is,

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau) d\tau \quad (1.44)$$

This property follows directly from Equation (1.42) by putting $f = 0$.

Property 2

The mean-square value of a stationary process equals the total area under the graph of the power spectral density; that is,

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f) df \quad (1.45)$$

This property follows directly from Equation (1.43) by putting $\tau = 0$ and noting that $R_X(0) = E[X^2(t)]$.

Property 3

The power spectral density of a stationary process is always nonnegative; that is,

$$S_X(f) \geq 0 \quad \text{for all } f \quad (1.46)$$

This property is an immediate consequence of the fact that, in Equation (1.41), the mean-square value $E[Y^2(t)]$ must always be nonnegative.

Property 4

The power spectral density of a real-valued random process is an even function of frequency; that is,

$$S_X(-f) = S_X(f) \quad (1.47)$$

This property is readily obtained by substituting $-f$ for f in Equation (1.42):

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(j2\pi f\tau) d\tau$$

Next, substituting $-\tau$ for τ , and recognizing that $R_X(-\tau) = R_X(\tau)$, we get

$$S_X(-f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau = S_X(f)$$

which is the desired result.

Property 5

The power spectral density, appropriately normalized, has the properties usually associated with a probability density function.

The normalization we have in mind here is with respect to the total area under the graph of the power spectral density (i.e., the mean-square value of the process). Consider then the function

$$p_X(f) = \frac{S_X(f)}{\int_{-\infty}^{\infty} S_X(f) df} \quad (1.48)$$

In light of Properties 2 and 3, we note that $p_X(f) \geq 0$ for all f . Moreover, the total area under the function $p_X(f)$ is unity. Hence, the normalized form of the power spectral density, as defined in Equation (1.48), behaves similar to a probability density function.

▶ EXAMPLE 1.5 Sinusoidal Wave with Random Phase (continued)

Consider the random process $X(t) = A \cos(2\pi f_c t + \Theta)$, where Θ is a uniformly distributed random variable over the interval $[-\pi, \pi]$. The autocorrelation function of this random process is given by Equation (1.17), which is reproduced here for convenience:

$$R_X(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau)$$

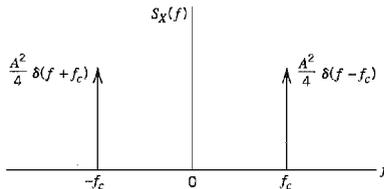


FIGURE 1.10 Power spectral density of sine wave with random phase; $\delta(f)$ denotes the delta function at $f = 0$.

Let $\delta(f)$ denote the *delta function* at $f = 0$; for the definition of the delta function and its properties, see Appendix 2. Taking the Fourier transform of both sides of the relation defining $R_X(\tau)$, we find that the power spectral density of the sinusoidal process $X(t)$ is

$$S_X(f) = \frac{A^2}{4} [\delta(f - f_c) + \delta(f + f_c)] \quad (1.49)$$

which consists of a pair of delta functions weighted by the factor $A^2/4$ and located at $\pm f_c$, as illustrated in Figure 1.10. We note that the total area under a delta function is one. Hence, the total area under $S_X(f)$ is equal to $A^2/2$, as expected. ◀

▶ EXAMPLE 1.6 Random Binary Wave (continued)

Consider again a random binary wave consisting of a sequence of 1s and 0s represented by the values $+A$ and $-A$, respectively. In Example 1.3 we showed that the autocorrelation function of this random process has a triangular waveform, as shown by

$$R_X(\tau) = \begin{cases} A^2 \left(1 - \frac{|\tau|}{T}\right), & |\tau| < T \\ 0, & |\tau| \geq T \end{cases}$$

The power spectral density of the process is therefore

$$S_X(f) = \int_{-T}^T A^2 \left(1 - \frac{|\tau|}{T}\right) \exp(-j2\pi f\tau) d\tau$$

Using the Fourier transform of a triangular function (see Table A6.3), we obtain

$$S_X(f) = A^2 T \operatorname{sinc}^2(fT) \quad (1.50)$$

which is plotted in Figure 1.11. Here again we see that the power spectral density is nonnegative for all f and that it is an even function of f . Noting that $R_X(0) = A^2$ and using Property 2, we find that the total area under $S_X(f)$, or the average power of the random binary wave described here, is A^2 , which is intuitively satisfying. ◀

The result of Equation (1.50) may be generalized as follows. We note that the *energy spectral density* (i.e., the squared magnitude of the Fourier transform) of a rectangular pulse $g(t)$ of amplitude A and duration T is given by

$$\mathcal{E}_g(f) = A^2 T^2 \operatorname{sinc}^2(fT) \quad (1.51)$$

We may therefore rewrite Equation (1.50) in terms of $\mathcal{E}_g(f)$ simply as

$$S_X(f) = \frac{\mathcal{E}_g(f)}{T} \quad (1.52)$$

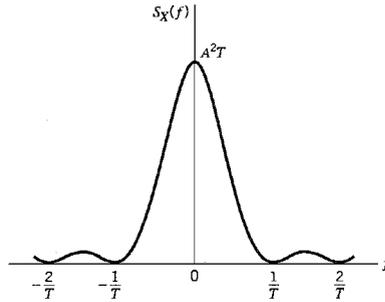


FIGURE 1.11 Power spectral density of random binary wave.

Equation (1.52) states that for a random binary wave in which binary symbols 1 and 0 are represented by pulses $g(t)$ and $-g(t)$, respectively, the power spectral density $S_X(f)$ is equal to the energy spectral density $\mathcal{E}_g(f)$ of the *symbol shaping pulse* $g(t)$, divided by the *symbol duration* T .

EXAMPLE 1.7 Mixing of a Random Process with a Sinusoidal Process

A situation that often arises in practice is that of *mixing* (i.e., multiplication) of a stationary process $X(t)$ with a sinusoidal wave $\cos(2\pi f_c t + \Theta)$, where the phase Θ is a random variable that is uniformly distributed over the interval $[0, 2\pi]$. The addition of the random phase Θ in this manner merely recognizes the fact that the time origin is arbitrarily chosen when $X(t)$ and $\cos(2\pi f_c t + \Theta)$ come from physically independent sources, as is usually the case. We are interested in determining the power spectral density of the random process $Y(t)$, defined by

$$Y(t) = X(t) \cos(2\pi f_c t + \Theta) \quad (1.53)$$

Using the definition of autocorrelation function of a stationary process and noting that the random variable Θ is independent of $X(t)$, we find that the autocorrelation function of $Y(t)$ is given by

$$\begin{aligned} R_Y(\tau) &= E[Y(t + \tau)Y(t)] \\ &= E[X(t + \tau) \cos(2\pi f_c t + 2\pi f_c \tau + \Theta) X(t) \cos(2\pi f_c t + \Theta)] \\ &= E[X(t + \tau)X(t)] E[\cos(2\pi f_c t + 2\pi f_c \tau + \Theta) \cos(2\pi f_c t + \Theta)] \\ &= \frac{1}{2} R_X(\tau) E[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta)] \\ &= \frac{1}{2} R_X(\tau) \cos(2\pi f_c \tau) \end{aligned} \quad (1.54)$$

Because the power spectral density is the Fourier transform of the autocorrelation function, we find that the power spectral densities of the random processes $X(t)$ and $Y(t)$ are related as follows:

$$S_Y(f) = \frac{1}{4} [S_X(f - f_c) + S_X(f + f_c)] \quad (1.55)$$

According to Equation (1.55), the power spectral density of the random process $Y(t)$ defined in Equation (1.53) is obtained as follows: We shift the given power spectral density $S_X(f)$ of random process $X(t)$ to the right by f_c , shift it to the left by f_c , add the two shifted power spectra, and divide the result by 4. ◀

■ RELATION AMONG THE POWER SPECTRAL DENSITIES OF THE INPUT AND OUTPUT RANDOM PROCESSES

Let $S_Y(f)$ denote the power spectral density of the output random process $Y(t)$ obtained by passing the random process $X(t)$ through a linear filter of frequency response $H(f)$. Then, recognizing by definition that the power spectral density of a random process is equal to the Fourier transform of its autocorrelation function and using Equation (1.32), we obtain

$$\begin{aligned} S_Y(f) &= \int_{-\infty}^{\infty} R_Y(\tau) \exp(-j2\pi f\tau) d\tau \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau - \tau_1 + \tau_2) \exp(-j2\pi f\tau) d\tau_1 d\tau_2 d\tau \end{aligned} \quad (1.56)$$

Let $\tau - \tau_1 + \tau_2 = \tau_0$, or, equivalently, $\tau = \tau_0 + \tau_1 - \tau_2$. Then by making this substitution in Equation (1.56), we find that $S_Y(f)$ may be expressed as the product of three terms: the frequency response $H(f)$ of the filter, the complex conjugate of $H(f)$, and the power spectral density $S_X(f)$ of the input random process $X(t)$. We may thus simplify Equation (1.56) as

$$S_Y(f) = H(f)H^*(f)S_X(f) \quad (1.57)$$

Finally, since $|H(f)|^2 = H(f)H^*(f)$, we find that the relationship among the power spectral densities of the input and output random processes is expressed in the frequency domain by writing

$$S_Y(f) = |H(f)|^2 S_X(f) \quad (1.58)$$

Equation (1.58) states that *the power spectral density of the output process $Y(t)$ equals the power spectral density of the input process $X(t)$ multiplied by the squared magnitude response of the filter*. By using this relation, we can therefore determine the effect of passing a random process through a stable, linear, time-invariant, filter. In computational terms, Equation (1.58) is usually easier to handle than its time-domain counterpart of Equation (1.32), involving the autocorrelation function.

■ RELATION AMONG THE POWER SPECTRAL DENSITY AND THE MAGNITUDE SPECTRUM OF A SAMPLE FUNCTION

We now wish to relate the power spectral density $S_X(f)$ directly to the spectral properties of a sample function $x(t)$ of a stationary process $X(t)$ that is ergodic. For the sample function $x(t)$ to be Fourier transformable, however, it must be absolutely integrable; that is

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty \quad (1.59)$$

This condition can never be satisfied by any stationary sample function $x(t)$ of infinite duration. In order to use the Fourier transform technique, we consider a truncated segment of $x(t)$, defined over the observation interval $-T \leq t \leq T$, say. Thus, using $X(f, T)$ to denote the Fourier transform of the truncated sample function so defined, we may write

$$X(f, T) = \int_{-T}^T x(t) \exp(-j2\pi ft) dt \quad (1.60)$$

Assuming that the process $x(t)$ is also ergodic, we may evaluate the autocorrelation function $R_X(\tau)$ of $X(t)$ using the time-average formula (see Section 1.5)

$$R_X(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t + \tau)x(t) dt \quad (1.61)$$

It is customary to view the sample function $x(t)$ as a *power signal* (i.e., a signal with finite average power). Hence, we may formulate the following Fourier-transform pair:

$$\frac{1}{2T} \int_{-T}^T x(t + \tau)x(t) dt \doteq \frac{1}{2T} |X(f, T)|^2 \quad (1.62)$$

The parameter on the left-hand side is a time-averaged autocorrelation function. The parameter on the right-hand side is called the *periodogram*, whose dimensions are the same as those of the power spectral density. This terminology is a misnomer, however, since the periodogram is a function of frequency, not period. Nevertheless, it has wide usage. The quantity was first used by statisticians to look for periodicities such as seasonal trends in data.

Using the formula for the inverse Fourier transform in the Fourier-transform pair of Equation (1.62), we may express the time-averaged autocorrelation function of the sample function $x(t)$ in terms of the periodogram as

$$\frac{1}{2T} \int_{-T}^T x(t + \tau)x(t) dt = \int_{-\infty}^{\infty} \frac{1}{2T} |X(f, T)|^2 \exp(j2\pi fT) df \quad (1.63)$$

Hence, substituting Equation (1.63) into (1.61), we get

$$R_X(\tau) = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{1}{2T} |X(f, T)|^2 \exp(j2\pi f\tau) df \quad (1.64)$$

For a fixed value of the frequency f , the periodogram is a random variable in that its value varies in a random manner from one sample function of the random process to another. Thus, for a given sample function $x(t)$, the periodogram does not converge in any statistical sense to a limiting value as T tends to infinity. As such, it would be incorrect to interchange the order of the integration and limiting operations in Equation (1.64). Suppose, however, that we take the expectation of both sides of Equation (1.64) over the ensemble of all sample functions of the random process and recognize that for an ergodic process the autocorrelation function $R_X(\tau)$ is unchanged by such an operation. Then, since each sample function of an ergodic process eventually takes on nearly all the modes of behavior of each other sample function, we may thus write

$$R_X(\tau) = \lim_{T \rightarrow \infty} \int_{-\infty}^{\infty} \frac{1}{2T} E[|X(f, T)|^2] \exp(j2\pi f\tau) df \quad (1.65)$$

Now we may interchange the order of the integration and limiting operations and so obtain

$$R_X(\tau) = \int_{-\infty}^{\infty} \left\{ \lim_{T \rightarrow \infty} \frac{1}{2T} E[|X(f, T)|^2] \right\} \exp(j2\pi f\tau) df \quad (1.66)$$

Hence, comparing Equations (1.66) and (1.43), we obtain the desired relation between the power spectral density $S_X(f)$ of an ergodic process and the squared magnitude spectrum $|X(f, T)|^2$ of a truncated sample function of the process:

$$\begin{aligned} S_X(f) &= \lim_{T \rightarrow \infty} \frac{1}{2T} E[|X(f, T)|^2] \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} E \left[\left| \int_{-T}^T x(t) \exp(-j2\pi ft) dt \right|^2 \right] \end{aligned} \quad (1.67)$$

It is important to note that in Equation (1.67) it is not possible to let $T \rightarrow \infty$ before taking the expectation. Equation (1.67) provides the mathematical basis for *estimating*⁴ the power spectral density of an ergodic random process, given a sample function $x(t)$ of the process observed over the interval $[-T, T]$.

■ CROSS-SPECTRAL DENSITIES

Just as the power spectral density provides a measure of the frequency distribution of a single random process, cross-spectral densities provide a measure of the frequency interrelationship between two random processes. In particular, let $X(t)$ and $Y(t)$ be two jointly stationary processes with their cross-correlation functions denoted by $R_{XY}(\tau)$ and $R_{YX}(\tau)$. We then define the *cross-spectral densities* $S_{XY}(f)$ and $S_{YX}(f)$ of this pair of random processes to be the Fourier transforms of their respective cross-correlation functions, as shown by

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) \exp(-j2\pi f\tau) d\tau \quad (1.68)$$

and

$$S_{YX}(f) = \int_{-\infty}^{\infty} R_{YX}(\tau) \exp(-j2\pi f\tau) d\tau \quad (1.69)$$

The cross-correlation functions and cross-spectral densities thus form Fourier-transform pairs. Accordingly, using the formula for inverse Fourier transformation we may also write

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} S_{XY}(f) \exp(j2\pi f\tau) df \quad (1.70)$$

and

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} S_{YX}(f) \exp(j2\pi f\tau) df \quad (1.71)$$

The cross-spectral densities $S_{XY}(f)$ and $S_{YX}(f)$ are not necessarily real functions of the frequency f . However, substituting the relationship

$$R_{XY}(\tau) = R_{YX}(-\tau)$$

into Equation (1.68) and then using Equation (1.69) we find that $S_{XY}(f)$ and $S_{YX}(f)$ are related by

$$S_{XY}(f) = S_{YX}(-f) = S_{YX}^*(f) \quad (1.72)$$

► EXAMPLE 1.8

Suppose that the random processes $X(t)$ and $Y(t)$ have zero mean, and they are individually stationary. Consider the sum random process

$$Z(t) = X(t) + Y(t)$$

The problem is to determine the power spectral density of $Z(t)$.

The autocorrelation function of $Z(t)$ is given by

$$\begin{aligned} R_Z(t, u) &= E[Z(t)Z(u)] \\ &= E[(X(t) + Y(t))(X(u) + Y(u))] \\ &= E[X(t)X(u)] + E[X(t)Y(u)] + E[Y(t)X(u)] + E[Y(t)Y(u)] \\ &= R_X(t, u) + R_{XY}(t, u) + R_{YX}(t, u) + R_Y(t, u) \end{aligned}$$

Defining $\tau = t - u$, we may therefore write

$$R_Z(\tau) = R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau) \tag{1.73}$$

when the random processes $X(t)$ and $Y(t)$ are also jointly stationary. Accordingly, taking the Fourier transform of both sides of Equation (1.73), we get

$$S_Z(f) = S_X(f) + S_{XY}(f) + S_{YX}(f) + S_Y(f) \tag{1.74}$$

We thus see that the cross-spectral densities $S_{XY}(f)$ and $S_{YX}(f)$ represent the spectral components that must be added to the individual power spectral densities of a pair of correlated random processes in order to obtain the power spectral density of their sum.

When the stationary processes $X(t)$ and $Y(t)$ are uncorrelated, the cross-spectral densities $S_{XY}(f)$ and $S_{YX}(f)$ are zero, and so Equation (1.74) reduces as follows:

$$S_Z(f) = S_X(f) + S_Y(f) \tag{1.75}$$

We may generalize this latter result by stating that when there is a multiplicity of zero-mean stationary processes that are uncorrelated with each other, the power spectral density of their sum is equal to the sum of their individual power spectral densities. ◀

▶ **EXAMPLE 1.9**

Consider next the problem of passing two jointly stationary processes through a pair of separate, stable, linear, time-invariant filters, as shown in Figure 1.12. In particular, suppose that the random process $X(t)$ is the input to the filter of impulse response $b_1(t)$ and that the random process $Y(t)$ is the input to the filter of impulse response $b_2(t)$. Let $V(t)$ and $Z(t)$ denote the random processes at the respective filter outputs. The cross-correlation function of $V(t)$ and $Z(t)$ is therefore

$$\begin{aligned} R_{VZ}(t, u) &= E[V(t)Z(u)] \\ &= E\left[\int_{-\infty}^{\infty} b_1(\tau_1)X(t - \tau_1) d\tau_1 \int_{-\infty}^{\infty} b_2(\tau_2)Y(u - \tau_2) d\tau_2\right] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b_1(\tau_1)b_2(\tau_2)E[X(t - \tau_1)Y(u - \tau_2)] d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b_1(\tau_1)b_2(\tau_2)R_{XY}(t - \tau_1, u - \tau_2) d\tau_1 d\tau_2 \end{aligned} \tag{1.76}$$

where $R_{XY}(t, u)$ is the cross-correlation function of $X(t)$ and $Y(t)$. Because the input random processes are jointly stationary (by hypothesis), we may set $\tau = t - u$ and so rewrite Equation (1.76) as follows:

$$R_{VZ}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b_1(\tau_1)b_2(\tau_2)R_{XY}(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2 \tag{1.77}$$

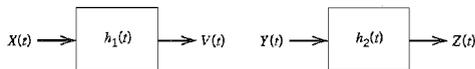


FIGURE 1.12 A pair of separate linear time-invariant filters.

Taking the Fourier transform of both sides of Equation (1.77) and using a procedure similar to that which led to the development of Equation (1.39), we finally get

$$S_{vz}(f) = H_1(f)H_2^*(f)S_{XY}(f) \quad (1.78)$$

where $H_1(f)$ and $H_2(f)$ are the frequency responses of the respective filters in Figure 1.12, and $H_2^*(f)$ is the complex conjugate of $H_2(f)$. This is the desired relationship between the cross-spectral density of the output processes and that of the input processes. ◀

1.8 Gaussian Process

The material we have presented on random processes up to this point in the discussion has been of a fairly general nature. In this section, we consider an important family of random processes known as Gaussian processes.⁵

Let us suppose that we observe a random process $X(t)$ for an interval that starts at time $t = 0$ and lasts until $t = T$. Suppose also that we weight the random process $X(t)$ by some function $g(t)$ and then integrate the product $g(t)X(t)$ over this observation interval, thereby obtaining a random variable Y defined by

$$Y = \int_0^T g(t)X(t) dt \quad (1.79)$$

We refer to Y as a *linear functional* of $X(t)$. The distinction between a function and a functional should be carefully noted. For example, the sum $Y = \sum_{i=1}^N a_i X_i$, where the a_i are constants and the X_i are random variables, is a *linear function* of the X_i ; for each observed set of values for the random variables X_i , we have a corresponding value for the random variable Y . On the other hand, in Equation (1.79) the value of the random variable Y depends on the course of the *argument function* $g(t)X(t)$ over the entire observation interval from 0 to T . Thus a functional is a quantity that depends on the entire course of one or more functions rather than on a number of discrete variables. In other words, the domain of a functional is a set or space of admissible functions rather than a region of a coordinate space.

If in Equation (1.79) the weighting function $g(t)$ is such that the mean-square value of the random variable Y is finite, and if the random variable Y is a *Gaussian-distributed* random variable for every $g(t)$ in this class of functions, then the process $X(t)$ is said to be a *Gaussian process*. In other words, the process $X(t)$ is a Gaussian process if every linear functional of $X(t)$ is a Gaussian random variable.

We say that the random variable Y has a Gaussian distribution if its probability density function has the form

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{(y - \mu_Y)^2}{2\sigma_Y^2}\right] \quad (1.80)$$

where μ_Y is the mean and σ_Y^2 is the variance of the random variable Y . A plot of this probability density function is given in Figure 1.13 for the special case when the Gaussian random variable Y is *normalized* to have a mean μ_Y of zero and a variance σ_Y^2 of one, as shown by

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)$$

Such a normalized Gaussian distribution is commonly written as $\mathcal{N}(0, 1)$.

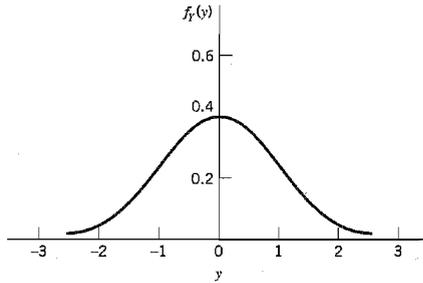


FIGURE 1.13 Normalized Gaussian distribution.

A Gaussian process has two main virtues. First, the Gaussian process has many properties that make analytic results possible; we will discuss these properties later in the section. Second, the random processes produced by physical phenomena are often such that a Gaussian model is appropriate. Furthermore, the use of a Gaussian model to describe the physical phenomena is usually confirmed by experiments. Thus the frequent occurrence of physical phenomena for which a Gaussian model is appropriate, together with the ease with which a Gaussian process is handled mathematically, make the Gaussian process very important in the study of communication systems.

■ CENTRAL LIMIT THEOREM

The *central limit theorem* provides the mathematical justification for using a Gaussian process as a model for a large number of different physical phenomena in which the observed random variable, at a particular instant of time, is the result of a large number of individual random events. To formulate this important theorem, let X_i , $i = 1, 2, \dots, N$, be a set of random variables that satisfies the following requirements:

1. The X_i are statistically independent.
2. The X_i have the same probability distribution with mean μ_X and variance σ_X^2 .

The X_i so described are said to constitute a set of *independently and identically distributed* (i.i.d.) random variables. Let these random variables be *normalized* as follows:

$$Y_i = \frac{1}{\sigma_X} (X_i - \mu_X), \quad i = 1, 2, \dots, N$$

so that we have

$$E[Y_i] = 0$$

and

$$\text{var}[Y_i] = 1$$

Define the random variable

$$V_N = \frac{1}{\sqrt{N}} \sum_{i=1}^N Y_i$$

The central limit theorem states that the probability distribution of V_N approaches a normalized Gaussian distribution $\mathcal{N}(0, 1)$ in the limit as the number of random variables N approaches infinity.

It is important to realize, however, that the central limit theorem gives only the “limiting” form of the probability distribution of the normalized random variable V_N as N approaches infinity. When N is finite, it is sometimes found that the Gaussian limit gives a relatively poor approximation for the actual probability distribution of V_N even though N may be quite large.

■ PROPERTIES OF A GAUSSIAN PROCESS

A Gaussian process has some useful properties that are described in the sequel.

Property 1

If a Gaussian process $X(t)$ is applied to a stable linear filter, then the random process $Y(t)$ developed at the output of the filter is also Gaussian.

This property is readily derived by using the definition of a Gaussian process based on Equation (1.79). Consider the situation depicted in Figure 1.8, where we have a linear time-invariant filter of impulse response $h(t)$, with the random process $X(t)$ as input and the random process $Y(t)$ as output. We assume that $X(t)$ is a Gaussian process. The random processes $Y(t)$ and $X(t)$ are related by the convolution integral

$$Y(t) = \int_0^T h(t - \tau)X(\tau) d\tau, \quad 0 \leq t < \infty \quad (1.81)$$

We assume that the impulse response $h(t)$ is such that the mean-square value of the output random process $Y(t)$ is finite for all t in the range $0 \leq t < \infty$ for which $Y(t)$ is defined. To demonstrate that the output process $Y(t)$ is Gaussian, we must show that any linear functional of it is a Gaussian random variable. That is, if we define the random variable

$$Z = \int_0^\infty g_Y(t) \int_0^T h(t - \tau)X(\tau) d\tau dt \quad (1.82)$$

then Z must be a Gaussian random variable for every function $g_Y(t)$, such that the mean-square value of Z is finite. Interchanging the order of integration in Equation (1.82), we get

$$Z = \int_0^T g(\tau)X(\tau) d\tau \quad (1.83)$$

where

$$g(\tau) = \int_0^\infty g_Y(t)h(t - \tau) d\tau \quad (1.84)$$

Since $X(t)$ is a Gaussian process by hypothesis, it follows from Equation (1.83) that Z must be a Gaussian random variable. We have thus shown that if the input $X(t)$ to a linear filter is a Gaussian process, then the output $Y(t)$ is also a Gaussian process. Note, however,

that although our proof was carried out assuming a time-invariant linear filter, this property is true for any arbitrary stable linear system.

Property 2

Consider the set of random variables or samples $X(t_1), X(t_2), \dots, X(t_n)$, obtained by observing a random process $X(t)$ at times t_1, t_2, \dots, t_n . If the process $X(t)$ is Gaussian, then this set of random variables is jointly Gaussian for any n , with their n -fold joint probability density function being completely determined by specifying the set of means

$$\mu_{X(t_i)} = E[X(t_i)], \quad i = 1, 2, \dots, n$$

and the set of covariance functions

$$C_X(t_k, t_i) = E[(X(t_k) - \mu_{X(t_k)})(X(t_i) - \mu_{X(t_i)})], \quad k, i = 1, 2, \dots, n$$

Let the n -by-1 vector \mathbf{X} denote the set of random variables $X(t_1), \dots, X(t_n)$ derived from the Gaussian process $X(t)$ by sampling it at times t_1, \dots, t_n . Let \mathbf{x} denote a value of \mathbf{X} . According to Property 2, the random vector \mathbf{X} has a *multivariate Gaussian distribution* defined in matrix form as

$$f_{X(t_1), \dots, X(t_n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \frac{1}{(2\pi)^{n/2} \Delta^{1/2}} \exp\left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right) \quad (1.85)$$

where the superscript T denotes transposition and

$$\begin{aligned} \boldsymbol{\mu} &= \text{mean vector} \\ &= [\mu_1, \mu_2, \dots, \mu_n]^T \\ \boldsymbol{\Sigma} &= \text{covariance matrix} \\ &= \{C_X(t_k, t_i)\}_{k,i=1}^n \\ \boldsymbol{\Sigma}^{-1} &= \text{inverse of covariance matrix} \\ \Delta &= \text{determinant of covariance matrix } \boldsymbol{\Sigma} \end{aligned}$$

Property 2 is frequently used as the definition of a Gaussian process. However, this definition is more difficult to use than that based on Equation (1.79) for evaluating the effects of filtering on a Gaussian process.

We may extend Property 2 to two (or more) random processes as follows. Consider the composite set of random variables $X(t_1), X(t_2), \dots, X(t_n), Y(u_1), Y(u_2), \dots, Y(u_m)$ obtained by observing a random process $X(t)$ at times $\{t_i, i = 1, 2, \dots, n\}$, and a second random process $Y(t)$ at times $\{u_k, k = 1, 2, \dots, m\}$. We say that the processes $X(t)$ and $Y(t)$ are *jointly Gaussian* if this composite set of random variables is jointly Gaussian for any n and m . Note that in addition to the mean and correlation functions of the random processes $X(t)$ and $Y(t)$ individually, we must also know the cross-covariance function

$$E[(X(t_i) - \mu_{X(t_i)})(Y(u_k) - \mu_{Y(u_k)})] = R_{XY}(t_i, u_k) - \mu_{X(t_i)}\mu_{Y(u_k)}$$

for any pair of observation instants (t_i, u_k) . This additional knowledge is embodied in the cross-correlation function, $R_{XY}(t_i, u_k)$, of the two processes $X(t)$ and $Y(t)$.

Property 3

If a Gaussian process is stationary, then the process is also strictly stationary.

This follows directly from Property 2.

Property 4

If the random variables $X(t_1), X(t_2), \dots, X(t_n)$, obtained by sampling a Gaussian process $X(t)$ at times t_1, t_2, \dots, t_n , are uncorrelated, that is,

$$E[(X(t_k) - \mu_{X(t_k)})(X(t_i) - \mu_{X(t_i)})] = 0, \quad i \neq k$$

then these random variables are statistically independent.

The uncorrelatedness of $X(t_1), \dots, X(t_n)$ means that the covariance matrix Σ is a diagonal matrix as shown by

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & & \mathbf{0} \\ & \sigma_2^2 & & \\ & & \ddots & \\ \mathbf{0} & & & \sigma_n^2 \end{bmatrix}$$

where

$$\sigma_i^2 = E[(X(t_i) - E[X(t_i)])^2], \quad i = 1, 2, \dots, n$$

Under this condition, the multivariate Gaussian distribution of Equation (1.85) simplifies to

$$f_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^n f_{X_i}(x_i)$$

where $X_i = X(t_i)$ and

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left(-\frac{(x_i - \mu_{X_i})^2}{2\sigma_i^2}\right)$$

In words, if the Gaussian random variables $X(t_1), \dots, X(t_n)$ are uncorrelated, then they are statistically independent, which, in turn, means that the joint probability density function of this set of random variables can be expressed as the product of the probability density functions of the individual random variables in the set.

1.9 Noise

The term *noise* is used customarily to designate unwanted signals that tend to disturb the transmission and processing of signals in communication systems and over which we have incomplete control. In practice, we find that there are many potential sources of noise in a communication system. The sources of noise may be external to the system (e.g., atmospheric noise, galactic noise, man-made noise), or internal to the system. The second category includes an important type of noise that arises from *spontaneous fluctuations* of current or voltage in electrical circuits.⁶ This type of noise represents a basic limitation on the transmission or detection of signals in communication systems involving the use of electronic devices. The two most common examples of spontaneous fluctuations in electrical circuits are *shot noise* and *thermal noise*, which are described in the sequel.

■ SHOT NOISE

Shot noise arises in electronic devices such as diodes and transistors because of the discrete nature of current flow in these devices. For example, in a *photodetector* circuit a current

pulse is generated every time an electron is emitted by the cathode due to incident light from a source of constant intensity. The electrons are naturally emitted at random times denoted by τ_k , where $-\infty < k < \infty$. It is assumed that the random emissions of electrons have been going on for a long time. Thus, the total current flowing through the photo-detector may be modeled as an infinite sum of current pulses, as shown by

$$X(t) = \sum_{k=-\infty}^{\infty} h(t - \tau_k) \quad (1.86)$$

where $h(t - \tau_k)$ is the current pulse generated at time τ_k . The process $X(t)$ defined by Equation (1.86) is a stationary process called *shot noise*.

The number of electrons, $N(t)$, emitted in the time interval $[0, t]$ constitutes a discrete stochastic process, the value of which increases by one each time an electron is emitted. Figure 1.14 shows a sample function of such a process. Let the mean value of the number of electrons, ν , emitted between times t and $t + t_0$ be defined by

$$E[\nu] = \lambda t_0 \quad (1.87)$$

The parameter λ is a constant called the *rate* of the process. The total number of electrons emitted in the interval $[t, t + t_0]$, that is,

$$\nu = N(t + t_0) - N(t)$$

follows a *Poisson distribution* with a mean value equal to λt_0 . In particular, the probability that k electrons are emitted in the interval $[t, t + t_0]$ is defined by

$$P(\nu = k) = \frac{(\lambda t_0)^k}{k!} e^{-\lambda t_0} \quad k = 0, 1, \dots \quad (1.88)$$

Unfortunately, a detailed statistical characterization of the shot-noise process $X(t)$ defined in Equation (1.86) is a difficult mathematical task. Here we simply quote the results pertaining to the first two moments of the process:

► The mean of $X(t)$ is

$$\mu_X = \lambda \int_{-\infty}^{\infty} h(t) dt \quad (1.89)$$

where λ is the rate of the process and $h(t)$ is the waveform of a current pulse.

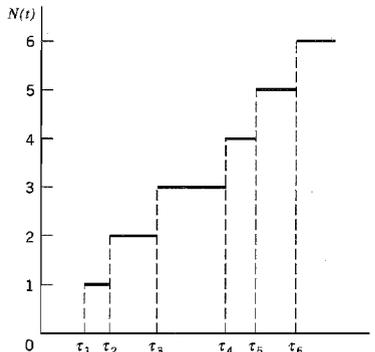


FIGURE 1.14 Sample function of a Poisson counting process.

► The autocovariance function of $X(t)$ is

$$C_X(\tau) = \lambda \int_{-\infty}^{\infty} h(t)h(t + \tau) dt \quad (1.90)$$

This second result is known as *Campbell's theorem*.

For the special case of a waveform $h(t)$ consisting of a rectangular pulse of amplitude A and duration T , the mean of the shot-noise process $X(t)$ is λAT , and its autocovariance function is

$$C_X(\tau) = \begin{cases} \lambda A^2(T - |\tau|), & |\tau| < T \\ 0, & |\tau| \geq T \end{cases}$$

which has a triangular form similar to that shown in Figure 1.7.

■ THERMAL NOISE

Thermal noise is the name given to the electrical noise arising from the random motion of electrons in a conductor. The mean-square value of the thermal noise voltage V_{TN} appearing across the terminals of a resistor, measured in a bandwidth of Δf Hertz, is, for all practical purposes, given by

$$E[V_{TN}^2] = 4kTR \Delta f \text{ volts}^2 \quad (1.91)$$

where k is *Boltzmann's constant* equal to 1.38×10^{-23} joules per degree Kelvin, T is the *absolute temperature* in degrees Kelvin, and R is the resistance in ohms. We may thus model a noisy resistor by the *Thévenin equivalent circuit* consisting of a noise voltage generator of mean-square value $E[V_{TN}^2]$ in series with a noiseless resistor, as in Figure 1.15a. Alternatively, we may use the *Norton equivalent circuit* consisting of a noise current generator in parallel with a noiseless conductance, as in Figure 1.15b. The mean-square value of the noise current generator is

$$\begin{aligned} E[I_{TN}^2] &= \frac{1}{R^2} E[V_{TN}^2] \\ &= 4kTG \Delta f \text{ amps}^2 \end{aligned} \quad (1.92)$$

where $G = 1/R$ is the conductance. It is also of interest to note that because the number of electrons in a resistor is very large and their random motions inside the resistor are statistically independent of each other, the central limit theorem indicates that thermal noise is Gaussian distributed with zero mean.

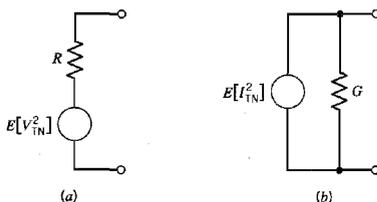


FIGURE 1.15 Models of a noisy resistor. (a) Thévenin equivalent circuit. (b) Norton equivalent circuit.

Noise calculations involve the transfer of power, and so we find that the use of the *maximum-power transfer theorem* is applicable to such calculations. This theorem states that the maximum possible power is transferred from a source of internal resistance R to a load of resistance R_l when $R_l = R$. Under this *matched condition*, the power produced by the source is divided equally between the internal resistance of the source and the load resistance, and the power delivered to the load is referred to as the *available power*. Applying the maximum-power transfer theorem to the Thévenin equivalent circuit of Figure 1.15a or the Norton equivalent circuit of Figure 1.15b, we find that a noisy resistor produces an *available noise power* equal to $kT \Delta f$ watts.

■ WHITE NOISE

The noise analysis of communication systems is customarily based on an idealized form of noise called *white noise*, the power spectral density of which is independent of the operating frequency. The adjective *white* is used in the sense that white light contains equal amounts of all frequencies within the visible band of electromagnetic radiation. We express the power spectral density of white noise, with a sample function denoted by $w(t)$, as

$$S_w(f) = \frac{N_0}{2} \quad (1.93)$$

which is illustrated in Figure 1.16a. The dimensions of N_0 are in watts per Hertz. The parameter N_0 is usually referenced to the input stage of the receiver of a communication system. It may be expressed as

$$N_0 = kT_e \quad (1.94)$$

where k is Boltzmann's constant and T_e is the *equivalent noise temperature* of the receiver.⁷ The equivalent noise temperature of a system is defined as *the temperature at which a noisy resistor has to be maintained such that, by connecting the resistor to the input of a noiseless version of the system, it produces the same available noise power at the output of the system as that produced by all the sources of noise in the actual system*. The important feature of the equivalent noise temperature is that it depends only on the parameters of the system.

Since the autocorrelation function is the inverse Fourier transform of the power spectral density, it follows that for white noise

$$R_w(\tau) = \frac{N_0}{2} \delta(\tau) \quad (1.95)$$

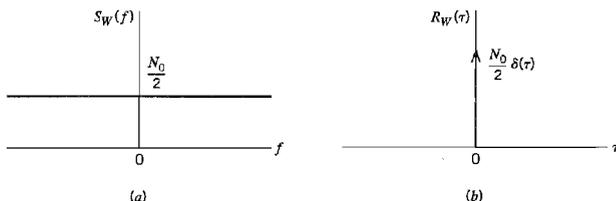


FIGURE 1.16 Characteristics of white noise. (a) Power spectral density. (b) Autocorrelation function.

That is, the autocorrelation function of white noise consists of a delta function weighted by the factor $N_0/2$ and occurring at $\tau = 0$, as in Figure 1.16*b*. We note that $R_w(\tau)$ is zero for $\tau \neq 0$. Accordingly, any two different samples of white noise, no matter how closely together in time they are taken, are uncorrelated. If the white noise $w(t)$ is also Gaussian, then the two samples are statistically independent. In a sense, white Gaussian noise represents the ultimate in "randomness."

Strictly speaking, white noise has infinite average power and, as such, it is not physically realizable. Nevertheless, white noise has simple mathematical properties exemplified by Equations (1.93) and (1.95), which make it useful in statistical system analysis.

The utility of a white noise process is parallel to that of an impulse function or delta function in the analysis of linear systems. Just as we may observe the effect of an impulse only after it has been passed through a system with a finite bandwidth, so it is with white noise whose effect is observed only after passing through a similar system. We may state, therefore, that as long as the bandwidth of a noise process at the input of a system is appreciably larger than that of the system itself, then we may model the noise process as white noise.

► EXAMPLE 1.10 Ideal Low-Pass Filtered White Noise

Suppose that a white Gaussian noise $w(t)$ of zero mean and power spectral density $N_0/2$ is applied to an ideal low-pass filter of bandwidth B and passband magnitude response of one. The power spectral density of the noise $n(t)$ appearing at the filter output is therefore (see Figure 1.17)

$$S_N(f) = \begin{cases} \frac{N_0}{2}, & -B < f < B \\ 0, & |f| > B \end{cases} \quad (1.96)$$

The autocorrelation function of $n(t)$ is the inverse Fourier transform of the power spectral density shown in Figure 1.17*a*:

$$\begin{aligned} R_N(\tau) &= \int_{-B}^B \frac{N_0}{2} \exp(j2\pi f\tau) df \\ &= N_0 B \operatorname{sinc}(2B\tau) \end{aligned} \quad (1.97)$$

This autocorrelation function is plotted in Figure 1.17*b*. We see that $R_N(\tau)$ has its maximum value of $N_0 B$ at the origin, and it passes through zero at $\tau = \pm k/2B$, where $k = 1, 2, 3, \dots$

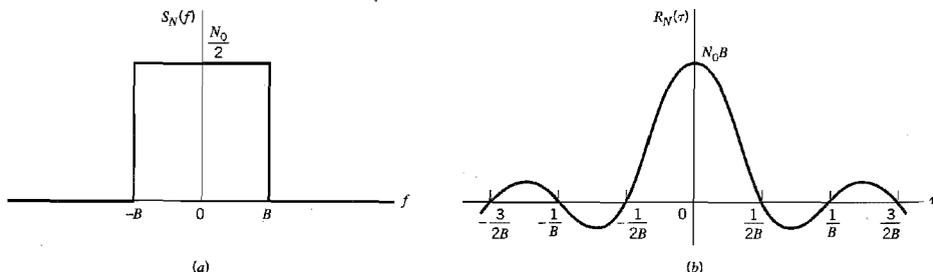


FIGURE 1.17 Characteristics of low-pass filtered white noise. (a) Power spectral density. (b) Autocorrelation function.

Since the input noise $w(t)$ is Gaussian (by hypothesis), it follows that the band-limited noise $n(t)$ at the filter output is also Gaussian. Suppose now that $n(t)$ is sampled at the rate of $2B$ times per second. From Figure 1.17b, we see that the resulting noise samples are uncorrelated and, being Gaussian, they are statistically independent. Accordingly, the joint probability density function of a set of noise samples obtained in this way is equal to the product of the individual probability density functions. Note that each such noise sample has a mean of zero and variance of N_0B . ◀

▶ **EXAMPLE 1.11 Correlation of White Noise with a Sinusoidal Wave**

Consider the sample function

$$w'(t) = \sqrt{\frac{2}{T}} \int_0^T w(t) \cos(2\pi f_c t) dt \quad (1.98)$$

which is the output of a correlator with white Gaussian noise $w(t)$ and sinusoidal wave $\sqrt{2/T} \cos(2\pi f_c t)$ as inputs; the scaling factor $\sqrt{2/T}$ is included here to make the sinusoidal wave input have unit energy over the interval $0 \leq t \leq T$. (This problem was encountered in the Background and Preview chapter but was not elaborated on at that time.) With the noise $w(t)$ having zero mean, it immediately follows that the correlator output $w'(t)$ has zero mean, too. The variance of the correlator output is defined by

$$\begin{aligned} \sigma^2 &= E \left[\frac{2}{T} \int_0^T \int_0^T w(t_1) \cos(2\pi f_c t_1) w(t_2) \cos(2\pi f_c t_2) dt_1 dt_2 \right] \\ &= \frac{2}{T} \int_0^T \int_0^T E[w(t_1)w(t_2)] \cos(2\pi f_c t_1) \cos(2\pi f_c t_2) dt_1 dt_2 \\ &= \frac{2}{T} \int_0^T \int_0^T R_w(t_1, t_2) \cos(2\pi f_c t_1) \cos(2\pi f_c t_2) dt_1 dt_2 \end{aligned}$$

where $R_w(t_1, t_2)$ is the autocorrelation function of the white noise $w(t)$. But from Equation (1.95):

$$R_w(t_1, t_2) = \frac{N_0}{2} \delta(t_1 - t_2)$$

where $N_0/2$ is the power spectral density of the white noise $w(t)$. Accordingly, we may simplify the expression for the variance σ^2 as

$$\sigma^2 = \frac{N_0}{2} \cdot \frac{2}{T} \int_0^T \int_0^T \delta(t_1 - t_2) \cos(2\pi f_c t_1) \cos(2\pi f_c t_2) dt_1 dt_2$$

We now invoke the *sifting property* of the delta function, namely,

$$\int_{-\infty}^{\infty} g(t) \delta(t) dt = g(0)$$

where $g(t)$ is a continuous function of time, assuming the value $g(0)$ at time $t = 0$. Hence, we may further simplify σ^2 as

$$\begin{aligned} \sigma^2 &= \frac{N_0}{2} \cdot \frac{2}{T} \int_0^T \cos^2(2\pi f_c t) dt \\ &= \frac{N_0}{2} \end{aligned} \quad (1.99)$$

where it is assumed that the frequency f_c of the sinusoidal wave input is an integer multiple of the reciprocal of T . ◀

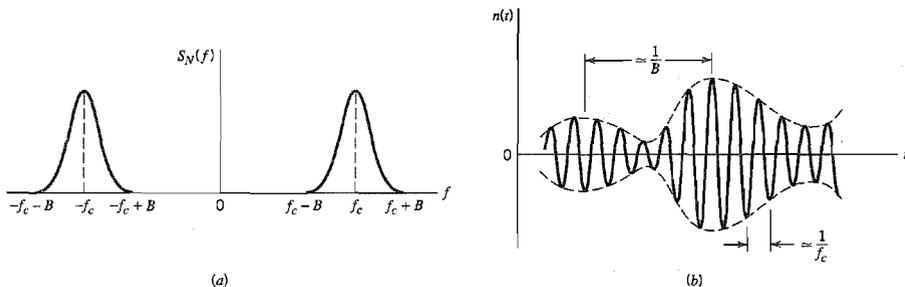


FIGURE 1.18 (a) Power spectral density of narrowband noise. (b) Sample function of narrowband noise.

1.10 Narrowband Noise

The receiver of a communication system usually includes some provision for *preprocessing* the received signal. The preprocessing may take the form of a narrowband filter whose bandwidth is just large enough to pass the modulated component of the received signal essentially undistorted but not so large as to admit excessive noise through the receiver. The noise process appearing at the output of such a filter is called *narrowband noise*. With the spectral components of narrowband noise concentrated about some midband frequency $\pm f_c$ as in Figure 1.18a, we find that a sample function $n(t)$ of such a process appears somewhat similar to a sine wave of frequency f_c , which undulates slowly in both amplitude and phase, as illustrated in Figure 1.18b.

To analyze the effects of narrowband noise on the performance of a communication system, we need a mathematical representation of it. Depending on the application of interest, there are two specific representations of narrowband noise:

1. The narrowband noise is defined in terms of a pair of components called the *in-phase* and *quadrature* components.
2. The narrowband noise is defined in terms of two other components called the *envelope* and *phase*.

These two representations are described in what follows. For now it suffices to say that given the in-phase and quadrature components, we may determine the envelope and phase components, and vice versa. Moreover, in their own individual ways, the two representations are not only basic to the noise analysis of communication systems but also to the characterization of narrowband noise itself.

1.11 Representation of Narrowband Noise in Terms of In-Phase and Quadrature Components

Consider a narrowband noise $n(t)$ of bandwidth $2B$ centered on frequency f_c , as illustrated in Figure 1.18. In light of the theory of band-pass signals and systems presented in Appendix 2, we may represent $n(t)$ in the canonical (standard) form:

$$n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \quad (1.100)$$

where $n_I(t)$ is called the *in-phase component* of $n(t)$, and $n_Q(t)$ is called the *quadrature component* of $n(t)$. Both $n_I(t)$ and $n_Q(t)$ are low-pass signals. Except for the midband frequency f_c , these two components are fully representative of the narrowband noise $n(t)$.

Given the narrowband noise $n(t)$, we may extract its in-phase and quadrature components using the scheme shown in Figure 1.19a. It is assumed that the two low-pass filters used in this scheme are ideal, each having a bandwidth equal to B (i.e., one-half the bandwidth of the narrowband noise $n(t)$). The scheme of Figure 1.19a follows from the representation of Equation (1.100). We may, of course, use this equation directly to generate the narrowband noise $n(t)$, given its in-phase and quadrature components, as shown in Figure 1.19b. The schemes of Figures 1.19a and 1.19b may thus be viewed as narrowband noise *analyzer* and *synthesizer*, respectively.

The in-phase and quadrature components of a narrowband noise have important properties that are summarized here:

1. The in-phase component $n_I(t)$ and quadrature component $n_Q(t)$ of narrowband noise $n(t)$ have zero mean.
2. If the narrowband noise $n(t)$ is Gaussian, then its in-phase component $n_I(t)$ and quadrature component $n_Q(t)$ are jointly Gaussian.
3. If the narrowband noise $n(t)$ is stationary, then its in-phase component $n_I(t)$ and quadrature component $n_Q(t)$ are jointly stationary.
4. Both the in-phase component $n_I(t)$ and quadrature component $n_Q(t)$ have the same power spectral density, which is related to the power spectral density $S_N(f)$ of the narrowband noise $n(t)$ as

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c), & -B \leq f \leq B \\ 0, & \text{otherwise} \end{cases} \quad (1.101)$$

where it is assumed that $S_N(f)$ occupies the frequency interval $f_c - B \leq |f| \leq f_c + B$, and $f_c > B$.

5. The in-phase component $n_I(t)$ and quadrature component $n_Q(t)$ have the same variance as the narrowband noise $n(t)$.
6. The cross-spectral density of the in-phase and quadrature components of narrowband noise $n(t)$ is purely imaginary, as shown by

$$\begin{aligned} S_{N_I N_Q}(f) &= -S_{N_Q N_I}(f) \\ &= \begin{cases} j[S_N(f + f_c) - S_N(f - f_c)], & -B \leq f \leq B \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (1.102)$$

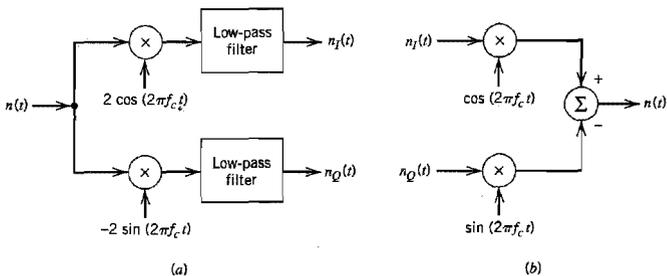


FIGURE 1.19 (a) Extraction of in-phase and quadrature components of a narrowband process. (b) Generation of a narrowband process from its in-phase and quadrature components.

7. If the narrowband noise $n(t)$ is Gaussian and its power spectral density $S_N(f)$ is symmetric about the mid-band frequency f_c , then the in-phase component $n_I(t)$ and quadrature component $n_Q(t)$ are statistically independent.

For further discussions of these properties, the reader is referred to Problems 1.28 and 1.29.

► EXAMPLE 1.12 Ideal Band-Pass Filtered White Noise

Consider a white Gaussian noise of zero mean and power spectral density $N_0/2$, which is passed through an ideal band-pass filter of passband magnitude response equal to one, mid-band frequency f_c , and bandwidth $2B$. The power spectral density characteristic of the filtered noise $n(t)$ will therefore be as shown in Figure 1.20a. The problem is to determine the autocorrelation functions of $n(t)$ and its in-phase and quadrature components.

The autocorrelation function of $n(t)$ is the inverse Fourier transform of the power spectral density characteristic shown in Figure 1.20a:

$$\begin{aligned} R_N(\tau) &= \int_{-f_c-B}^{-f_c+B} \frac{N_0}{2} \exp(j2\pi f\tau) df + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} \exp(j2\pi f\tau) df \\ &= N_0B \operatorname{sinc}(2B\tau) [\exp(-j2\pi f_c\tau) + \exp(j2\pi f_c\tau)] \\ &= 2N_0B \operatorname{sinc}(2B\tau) \cos(2\pi f_c\tau) \end{aligned} \quad (1.103)$$

which is plotted in Figure 1.20b.

The spectral density characteristic of Figure 1.20a is symmetric about $\pm f_c$. Therefore, we find that the corresponding spectral density characteristic of the in-phase noise component

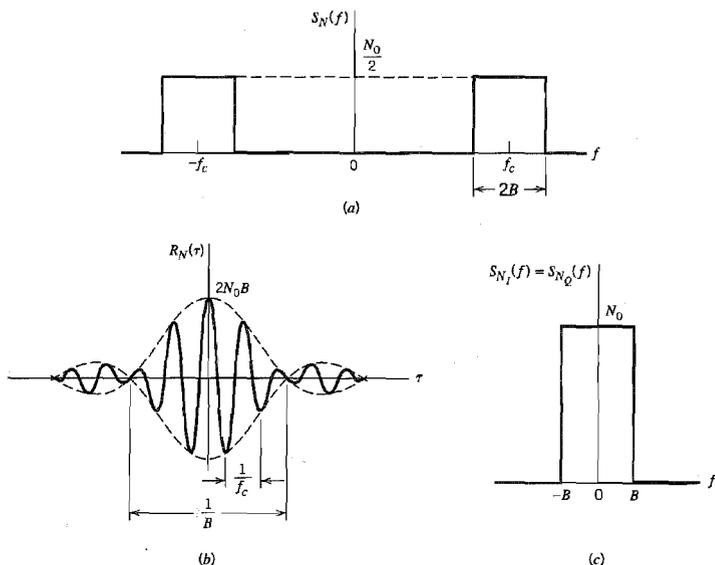


FIGURE 1.20 Characteristics of ideal band-pass filtered white noise. (a) Power spectral density. (b) Autocorrelation function. (c) Power spectral density of in-phase and quadrature components.

$n_I(t)$ or the quadrature noise component $n_Q(t)$ is as shown in Figure 1.21c. The autocorrelation function of $n_I(t)$ or $n_Q(t)$ is therefore (see Example 1.10):

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = 2N_0B \operatorname{sinc}(2B\tau) \quad (1.104)$$

1.12 Representation of Narrowband Noise in Terms of Envelope and Phase Components

In Section 1.11 we considered the representation of a narrowband noise $n(t)$ in terms of its in-phase and quadrature components. We may also represent the noise $n(t)$ in terms of its envelope and phase components as follows:

$$n(t) = r(t) \cos[2\pi f_c t + \psi(t)] \quad (1.105)$$

where

$$r(t) = [n_I^2(t) + n_Q^2(t)]^{1/2} \quad (1.106)$$

and

$$\psi(t) = \tan^{-1} \left[\frac{n_Q(t)}{n_I(t)} \right] \quad (1.107)$$

The function $r(t)$ is called the *envelope* of $n(t)$, and the function $\psi(t)$ is called the *phase* of $n(t)$.

The envelope $r(t)$ and phase $\psi(t)$ are both sample functions of low-pass random processes. As illustrated in Figure 1.18b, the time interval between two successive peaks of the envelope $r(t)$ is approximately $1/B$, where $2B$ is the bandwidth of the narrowband noise $n(t)$.

The probability distributions of $r(t)$ and $\psi(t)$ may be obtained from those of $n_I(t)$ and $n_Q(t)$ as follows. Let N_I and N_Q denote the random variables obtained by observing (at some fixed time) the random processes represented by the sample functions $n_I(t)$ and $n_Q(t)$, respectively. We note that N_I and N_Q are independent Gaussian random variables of zero mean and variance σ^2 , and so we may express their joint probability density function by

$$f_{N_I, N_Q}(n_I, n_Q) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{n_I^2 + n_Q^2}{2\sigma^2}\right) \quad (1.108)$$

Accordingly, the probability of the joint event that N_I lies between n_I and $n_I + dn_I$ and that N_Q lies between n_Q and $n_Q + dn_Q$ (i.e., the pair of random variables N_I and N_Q lies jointly inside the shaded area of Figure 1.21a) is given by

$$f_{N_I, N_Q}(n_I, n_Q) dn_I dn_Q = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{n_I^2 + n_Q^2}{2\sigma^2}\right) dn_I dn_Q \quad (1.109)$$

Define the transformation (see Figure 1.21a)

$$n_I = r \cos \psi \quad (1.110)$$

$$n_Q = r \sin \psi \quad (1.111)$$

In a limiting sense, we may equate the two incremental areas shown shaded in Figures 1.21a and 1.21b and thus write

$$dn_I dn_Q = r dr d\psi \quad (1.112)$$

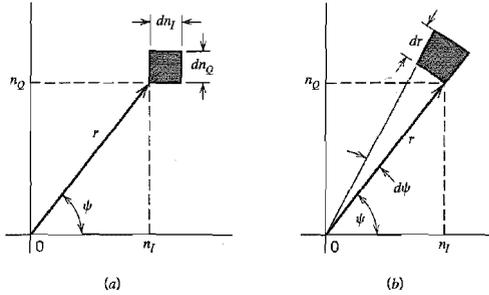


FIGURE 1.21 Illustrating the coordinate system for representation of narrowband noise: (a) in terms of in-phase and quadrature components, and (b) in terms of envelope and phase.

Now, let R and Ψ denote the random variables obtained by observing (at some time t) the random processes represented by the envelope $r(t)$ and phase $\psi(t)$, respectively. Then, substituting Equations (1.110)–(1.112) into (1.109), we find that the probability of the random variables R and Ψ lying jointly inside the shaded area of Figure 1.21b is equal to

$$\frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr d\psi$$

That is, the joint probability density function of R and Ψ is

$$f_{R,\Psi}(r, \psi) = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) \quad (1.113)$$

This probability density function is independent of the angle ψ , which means that the random variables R and Ψ are statistically independent. We may thus express $f_{R,\Psi}(r, \psi)$ as the product of $f_R(r)$ and $f_\Psi(\psi)$. In particular, the random variable Ψ representing phase is *uniformly distributed* inside the range 0 to 2π , as shown by

$$f_\Psi(\psi) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \psi \leq 2\pi \\ 0, & \text{elsewhere} \end{cases} \quad (1.114)$$

This leaves the probability density function of the random variable R as

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right), & r \geq 0 \\ 0, & \text{elsewhere} \end{cases} \quad (1.115)$$

where σ^2 is the variance of the original narrowband noise $n(t)$. A random variable having the probability density function of Equation (1.115) is said to be *Rayleigh distributed*.⁸

For convenience of graphical presentation, let

$$v = \frac{r}{\sigma} \quad (1.116)$$

$$f_V(v) = \sigma f_R(r) \quad (1.117)$$

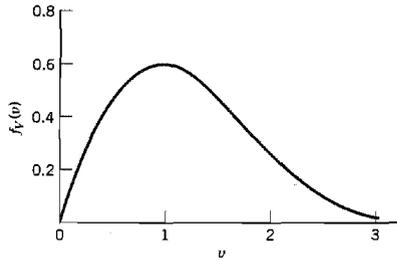


FIGURE 1.22 Normalized Rayleigh distribution.

Then we may rewrite the Rayleigh distribution of Equation (1.115) in the *normalized* form

$$f_v(v) = \begin{cases} v \exp\left(-\frac{v^2}{2}\right), & v \geq 0 \\ 0, & \text{elsewhere} \end{cases} \quad (1.118)$$

Equation (1.118) is plotted in Figure 1.22. The peak value of the distribution $f_v(v)$ occurs at $v=1$ and is equal to 0.607. Note also that, unlike the Gaussian distribution, the Rayleigh distribution is zero for negative values of v . This is because the envelope $r(t)$ can assume only nonnegative values.

1.13 Sine Wave Plus Narrowband Noise

Suppose next that we add the sinusoidal wave $A \cos(2\pi f_c t)$ to the narrowband noise $n(t)$, where A and f_c are both constants. We assume that the frequency of the sinusoidal wave is the same as the nominal carrier frequency of the noise. A sample function of the sinusoidal wave plus noise is then expressed by

$$x(t) = A \cos(2\pi f_c t) + n(t) \quad (1.119)$$

Representing the narrowband noise $n(t)$ in terms of its in-phase and quadrature components, we may write

$$x(t) = n'_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \quad (1.120)$$

where

$$n'_I(t) = A + n_I(t) \quad (1.121)$$

We assume that $n(t)$ is Gaussian with zero mean and variance σ^2 . Accordingly, we may state the following:

1. Both $n'_I(t)$ and $n_Q(t)$ are Gaussian and statistically independent.
2. The mean of $n'_I(t)$ is A and that of $n_Q(t)$ is zero.
3. The variance of both $n'_I(t)$ and $n_Q(t)$ is σ^2 .

We may therefore express the joint probability density function of the random variables N'_I and N_Q , corresponding to $n'_I(t)$ and $n_Q(t)$, as follows:

$$f_{N'_I, N_Q}(n'_I, n_Q) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(n'_I - A)^2 + n_Q^2}{2\sigma^2}\right] \quad (1.122)$$

Let $r(t)$ denote the envelope of $x(t)$ and $\psi(t)$ denote its phase. From Equation (1.120), we thus find that

$$r(t) = \{[n'_1(t)]^2 + n_{\odot}^2(t)\}^{1/2} \quad (1.123)$$

and

$$\psi(t) = \tan^{-1} \left[\frac{n_{\odot}(t)}{n'_1(t)} \right] \quad (1.124)$$

Following a procedure similar to that described in Section 1.12 for the derivation of the Rayleigh distribution, we find that the joint probability density function of the random variables R and Ψ , corresponding to $r(t)$ and $\psi(t)$ for some fixed time t , is given by

$$f_{R,\Psi}(r, \psi) = \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2 + A^2 - 2Ar \cos \psi}{2\sigma^2}\right) \quad (1.125)$$

We see that in this case, however, we cannot express the joint probability density function $f_{R,\Psi}(r, \psi)$ as a product $f_R(r)f_{\Psi}(\psi)$. This is because we now have a term involving the values of both random variables multiplied together as $r \cos \psi$. Hence, R and Ψ are dependent random variables for nonzero values of the amplitude A of the sinusoidal wave component.

We are interested, in particular, in the probability density function of R . To determine this probability density function, we integrate Equation (1.125) over all possible values of ψ obtaining the marginal density

$$\begin{aligned} f_R(r) &= \int_0^{2\pi} f_{R,\Psi}(r, \psi) d\psi \\ &= \frac{r}{2\pi\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) \int_0^{2\pi} \exp\left(\frac{Ar}{\sigma^2} \cos \psi\right) d\psi \end{aligned} \quad (1.126)$$

The integral in the right-hand side of Equation (1.126) can be identified in terms of the defining integral for the *modified Bessel function of the first kind of zero order* (see Appendix 3); that is,

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(x \cos \psi) d\psi \quad (1.127)$$

Thus, letting $x = Ar/\sigma^2$, we may rewrite Equation (1.126) in the compact form:

$$f_R(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2 + A^2}{2\sigma^2}\right) I_0\left(\frac{Ar}{\sigma^2}\right) \quad (1.128)$$

This relation is called the *Rician distribution*.⁹

As with the Rayleigh distribution, the graphical presentation of the Rician distribution is simplified by putting

$$v = \frac{r}{\sigma} \quad (1.129)$$

$$a = \frac{A}{\sigma} \quad (1.130)$$

$$f_V(v) = \sigma f_R(r) \quad (1.131)$$

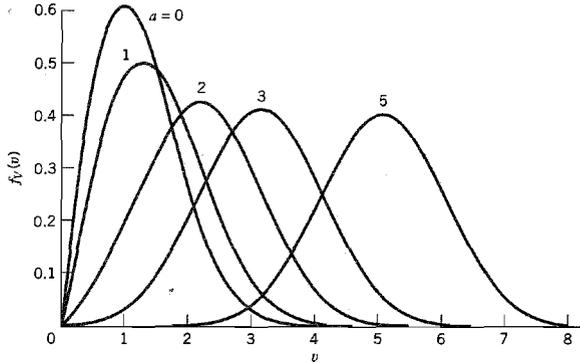


FIGURE 1.23 Normalized Rician distribution.

Then we may express the Rician distribution of Equation (1.128) in the *normalized form*

$$f_v(v) = v \exp\left(-\frac{v^2 + a^2}{2}\right) I_0(av) \quad (1.132)$$

which is plotted in Figure 1.23 for the values 0, 1, 2, 3, 5, of the parameter a . Based on these curves, we may make the following observations:

1. When a is zero, the Rician distribution reduces to the Rayleigh distribution.
2. The envelope distribution is approximately Gaussian in the vicinity of $v = a$ when a is large, that is, when the sine-wave amplitude A is large compared with σ , the square root of the average power of the noise $n(t)$.

1.14 Computer Experiments: Flat-Fading Channel

In this section we use computer simulations to study a *multipath channel* characterized by *Rayleigh fading*, examples of which arise in wireless communications and long-range radio transmission via the ionosphere. Fading occurs because of interference between different versions of the transmitted signal, which reach the receiver at correspondingly different times. The net result is that the received signal can vary widely in both amplitude and phase. Under certain conditions, the statistical time-varying nature of the received signal's envelope is closely described by a Rayleigh distribution as demonstrated herein.

Figure 1.24 presents a model of a multipath channel. It consists of a large collection of *scatterers* randomly positioned in space, whereby a single incident beam is converted into a correspondingly large number of scattered beams at the receiving antenna. The transmitted signal is set equal to $A \cos(2\pi f_c t)$. It is assumed that all the scattered beams travel at the same mean velocity. However, they differ from each other in amplitude and phase by virtue of differences in path loss and path delay. Thus the k th scattered beam is given by $A_k \cos(2\pi f_c t + \Theta_k)$, where the amplitude A_k and phase Θ_k are random variables that vary slowly with time. Moreover, the Θ_k are all independent of one another and

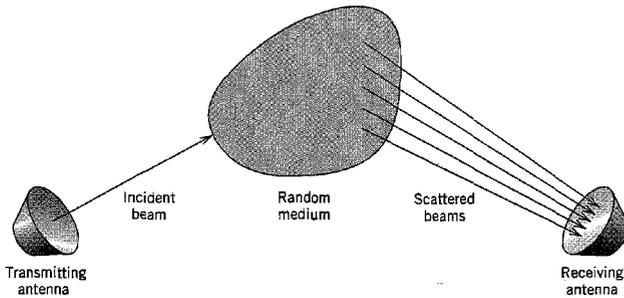


FIGURE 1.24 Model of a multipath channel.

uniformly distributed inside the interval $[0, 2\pi]$. The type of fading exhibited by the multipath channel described herein is referred to as “flat fading” because the spectral characteristics of the transmitted signal are completely preserved at the channel output. However, the strength of the channel output changes with time due to random fluctuations in the gain of the channel caused by the multipath phenomenon.

Summing the contributions of all the scatterers, assumed to be N in number, we may express the random process representing the received signal as

$$X(t) = \sum_{k=1}^N A_k \cos(2\pi f_c t + \Theta_k) \quad (1.133)$$

which may be rewritten in the equivalent form

$$X(t) = X_I \cos(2\pi f_c t) - X_Q \sin(2\pi f_c t) \quad (1.134)$$

where the X_I and X_Q are respectively defined by

$$X_I = \sum_{k=1}^N A_k \cos \Theta_k \quad (1.135)$$

and

$$X_Q = \sum_{k=1}^N A_k \sin \Theta_k \quad (1.136)$$

For convenience of presentation and without loss of generality, we may assume that A_k lies in the closed interval $[-1, 1]$ for all k .

Experiment 1. Gaussian Distributions

From the central limit theorem we note that as the number of scatterers, N , approaches infinity, both X_I and X_Q should approach Gaussian random variables. To test the validity of this statement, the probability distributions of the in-phase component X_I and quadrature component X_Q are computed for $N = 10, 100, 1000$, and $10,000$. To test the validity of the central limit theorem, we need a measure of the goodness-of-fit that tests the equivalence of the measured probability distribution of the sampled data for varying

N to the theoretical Gaussian distribution. One way of performing such a test is to use central moments of a distribution (up to order 4) to define the following two parameters:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \tag{1.137}$$

and

$$\beta_2 = \frac{\mu_4}{\mu_2^2} \tag{1.138}$$

where μ_2 , μ_3 , and μ_4 are the second, third, and fourth central moments, respectively. The parameters β_1 and β_2 together provide a measure of the *skewness* of the distribution under test. The closer the values β_1 and β_2 for the measured distribution are to the corresponding ones for the theoretical distribution, the better is the goodness-of-fit. For a Gaussian random variable X of mean μ_X and variance σ_X^2 we have

$$\begin{aligned} \mu_2 &= \sigma_X^2 \\ \mu_3 &= 0 \\ \mu_4 &= 3\sigma_X^4 \end{aligned}$$

which yield

$$\beta_1 = 0$$

and

$$\beta_2 = 3$$

Table 1.1 presents the values of β_1 and β_2 computed for both the in-phase component X_I and quadrature component X_Q for varying N . Comparing these values with the corresponding ones for a Gaussian distribution, we clearly see that as the number of scatterers, N , increases the distributions of both X_I and X_Q do approach a zero-mean Gaussian distribution in accordance with the central limit theorem.

TABLE 1.1 β Values for in-phase and quadrature components

(a) Measured Distribution

		Number of Scatterers, N			
		10	100	1000	10,000
In-phase component X_I	β_1	0.2443	0.0255	0.0065	0.0003
	β_2	2.1567	2.8759	2.8587	3.0075
Quadrature component X_Q	β_1	0.0874	0.0017	0.0004	0.0000
	β_2	1.9621	2.7109	3.1663	3.0135

(b) Theoretical Distribution: Gaussian

$$\begin{aligned} \beta_1 &= 0 \\ \beta_2 &= 3 \end{aligned}$$

Experiment 2. Rayleigh Distribution

In Equation (1.134) the random process $X(t)$ is expressed in terms of its in-phase and quadrature components. Equivalently, we may express $X(t)$ in terms of its envelope and phase as

$$X(t) = R \cos(2\pi f_c t + \Psi) \quad (1.139)$$

where

$$R = \sqrt{X_I^2 + X_Q^2} \quad (1.140)$$

and

$$\Psi = \tan^{-1}\left(\frac{X_Q}{X_I}\right) \quad (1.141)$$

Note that in the experiments considered here the in-phase component X_I , quadrature component X_Q , envelope R , and phase Ψ are all independent of time.

If X_I and X_Q approach Gaussian random variables for increasing N , then from the theory presented in Section 1.12 we note that the envelope R will approach a Rayleigh distribution, and the phase Ψ will approach a uniform distribution. In Figure 1.25 we present the actual probability density function of r for data generated for the case of $N = 10,000$, with 100 histograms and 100 ensemble averages being computed. This figure also includes the theoretical curve. There is close agreement between these two curves, substantiating the assertion that the envelope R of the received signal approaches a Rayleigh distribution.

Figure 1.26 illustrates the effect of Rayleigh fading on the waveform of the received signal $x(t)$, a sample function of $X(t)$, for the case of a sinusoidal transmitted signal with unit amplitude (i.e., $A = 1$) and frequency $f_c = 1$ MHz. Specifically, the transmitted signal and the corresponding received signal are shown in parts *a* and *b* of Figure 1.26, respectively. Comparing these two waveforms, we see that transmission through the multipath

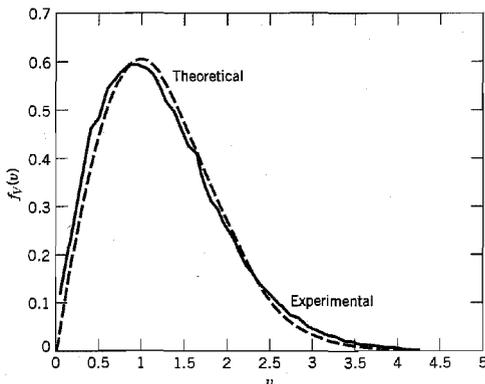


FIGURE 1.25 Probability density function of the envelope of random process $X(t)$; comparing theory and experiment.

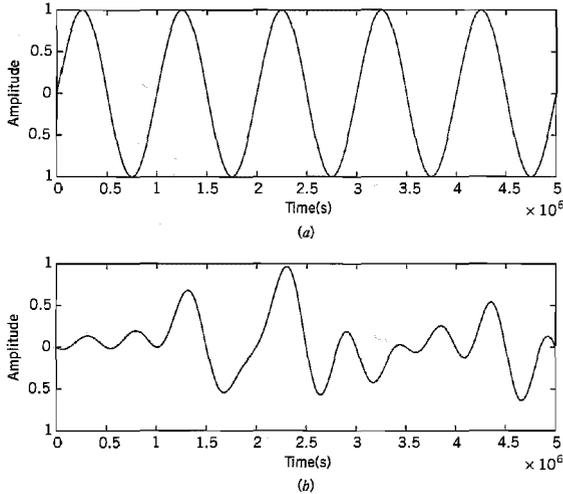


FIGURE 1.26 Effect of Rayleigh fading on a sinusoidal wave. (a) Input sinusoidal wave. (b) Waveform of the resulting signal.

channel of Figure 1.24 has resulted in a received signal whose amplitude and phase components vary randomly with time, as expected.

1.15 Summary and Discussion

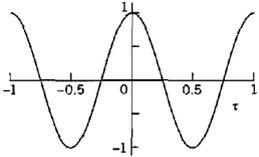
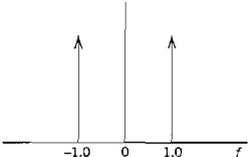
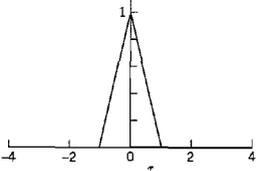
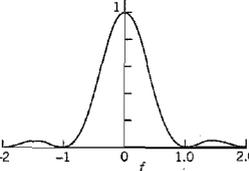
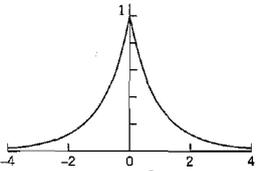
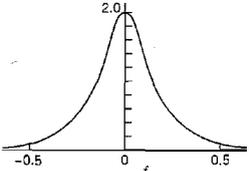
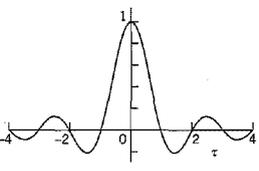
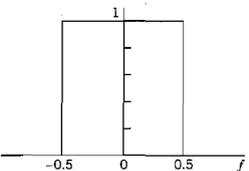
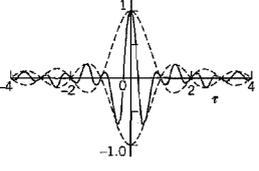
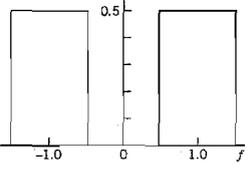
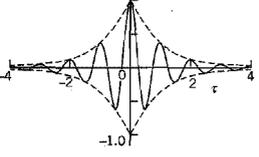
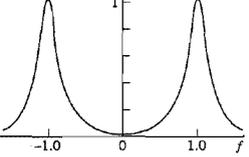
Much of the material presented in this chapter has dealt with the characterization of a particular class of random processes known to be stationary and ergodic. The implication of (wide-sense) stationarity is that we may develop a partial description of a random process in terms of two ensemble-averaged parameters: (1) a mean that is independent of time, and (2) an autocorrelation function that depends only on the difference between the times at which two observations of the process are made.¹⁰ Ergodicity enables us to use time averages as “estimates” of these parameters. The time averages are computed using a sample function (i.e., single realization) of the random process.

Another important parameter of a random process is the power spectral density. The autocorrelation function and the power spectral density constitute a Fourier-transform pair. The formulas that define the power spectral density in terms of the autocorrelation function and vice versa are known as the Einstein-Wiener-Khinchine relations.

In Table 1.2 we present a graphical summary of the autocorrelation functions and power spectral densities of important random processes. All the processes described in this table are assumed to have zero mean and unit variance. This table should give the reader a feeling for (1) the interplay between the autocorrelation function and power spectral density of a random process, and (2) the role of linear filtering in shaping the autocorrelation function or, equivalently, the power spectral density of a white noise process.

The latter part of the chapter dealt with a noise process that is Gaussian and narrowband, which is the kind of filtered noise encountered at the front end of an idealized form of communication receiver. Gaussianity means that the random variable obtained by

TABLE 1.2 Graphical summary of autocorrelation functions and power spectral densities of random processes of zero mean and unit variance

Type of Process, $X(t)$	Autocorrelation Function, $R_X(\tau)$	Power Spectral Density, $S_X(f)$
Sinusoidal process of unit frequency and random phase		
Random binary wave of unit symbol-duration		
RC low-pass filtered white noise		
Ideal low-pass filtered white noise		
Ideal band-pass filtered white noise		
RLC-filtered white noise		

observing the output of the filter at some fixed time has a Gaussian distribution. The narrowband nature of the noise means that it may be represented in terms of an in-phase and a quadrature component. These two components are both low-pass, Gaussian processes, each with zero mean and a variance equal to that of the original narrowband noise. Alternatively, a Gaussian narrowband noise may be represented in terms of a Rayleigh-distributed envelope and a uniformly distributed phase. Each of these representations has its own specific area of application, as shown in subsequent chapters of the book.

NOTES AND REFERENCES

1. For a rigorous treatment of random processes, see the classic books of Doob (1953), Loève (1963), and Cramér and Leadbetter (1967).
2. There is another important class of random processes commonly encountered in practice, the mean and autocorrelation function of which exhibit *periodicity*, as in

$$\begin{aligned}\mu_X(t_1 + T) &= \mu_X(t_1) \\ R_X(t_1 + T, t_2 + T) &= R_X(t_1, t_2)\end{aligned}$$

for all t_1 and t_2 . A random process $X(t)$ satisfying this pair of conditions is said to be *cyclostationary* (in the wide sense). Modeling the process $X(t)$ as cyclostationary adds a new dimension, namely, period T to the partial description of the process. Examples of cyclostationary processes include a television signal obtained by raster-scanning a random video field, and a modulated process obtained by varying the amplitude, phase, or frequency of a sinusoidal carrier. For detailed discussion of cyclostationary processes, see Franks (1969), pp. 204–214, and the paper by Gardner and Franks (1975).

3. Traditionally, Equations (1.42) and (1.43) have been referred to in the literature as the Wiener-Khinchine relations in recognition of pioneering work done by Norbert Wiener and A. I. Khinchine; for their original papers, see Wiener (1930) and Khinchine (1934). A discovery of a forgotten paper by Albert Einstein on time-series analysis (delivered at the Swiss Physical Society's February 1914 meeting in Basel) reveals that Einstein had discussed the autocorrelation function and its relationship to the spectral content of a time series many years before Wiener and Khinchine. An English translation of Einstein's paper is reproduced in the *IEEE ASSP Magazine*, vol. 4, October 1987. This particular issue also contains articles by W. A. Gardner and A. M. Yaglom, which elaborate on Einstein's original work.
4. For further details of power spectrum estimation, see Blackman and Tukey (1958), Box and Jenkins (1976), Marple (1987), and Kay (1988).
5. The Gaussian distribution and associated Gaussian process are named after the great mathematician C. F. Gauss. At age 18, Gauss invented *the method of least squares* for finding the best value of a sequence of measurements of some quantity. Gauss later used the method of least squares in fitting orbits of planets to data measurements, a procedure that was published in 1809 in his book entitled *Theory of Motion of the Heavenly Bodies*. In connection with the error of observation, he developed the *Gaussian distribution*. This distribution is also known as the *normal distribution*. Partly for historical reasons, mathematicians commonly use the term normal, while engineers and physicists commonly use the term Gaussian.
6. For a detailed treatment of electrical noise, see Van der Ziel (1970) and the collection of papers edited by Gupta (1977).

An introductory treatment of shot noise is presented in Helstrom (1990). For a more detailed treatment, see the paper by Yue, Lukanani, and Rice (1978).

Thermal noise was first studied experimentally by J. B. Johnson in 1928, and for this reason it is sometimes referred to as the *Johnson noise*. Johnson's experiments were confirmed theoretically by Nyquist (1928).

7. The noisiness of a receiver may also be measured in terms of the so-called *noise figure*. The relationship between the noise figure and the equivalent noise temperature is developed in Chapter 8.
8. The Rayleigh distribution is named after the English physicist J. W. Strutt, Lord Rayleigh.
9. The Rician distribution is named in honor of Stephen O. Rice for the original contribution reported in a pair of papers published in 1944 and 1945, which are reproduced in Wax (1954).
10. The statistical characterization of communication systems presented in this book is confined to the first two moments, mean and autocorrelation function (equivalently, autocovariance function) of the pertinent random process. However, when a random process is transmitted through a nonlinear system, valuable information is contained in higher-order moments of the resulting output process. The parameters used to characterize higher-order moments in the time domain are called *cumulants*, and their multidimensional Fourier transforms are called *polyspectra*. For a discussion of higher-order cumulants and polyspectra and their estimation, see the paper by Nikiyas and Raghuveer (1987).

PROBLEMS

Stationarity and Ergodicity

- 1.1 Consider a random process $X(t)$ defined by

$$X(t) = \sin(2\pi f_c t)$$

in which the frequency f_c is a random variable uniformly distributed over the interval $[0, W]$. Show that $X(t)$ is nonstationary. *Hint:* Examine specific sample functions of the random process $X(t)$ for the frequency $f = W/4$, $W/2$, and W , say.

- 1.2 Consider the sinusoidal process

$$X(t) = A \cos(2\pi f_c t)$$

where the frequency f_c is constant and the amplitude A is uniformly distributed:

$$f_A(a) = \begin{cases} 1, & 0 \leq a \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine whether or not this process is strictly stationary.

- 1.3 A random process $X(t)$ is defined by

$$X(t) = A \cos(2\pi f_c t)$$

where A is a Gaussian-distributed random variable of zero mean and variance σ_A^2 . This random process is applied to an ideal integrator, producing the output

$$Y(t) = \int_0^t X(\tau) d\tau$$

- (a) Determine the probability density function of the output $Y(t)$ at a particular time t_k .
 - (b) Determine whether or not $Y(t)$ is stationary.
 - (c) Determine whether or not $Y(t)$ is ergodic.
- 1.4 Let X and Y be statistically independent Gaussian-distributed random variables, each with zero mean and unit variance. Define the Gaussian process

$$Z(t) = X \cos(2\pi t) + Y \sin(2\pi t)$$

- (a) Determine the joint probability density function of the random variables $Z(t_1)$ and $Z(t_2)$ obtained by observing $Z(t)$ at times t_1 and t_2 , respectively.
- (b) Is the process $Z(t)$ stationary? Why?

Correlation and Spectral Density Functions

- 1.5 Prove the following two properties of the autocorrelation function $R_X(\tau)$ of a random process $X(t)$:
- (a) If $X(t)$ contains a DC component equal to A , then $R_X(\tau)$ will contain a constant component equal to A^2 .
 - (b) If $X(t)$ contains a sinusoidal component, then $R_X(\tau)$ will also contain a sinusoidal component of the same frequency.
- 1.6 The square wave $x(t)$ of Figure P1.6 of constant amplitude A , period T_0 , and delay t_d , represents the sample function of a random process $X(t)$. The delay is random, described by the probability density function

$$f_{T_d}(t_d) = \begin{cases} \frac{1}{T_0}, & -\frac{1}{2} T_0 \leq t_d \leq \frac{1}{2} T_0 \\ 0, & \text{otherwise} \end{cases}$$

- (a) Determine the probability density function of the random variable $X(t_k)$ obtained by observing the random process $X(t)$ at time t_k .
- (b) Determine the mean and autocorrelation function of $X(t)$ using ensemble-averaging.
- (c) Determine the mean and autocorrelation function of $X(t)$ using time-averaging.
- (d) Establish whether or not $X(t)$ is stationary. In what sense is it ergodic?

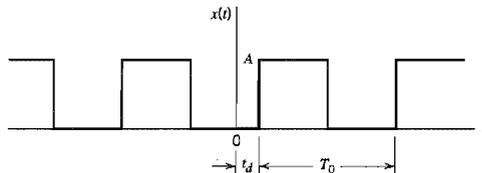


FIGURE P1.6

- 1.7 A binary wave consists of a random sequence of symbols 1 and 0, similar to that described in Example 1.3, with one basic difference: symbol 1 is now represented by a pulse of amplitude A volts and symbol 0 is represented by zero volts. All other parameters are the same as before. Show that for this new random binary wave $X(t)$:

(a) The autocorrelation function is

$$R_X(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4} \left(1 - \frac{|\tau|}{T}\right), & |\tau| < T \\ \frac{A^2}{4}, & |\tau| \geq T \end{cases}$$

(b) The power spectral density is

$$S_X(f) = \frac{A^2}{4} \delta(f) + \frac{A^2 T}{4} \text{sinc}^2(fT)$$

What is the percentage power contained in the DC component of the binary wave?

1.8 A random process $Y(t)$ consists of a DC component of $\sqrt{3/2}$ volts, a periodic component $g(t)$, and a random component $X(t)$. The autocorrelation function of $Y(t)$ is shown in Figure P1.8.

- (a) What is the average power of the periodic component $g(t)$?
 (b) What is the average power of the random component $X(t)$?

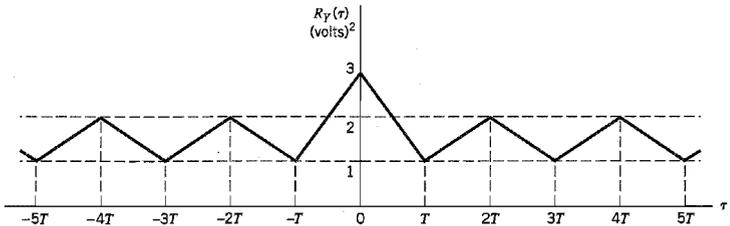


FIGURE P1.8

- 1.9 Consider a pair of stationary processes $X(t)$ and $Y(t)$. Show that the cross-correlations $R_{XY}(\tau)$ and $R_{YX}(\tau)$ of these processes have the following properties:
- (a) $R_{XY}(\tau) = R_{YX}(-\tau)$
 (b) $|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$
 where $R_X(\tau)$ and $R_Y(\tau)$ are the autocorrelation functions of $X(t)$ and $Y(t)$, respectively.
- 1.10 Consider two linear filters connected in cascade as in Figure P1.10. Let $X(t)$ be a stationary process with autocorrelation function $R_X(\tau)$. The random process appearing at the first filter output is $V(t)$ and that at the second filter output is $Y(t)$.
- (a) Find the autocorrelation function of $Y(t)$.
 (b) Find the cross-correlation function $R_{VY}(\tau)$ of $V(t)$ and $Y(t)$.

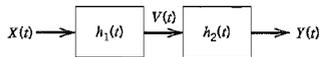


FIGURE P1.10

- 1.11 A stationary process $X(t)$ is applied to a linear time-invariant filter of impulse response $h(t)$, producing an output $Y(t)$.
- (a) Show that the cross-correlation function $R_{YX}(\tau)$ of the output $Y(t)$ and the input $X(t)$ is equal to the impulse response $h(\tau)$ convolved with the autocorrelation function $R_X(\tau)$ of the input, as shown by

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} h(u)R_X(\tau - u) du$$

Show that the second cross-correlation function $R_{XY}(\tau)$ equals

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(-u)R_X(\tau - u) du$$

- (b) Find the cross-spectral densities $S_{YX}(f)$ and $S_{XY}(f)$.

- (c) Assuming that $X(t)$ is a white noise process with zero mean and power spectral density $N_0/2$, show that

$$R_{YX}(\tau) = \frac{N_0}{2} b(\tau)$$

Comment on the practical significance of this result.

- 1.12 The power spectral density of a random process $X(t)$ is shown in Figure P1.12. It consists of a delta function at $f = 0$ and a triangular component.
- Determine and sketch the autocorrelation function $R_X(\tau)$ of $X(t)$.
 - What is the DC power contained in $X(t)$?
 - What is the AC power contained in $X(t)$?
 - What sampling rates will give uncorrelated samples of $X(t)$? Are the samples statistically independent?

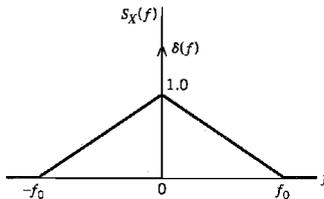


FIGURE P1.12

- 1.13 A pair of noise processes $n_1(t)$ and $n_2(t)$ are related by

$$n_2(t) = n_1(t) \cos(2\pi f_c t + \theta) - n_1(t) \sin(2\pi f_c t + \theta)$$

where f_c is a constant, and θ is the value of a random variable Θ whose probability density function is defined by

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta \leq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

The noise process $n_1(t)$ is stationary and its power spectral density is as shown in Figure P1.13. Find and plot the corresponding power spectral density of $n_2(t)$.

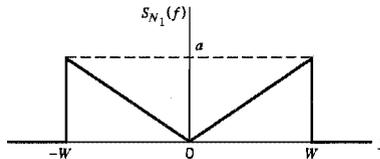


FIGURE P1.13

- 1.14 A random telegraph signal $X(t)$, characterized by the autocorrelation function

$$R_X(\tau) = \exp(-2\nu|\tau|)$$

where v is a constant, is applied to the low-pass RC filter of Figure P1.14. Determine the power spectral density and autocorrelation function of the random process at the filter output.

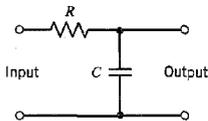


FIGURE P1.14

- 1.15 A *running integrator* is defined by

$$y(t) = \int_{t-T}^t x(\tau) d\tau$$

where $x(t)$ is the input, $y(t)$ is the output, and T is the integration period. Both $x(t)$ and $y(t)$ are sample functions of stationary processes $X(t)$ and $Y(t)$, respectively. Show that the power spectral density of the integrator output is related to that of the integrator input as

$$S_Y(f) = T^2 \operatorname{sinc}^2(fT) S_X(f)$$

- 1.16 A zero-mean stationary process $X(t)$ is applied to a linear filter whose impulse response is defined by a truncated exponential:

$$h(t) = \begin{cases} ae^{-at}, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Show that the power spectral density of the filter output $Y(t)$ is defined by

$$S_Y(f) = \frac{a^2}{a^2 + 4\pi^2 f^2} (1 - 2 \exp(-aT) \cos(2\pi fT) + \exp(-2aT)) S_X(f)$$

where $S_X(f)$ is the power spectral density of the filter input.

- 1.17 The output of an oscillator is described by

$$X(t) = A \cos(2\pi ft - \Theta)$$

where A is a constant, and f and Θ are independent random variables. The probability density function of Θ is defined by

$$f_{\Theta}(\theta) = \begin{cases} \frac{1}{2\pi}, & 0 \leq \theta \leq 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Find the power spectral density of $X(t)$ in terms of the probability density function of the frequency f . What happens to this power spectral density when the frequency f assumes a constant value?

Gaussian Processes

1.18 A stationary, Gaussian process $X(t)$ has zero mean and power spectral density $S_X(f)$. Determine the probability density function of a random variable obtained by observing the process $X(t)$ at some time t_k .

1.19 A Gaussian process $X(t)$ of zero mean and variance σ_X^2 is passed through a full-wave rectifier, which is described by the input-output relation of Figure P1.19. Show that the probability density function of the random variable $Y(t_k)$, obtained by observing the random process $Y(t)$ at the rectifier output at time t_k , is as follows:

$$f_{Y(t_k)}(y) = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{1}{\sigma_X} \exp\left(-\frac{y^2}{2\sigma_X^2}\right), & y \geq 0 \\ 0, & y < 0 \end{cases}$$

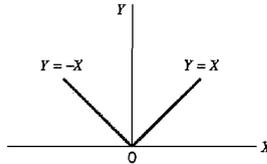


FIGURE P1.19

1.20 Let $X(t)$ be a zero-mean, stationary, Gaussian process with autocorrelation function $R_X(\tau)$. This process is applied to a square-law device, which is defined by the input-output relation

$$Y(t) = X^2(t)$$

where $Y(t)$ is the output.

- (a) Show that the mean of $Y(t)$ is $R_X(0)$.
- (b) Show that the autocovariance function of $Y(t)$ is $2R_X^2(\tau)$.

1.21 A stationary, Gaussian process $X(t)$ with mean μ_X and variance σ_X^2 is passed through two linear filters with impulse responses $h_1(t)$ and $h_2(t)$, yielding processes $Y(t)$ and $Z(t)$, as shown in Figure P1.21.

- (a) Determine the joint probability density function of the random variables $Y(t_1)$ and $Z(t_2)$.
- (b) What conditions are necessary and sufficient to ensure that $Y(t_1)$ and $Z(t_2)$ are statistically independent?

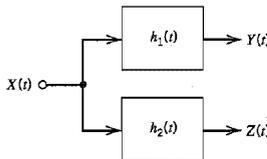


FIGURE P1.21

1.22 A stationary, Gaussian process $X(t)$ with zero mean and power spectral density $S_X(f)$ is applied to a linear filter whose impulse response $h(t)$ is shown in Figure P1.22. A sample Y is taken of the random process at the filter output at time T .

- Determine the mean and variance of Y .
- What is the probability density function of Y ?

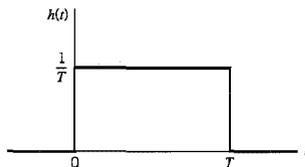


FIGURE P1.22

Noise

1.23 Consider a white Gaussian noise process of zero mean and power spectral density $N_0/2$ that is applied to the input of the high-pass RL filter shown in Figure P1.23.

- Find the autocorrelation function and power spectral density of the random process at the output of the filter.
- What are the mean and variance of this output?

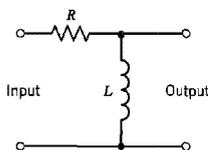


FIGURE P1.23

1.24 A white noise $w(t)$ of power spectral density $N_0/2$ is applied to a *Butterworth* low-pass filter of order n , whose magnitude response is defined by

$$|H(f)| = \frac{1}{[1 + (f/f_0)^{2n}]^{1/2}}$$

- Determine the noise equivalent bandwidth for this low-pass filter. (See Appendix 2 for the definition of noise equivalent bandwidth.)
 - What is the limiting value of the noise equivalent bandwidth as n approaches infinity?
- 1.25 The shot-noise process $X(t)$ defined by Equation (1.86) is stationary. Why?
- 1.26 White Gaussian noise of zero mean and power spectral density $N_0/2$ is applied to the filtering scheme shown in Figure P1.26a. The frequency responses of these two filters are shown in Figure P1.26b. The noise at the low-pass filter output is denoted by $n(t)$.
- Find the power spectral density and the autocorrelation function of $n(t)$.
 - Find the mean and variance of $n(t)$.

- (c) What is the rate at which $n(t)$ can be sampled so that the resulting samples are essentially uncorrelated?

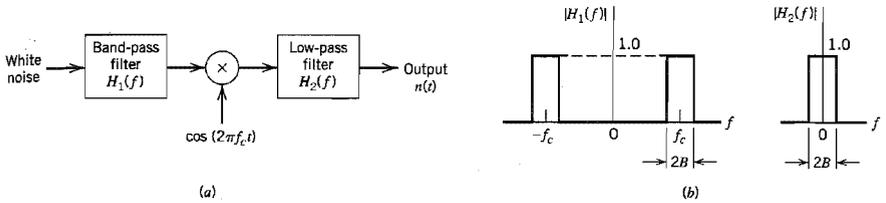


FIGURE P1.26

- 1.27 Let $X(t)$ be a stationary process with zero mean, autocorrelation function $R_X(\tau)$, and power spectral density $S_X(f)$. We are required to find a linear filter with impulse response $h(t)$, such that the filter output has the same statistical characteristics as $X(t)$ when the input is white noise of power spectral density $N_0/2$.
- Determine the condition which the impulse response $h(t)$ must satisfy to achieve this requirement.
 - What is the corresponding condition on the frequency response $H(f)$ of the filter?

Narrowband Noise

- 1.28 In the noise analyzer of Figure 1.19a, the low-pass filters are ideal with a bandwidth equal to one-half that of the narrowband noise $n(t)$ applied to the input. Using this scheme, derive the following results:
- Equation (1.101), defining the power spectral densities of the in-phase noise component $n_I(t)$ and quadrature noise component $n_Q(t)$ in terms of the power spectral density of $n(t)$.
 - Equation (1.102), defining the cross-spectral densities of $n_I(t)$ and $n_Q(t)$.
- 1.29 Assume that the narrowband noise $n(t)$ is Gaussian and its power spectral density $S_N(f)$ is symmetric about the midband frequency f_c . Show that the in-phase and quadrature components of $n(t)$ are statistically independent.
- 1.30 The power spectral density of a narrowband noise $n(t)$ is as shown in Figure P1.30. The carrier frequency is 5 Hz.
- Find the power spectral densities of the in-phase and quadrature components of $n(t)$.
 - Find their cross-spectral densities.

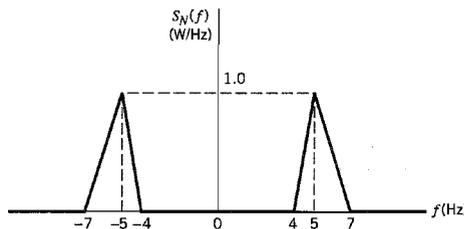


FIGURE P1.30

- 1.31 Consider a Gaussian noise $n(t)$ with zero mean and the power spectral density $S_N(f)$ shown in Figure P1.31.
- Find the probability density function of the envelope of $n(t)$.
 - What are the mean and variance of this envelope?

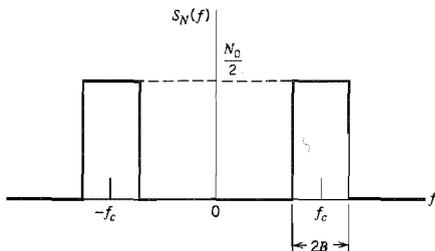


FIGURE P1.31

Computer Experiments

- 1.32 In this computer experiment we study the statistical characterization of a random process $X(t)$ defined by

$$X(t) = A \cos(2\pi f_c t + \Theta) + W(t)$$

where the phase Θ of the sinusoidal component is a uniformly distributed random variable over the interval $[-\pi, \pi]$, and $W(t)$ is a white Gaussian noise component of zero mean and power spectral density $N_0/2$. The two components of $X(t)$ are statistically independent; hence the autocorrelation function of $X(t)$ is

$$R_X(\tau) = \frac{A^2}{2} \cos(2\pi f_c \tau) + \frac{N_0}{2} \delta(\tau)$$

This equation shows that for $|\tau| > 0$ the autocorrelation function $R_X(\tau)$ has the same sinusoidal waveform as the signal component of $X(t)$.

The purpose of this computer experiment is to perform the computation of $R_X(\tau)$ using two different methods:

- Ensemble averaging.** Generate $M = 50$ randomly picked realizations of the process $X(t)$. Hence compute the product $x(t + \tau)x(t)$ for some fixed time t , where $x(t)$ is a realization of $X(t)$. Repeat the computation of $x(t + \tau)x(t)$ for the M realizations of $X(t)$, and thereby compute the average of these computations over M . Repeat this sequence of computations for different values of τ .
- Time averaging.** Compute the time-averaged autocorrelation function

$$R_x(\tau, T) = \frac{1}{2T} \int_{-T}^T x(t + \tau)x(t) dt$$

where $x(t)$ is a particular realization of $X(t)$, and $2T$ is the total observation interval. For this computation, use the Fourier-transform pair:

$$R_x(\tau, T) \rightleftharpoons \frac{1}{2T} |X_T(f)|^2$$

where $|X_T(f)|^2/2T$ is the periodogram of the process $X(t)$. Specifically, compute the Fourier transform $X_T(f)$ of the time-windowed function

$$x_T(t) = \begin{cases} x(t), & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Hence compute the inverse Fourier transform of $|X_T(f)|^2/2T$.

Compare the results of your computation of $R_X(\tau)$ using these two approaches.

- 1.33 In this computer experiment we continue the study of the multipath channel described in Section 1.14. Specifically, consider the situation where the received signal includes a line-of-sight component, as shown by

$$X(t) = \sum_{k=1}^N A_k \cos(2\pi f_c t + \Theta_k) + a \cos(2\pi f_c t)$$

where $a \cos(2\pi f_c t)$ is the directly received component. Following the material presented in Section 1.14, compute the envelope of $X(t)$ for $N = 10,000$, and $a = 0, 1, 2, 3, 5$. Compare your results with the Rician distribution studied in Section 1.13.

CONTINUOUS-WAVE MODULATION

In this chapter we study continuous-wave modulation, which is basic to the operation of analog communication systems. The chapter is divided into two related parts. In the first part we study the time-domain and frequency-domain descriptions of two basic families of continuous-wave modulation:

- ▶ *Amplitude modulation, in which the amplitude of a sinusoidal carrier is varied in accordance with an incoming message signal.*
- ▶ *Angle modulation, in which the instantaneous frequency or phase of the sinusoidal carrier is varied in accordance with the message signal.*

The second part of the chapter focuses on the effects of channel noise on the performance of the receivers pertaining to these modulation schemes.

Advantages and disadvantages of the different methods of continuous-wave modulation are highlighted in light of the material presented herein.

2.1 Introduction

The purpose of a communication system is to transmit *information-bearing signals* through a communication channel separating the transmitter from the receiver. Information-bearing signals are also referred to as *baseband signals*. The term *baseband* is used to designate the band of frequencies representing the original signal as delivered by a source of information. The proper use of the communication channel requires a shift of the range of baseband frequencies into other frequency ranges suitable for transmission, and a corresponding shift back to the original frequency range after reception. For example, a radio system must operate with frequencies of 30 kHz and upward, whereas the baseband signal usually contains frequencies in the audio frequency range, and so some form of frequency-band shifting must be used for the system to operate satisfactorily. A shift of the range of frequencies in a signal is accomplished by using *modulation*, which is defined as *the process by which some characteristic of a carrier is varied in accordance with a modulating wave (signal)*. A common form of the carrier is a *sinusoidal wave*, in which case we speak of a *continuous-wave modulation*¹ process. The baseband signal is referred to as the *modulating wave*, and the result of the modulation process is referred to as the *modulated wave*. Modulation is performed at the transmitting end of the communication system. At the receiving end of the system, we usually require the original baseband signal to be restored. This is accomplished by using a process known as *demodulation*, which is the reverse of the modulation process.

In basic signal-processing terms, we thus find that the transmitter of an analog communication system consists of a modulator and the receiver consists of a demodulator, as

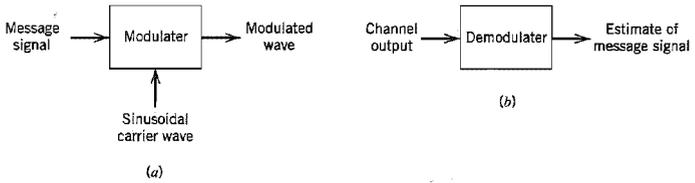


FIGURE 2.1 Components of a continuous-wave modulation system: (a) transmitter, and (b) receiver.

depicted in Figure 2.1. In addition to the signal received from the transmitter, the receiver input includes channel noise. The degradation in receiver performance due to channel noise is determined by the type of modulation used.

In this chapter we study two families of continuous-wave (CW) modulation systems, namely, *amplitude modulation* and *angle modulation*. In amplitude modulation, the amplitude of the sinusoidal carrier wave is varied in accordance with the baseband signal. In angle modulation, the angle of the sinusoidal carrier wave is varied in accordance with the baseband signal. Figure 2.2 displays the waveforms of amplitude-modulated and angle-modulated signals for the case of sinusoidal modulation. Parts (a) and (b) of the figure show the sinusoidal carrier and modulating waves, respectively. Parts (c) and (d) show the

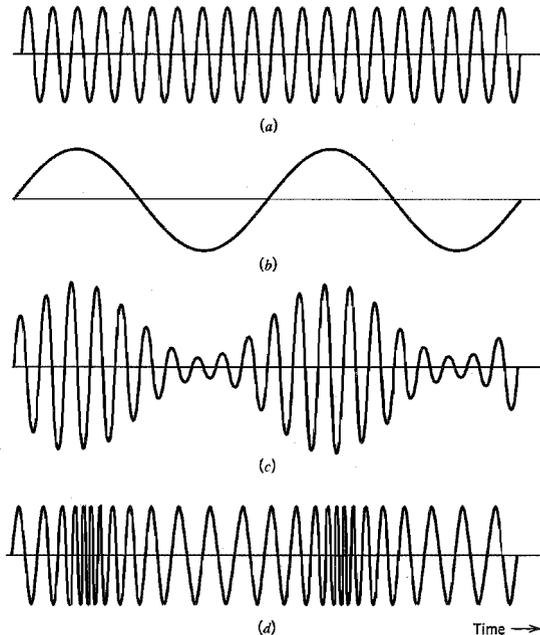


FIGURE 2.2 Illustrating AM and FM signals produced by a single tone. (a) Carrier wave. (b) Sinusoidal modulating signal. (c) Amplitude-modulated signal. (d) Frequency-modulated signal.

corresponding amplitude-modulated and frequency-modulated waves, respectively; frequency modulation is a form of angle modulation. This figure clearly illustrates the basic differences between amplitude modulation and angle modulation, which are discussed in what follows.

2.2 Amplitude Modulation

Consider a *sinusoidal carrier wave* $c(t)$ defined by

$$c(t) = A_c \cos(2\pi f_c t) \quad (2.1)$$

where A_c is the *carrier amplitude* and f_c is the *carrier frequency*. To simplify the exposition without affecting the results obtained and conclusions reached, we have assumed that the phase of the carrier wave is zero in Equation (2.1). Let $m(t)$ denote the baseband signal that carries the specification of the message. The source of carrier wave $c(t)$ is physically independent of the source responsible for generating $m(t)$. *Amplitude modulation (AM)* is defined as a process in which the amplitude of the carrier wave $c(t)$ is varied about a mean value, linearly with the baseband signal $m(t)$. An amplitude-modulated (AM) wave may thus be described, in its most general form, as a function of time as follows:

$$s(t) = A_c[1 + k_a m(t)] \cos(2\pi f_c t) \quad (2.2)$$

where k_a is a constant called the *amplitude sensitivity* of the modulator responsible for the generation of the modulated signal $s(t)$. Typically, the carrier amplitude A_c and the message signal $m(t)$ are measured in volts, in which case k_a is measured in volt^{-1} .

Figure 2.3a shows a baseband signal $m(t)$, and Figures 2.3b and 2.3c show the corresponding AM wave $s(t)$ for two values of amplitude sensitivity k_a and a carrier amplitude $A_c = 1$ volt. We observe that the *envelope* of $s(t)$ has essentially the same shape as the baseband signal $m(t)$ provided that two requirements are satisfied:

1. The amplitude of $k_a m(t)$ is always less than unity, that is,

$$|k_a m(t)| < 1 \quad \text{for all } t \quad (2.3)$$

This condition is illustrated in Figure 2.3b; it ensures that the function $1 + k_a m(t)$ is always positive, and since an envelope is a positive function, we may express the envelope of the AM wave $s(t)$ of Equation (2.2) as $A_c[1 + k_a m(t)]$. When the amplitude sensitivity k_a of the modulator is large enough to make $|k_a m(t)| > 1$ for any t , the carrier wave becomes *overmodulated*, resulting in carrier phase reversals whenever the factor $1 + k_a m(t)$ crosses zero. The modulated wave then exhibits *envelope distortion*, as in Figure 2.3c. It is therefore apparent that by avoiding overmodulation, a one-to-one relationship is maintained between the envelope of the AM wave and the modulating wave for all values of time—a useful feature, as we shall see later on. The absolute maximum value of $k_a m(t)$ multiplied by 100 is referred to as the *percentage modulation*.

2. The carrier frequency f_c is much greater than the highest frequency component W of the message signal $m(t)$, that is

$$f_c \gg W \quad (2.4)$$

We call W the *message bandwidth*. If the condition of Equation (2.4) is not satisfied, an envelope cannot be visualized (and therefore detected) satisfactorily.

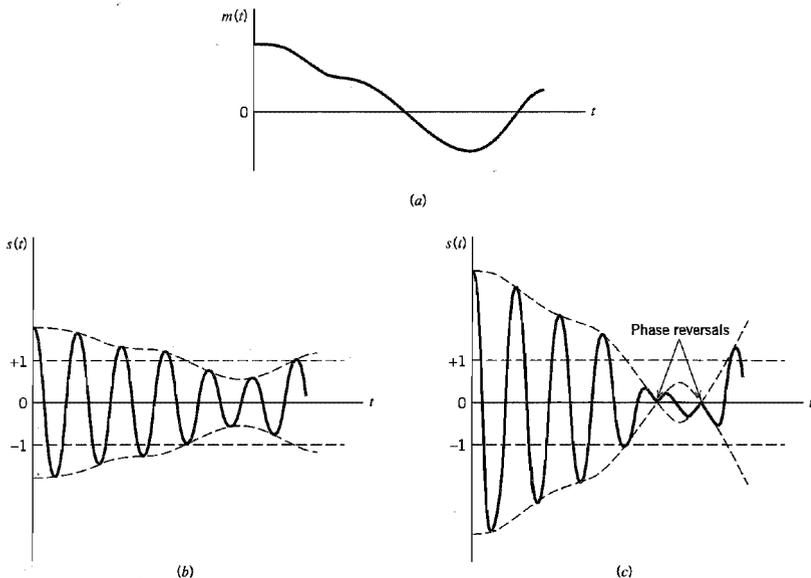


FIGURE 2.3 Illustrating the amplitude modulation process. (a) Baseband signal $m(t)$. (b) AM wave for $|k_a m(t)| < 1$ for all t . (c) AM wave for $|k_a m(t)| > 1$ for some t .

From Equation (2.2), we find that the Fourier transform of the AM wave $s(t)$ is given by

$$S(f) = \frac{A_c}{2} [\delta(f - f_c) + \delta(f + f_c)] + \frac{k_a A_c}{2} [M(f - f_c) + M(f + f_c)] \quad (2.5)$$

Suppose that the baseband signal $m(t)$ is band-limited to the interval $-\mathcal{W} \leq f \leq \mathcal{W}$, as in Figure 2.4a. The shape of the spectrum shown in this figure is intended for the purpose of illustration only. We find from Equation (2.5) that the spectrum $S(f)$ of the AM wave is as shown in Figure 2.4b for the case when $f_c > \mathcal{W}$. This spectrum consists of two delta functions weighted by the factor $A_c/2$ and occurring at $\pm f_c$, and two versions of the baseband spectrum translated in frequency by $\pm f_c$ and scaled in amplitude by $k_a A_c/2$. From the spectrum of Figure 2.4b, we note the following:

1. As a result of the modulation process, the spectrum of the message signal $m(t)$ for negative frequencies extending from $-\mathcal{W}$ to 0 becomes completely visible for positive (i.e., measurable) frequencies, provided that the carrier frequency satisfies the condition $f_c > \mathcal{W}$; herein lies the importance of the idea of “negative” frequencies.
2. For positive frequencies, the portion of the spectrum of an AM wave lying above the carrier frequency f_c is referred to as the *upper sideband*, whereas the symmetric portion below f_c is referred to as the *lower sideband*. For negative frequencies, the upper sideband is represented by the portion of the spectrum below $-f_c$ and the lower sideband by the portion above $-f_c$. The condition $f_c > \mathcal{W}$ ensures that the sidebands do not overlap.

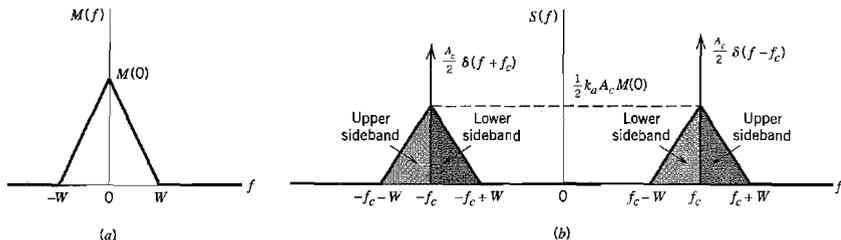


FIGURE 2.4 (a) Spectrum of baseband signal. (b) Spectrum of AM wave.

3. For positive frequencies, the highest frequency component of the AM wave equals $f_c + W$, and the lowest frequency component equals $f_c - W$. The difference between these two frequencies defines the *transmission bandwidth* B_T for an AM wave, which is exactly twice the message bandwidth W , that is,

$$B_T = 2W \quad (2.6)$$

■ VIRTUES AND LIMITATIONS OF AMPLITUDE MODULATION

Amplitude modulation is the oldest method of performing modulation. Its greatest virtue is the simplicity of implementation:

- ▶ In the transmitter, amplitude modulation is accomplished using a nonlinear device. For example, in the *switching modulator* discussed in Problem 2.3, the combined sum of the message signal and carrier wave is applied to a diode, with the carrier amplitude being large enough to swing across the characteristic curve of the diode. Fourier analysis of the voltage developed across a resistive load reveals the generation of an AM component, which may be extracted by means of a band-pass filter.
- ▶ In the receiver, amplitude demodulation is also accomplished using a nonlinear device. For example, we may use a simple and yet highly effective circuit known as the *envelope detector*, which is discussed in Problem 2.5. The circuit consists of a diode connected in series with the parallel combination of a capacitor and load resistor. Some version of this circuit is found in most commercial AM radio receivers. Provided that the carrier frequency is high enough and the percentage modulation is less than 100 percent, the demodulator output developed across the load resistor is nearly the same as the envelope of the incoming AM wave, hence the name “envelope detector.”

Recall, however, that transmitted power and channel bandwidth are our two primary communication resources, and they should be used efficiently. In this context, we find that the standard form of amplitude modulation defined in Equation (2.2) suffers from two major limitations:

1. *Amplitude modulation is wasteful of power.* The carrier wave $c(t)$ is completely independent of the information-bearing signal $m(t)$. The transmission of the carrier wave therefore represents a waste of power, which means that in amplitude modulation only a fraction of the total transmitted power is actually affected by $m(t)$.

2. *Amplitude modulation is wasteful of bandwidth.* The upper and lower sidebands of an AM wave are uniquely related to each other by virtue of their symmetry about the carrier frequency; hence, given the magnitude and phase spectra of either sideband, we can uniquely determine the other. This means that insofar as the transmission of information is concerned, only one sideband is necessary, and the communication channel therefore needs to provide only the same bandwidth as the baseband signal. In light of this observation, amplitude modulation is wasteful of bandwidth as it requires a transmission bandwidth equal to twice the message bandwidth.

To overcome these limitations, we must make certain modifications: suppress the carrier and modify the sidebands of the AM wave. These modifications naturally result in increased system complexity. In effect, we trade system complexity for improved use of communication resources. The basis of this trade-off is linear modulation, which is discussed in the next section. In a strict sense, full amplitude modulation does not qualify as linear modulation because of the presence of the carrier wave.

2.3 Linear Modulation Schemes

In its most general form, linear modulation is defined by

$$s(t) = s_I(t) \cos(2\pi f_c t) - s_Q(t) \sin(2\pi f_c t) \quad (2.7)$$

where $s_I(t)$ is the *in-phase component* of the modulated wave $s(t)$, and $s_Q(t)$ is its *quadrature component*. Equation (2.7) is recognized as the canonical representation of a narrowband signal, which is discussed in detail in Appendix 2. In linear modulation, both $s_I(t)$ and $s_Q(t)$ are low-pass signals that are linearly related to the message signal $m(t)$.

Indeed, depending on how these two components of $s(t)$ are defined, we may identify three types of linear modulation involving a single message signal:

1. *Double sideband-suppressed carrier (DSB-SC) modulation*, where only the upper and lower sidebands are transmitted.
2. *Single sideband (SSB) modulation*, where only one sideband (the lower sideband or the upper sideband) is transmitted.
3. *Vestigial sideband (VSB) modulation*, where only a vestige (i.e., trace) of one of the sidebands and a correspondingly modified version of the other sideband are transmitted.

Table 2.1 presents a summary of the definitions of these three special forms of linear modulation. There are two important points to note from Table 2.1:

1. The in-phase component $s_I(t)$ is solely dependent on the message signal $m(t)$.
2. The quadrature component $s_Q(t)$ is a *filtered* version of $m(t)$. The spectral modification of the modulated wave $s(t)$ is solely due to $s_Q(t)$.

To be more specific, the role of the quadrature component (if present) is merely to interfere with the in-phase component, so as to reduce or eliminate power in one of the sidebands of the modulated signal $s(t)$, depending on how the quadrature component is defined.

TABLE 2.1 Different forms of linear modulation

Type of Modulation	In-Phase Component $s_I(t)$	Quadrature Component $s_Q(t)$	Comments
DSB-SC	$m(t)$	0	$m(t)$ = message signal
SSB: ^a			
(a) Upper sideband transmitted	$\frac{1}{2}m(t)$	$\frac{1}{2}\hat{m}(t)$	$\hat{m}(t)$ = Hilbert transform of $m(t)$
(b) Lower sideband transmitted	$\frac{1}{2}m(t)$	$-\frac{1}{2}\hat{m}(t)$	
VSB:			
(a) Vestige of lower sideband transmitted	$\frac{1}{2}m(t)$	$\frac{1}{2}m'(t)$	$\left\{ \begin{array}{l} m'(t) = \text{output of the filter of} \\ \text{frequency response } H_Q(f) \\ \text{due to } m(t). \\ \text{For the definition of } H_Q(f), \\ \text{see Eq. (2.16)} \end{array} \right.$
(b) Vestige of upper sideband transmitted	$\frac{1}{2}m(t)$	$-\frac{1}{2}m'(t)$	

^aFor the mathematical description of single sideband modulation, see Problem 2.16.

■ **DOUBLE SIDEBAND-SUPPRESSED CARRIER (DSB-SC) MODULATION**

This form of linear modulation is generated by using a *product modulator* that simply multiplies the message signal $m(t)$ by the carrier wave $A_c \cos(2\pi f_c t)$, as illustrated in Figure 2.5a. Specifically, we write

$$s(t) = A_c m(t) \cos(2\pi f_c t) \tag{2.8}$$

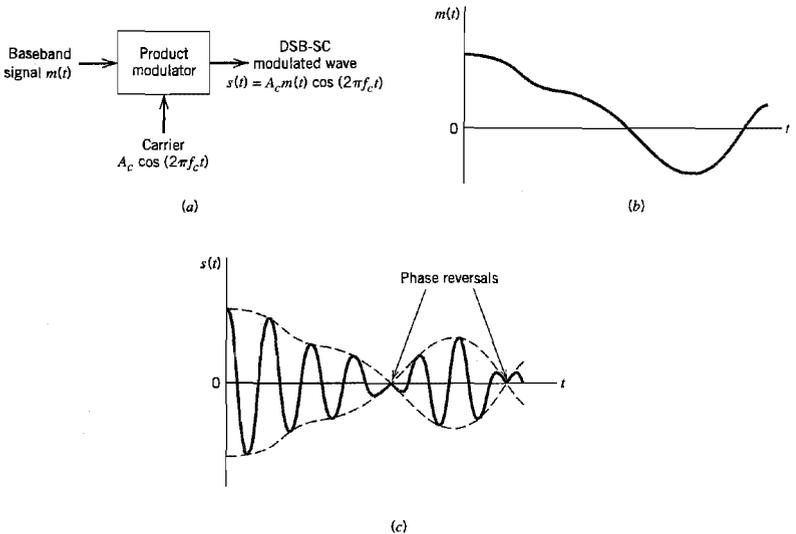


FIGURE 2.5 (a) Block diagram of product modulator. (b) Baseband signal. (c) DSB-SC modulated wave.

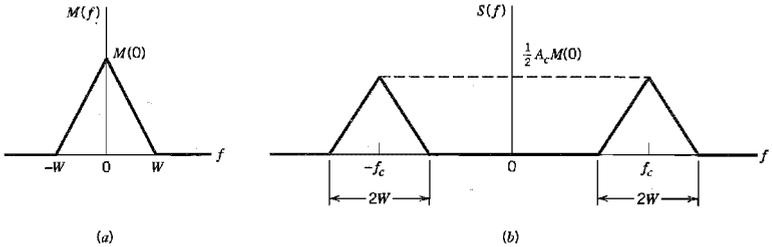


FIGURE 2.6 (a) Spectrum of baseband signal. (b) Spectrum of DSB-SC modulated wave.

Figure 2.5c shows the modulated signal $s(t)$ for the arbitrary message waveform of Figure 2.5b. The modulated signal $s(t)$ undergoes a *phase reversal* whenever the message signal $m(t)$ crosses zero. Consequently, the envelope of a DSB-SC modulated signal is different from the message signal; this is unlike the case of an AM wave that has a percentage modulation less than 100 percent.

From Equation (2.8), the Fourier transform of $s(t)$ is obtained as

$$S(f) = \frac{1}{2} A_c [M(f - f_c) + M(f + f_c)] \quad (2.9)$$

For the case when the baseband signal $m(t)$ is limited to the interval $-W \leq f \leq W$, as in Figure 2.6a, we thus find that the spectrum $S(f)$ of the DSB-SC wave $s(t)$ is as illustrated in Figure 2.6b. Except for a change in scale factor, the modulation process simply *translates* the spectrum of the baseband signal by $\pm f_c$. Of course, the transmission bandwidth required by DSB-SC modulation is the same as that for amplitude modulation, namely, $2W$.

☒ COHERENT DETECTION

The baseband signal $m(t)$ can be uniquely recovered from a DSB-SC wave $s(t)$ by first multiplying $s(t)$ with a locally generated sinusoidal wave and then low-pass filtering the product, as in Figure 2.7. It is assumed that the local oscillator signal is exactly coherent or synchronized, in both frequency and phase, with the carrier wave $c(t)$ used in the product modulator to generate $s(t)$. This method of demodulation is known as *coherent detection* or *synchronous demodulation*.

It is instructive to derive coherent detection as a special case of the more general demodulation process using a local oscillator signal of the same frequency but arbitrary phase difference ϕ , measured with respect to the carrier wave $c(t)$. Thus, denoting the local

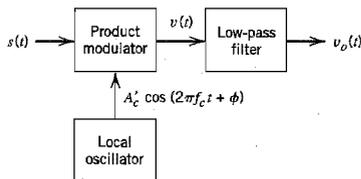


FIGURE 2.7 Coherent detector for demodulating DSB-SC modulated wave.

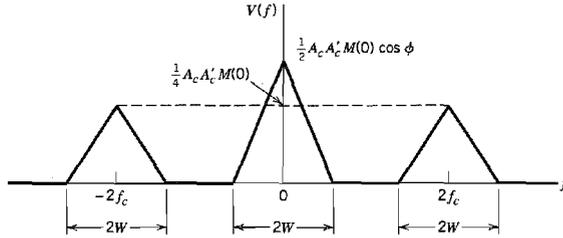


FIGURE 2.8 Illustrating the spectrum of a product modulator output with a DSB-SC modulated wave as input.

oscillator signal by $A'_c \cos(2\pi f_c t + \phi)$, and using Equation (2.8) for the DSB-SC wave $s(t)$, we find that the product modulator output in Figure 2.7 is

$$\begin{aligned} v(t) &= A'_c \cos(2\pi f_c t + \phi) s(t) \\ &= A_c A'_c \cos(2\pi f_c t) \cos(2\pi f_c t + \phi) m(t) \\ &= \frac{1}{2} A_c A'_c \cos(4\pi f_c t + \phi) m(t) + \frac{1}{2} A_c A'_c \cos \phi m(t) \end{aligned} \quad (2.10)$$

The first term in Equation (2.10) represents a DSB-SC modulated signal with a carrier frequency $2f_c$, whereas the second term is proportional to the baseband signal $m(t)$. This is further illustrated by the spectrum $V(f)$ shown in Figure 2.8, where it is assumed that the baseband signal $m(t)$ is limited to the interval $-W \leq f \leq W$. It is therefore apparent that the first term in Equation (2.10) is removed by the low-pass filter in Figure 2.7, provided that the cut-off frequency of this filter is greater than W but less than $2f_c - W$. This requirement is satisfied by choosing $f_c > W$. At the filter output we then obtain a signal given by

$$v_o(t) = \frac{1}{2} A_c A'_c \cos \phi m(t) \quad (2.11)$$

The demodulated signal $v_o(t)$ is therefore proportional to $m(t)$ when the phase error ϕ is a constant. The amplitude of this demodulated signal is maximum when $\phi = 0$, and it is minimum (zero) when $\phi = \pm\pi/2$. The zero demodulated signal, which occurs for $\phi = \pm\pi/2$, represents the *quadrature null effect* of the coherent detector. Thus the phase error ϕ in the local oscillator causes the detector output to be attenuated by a factor equal to $\cos \phi$. As long as the phase error ϕ is constant, the detector provides an undistorted version of the original baseband signal $m(t)$. In practice, however, we usually find that the phase error ϕ varies randomly with time, due to random variations in the communication channel. The result is that at the detector output, the multiplying factor $\cos \phi$ also varies randomly with time, which is obviously undesirable. Therefore, provision must be made in the system to maintain the local oscillator in the receiver in perfect synchronism, in both frequency and phase, with the carrier wave used to generate the DSB-SC modulated signal in the transmitter. The resulting system complexity is the price that must be paid for suppressing the carrier wave to save transmitter power.

■ COSTAS RECEIVER

One method of obtaining a practical synchronous receiver system, suitable for demodulating DSB-SC waves, is to use the *Costas receiver*² shown in Figure 2.9. This receiver

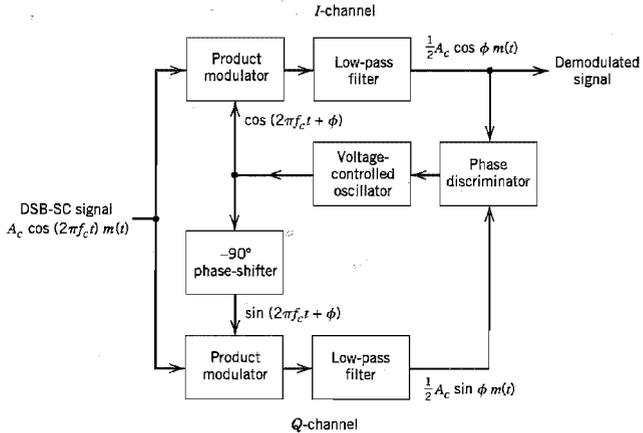


FIGURE 2.9 Costas receiver.

consists of two coherent detectors supplied with the same input signal, namely, the incoming DSB-SC wave $A_c \cos(2\pi f_c t)m(t)$, but with individual local oscillator signals that are in phase quadrature with respect to each other. The frequency of the local oscillator is adjusted to be the same as the carrier frequency f_c , which is assumed known *a priori*. The detector in the upper path is referred to as the *in-phase coherent detector* or *I-channel*, and that in the lower path is referred to as the *quadrature-phase coherent detector* or *Q-channel*. These two detectors are coupled together to form a *negative feedback* system designed in such a way as to maintain the local oscillator synchronous with the carrier wave.

To understand the operation of this receiver, suppose that the local oscillator signal is of the same phase as the carrier wave $A_c \cos(2\pi f_c t)$ used to generate the incoming DSB-SC wave. Under these conditions, we find that the *I-channel* output contains the desired demodulated signal $m(t)$, whereas the *Q-channel* output is zero due to the quadrature null effect of the *Q-channel*. Suppose next that the local oscillator phase drifts from its proper value by a small angle ϕ radians. The *I-channel* output will remain essentially unchanged, but there will now be some signal appearing at the *Q-channel* output, which is proportional to $\sin \phi \approx \phi$ for small ϕ . This *Q-channel* output will have the same polarity as the *I-channel* output for one direction of local oscillator phase drift and opposite polarity for the opposite direction of local oscillator phase drift. Thus, by combining the *I-* and *Q-channel* outputs in a *phase discriminator* (which consists of a multiplier followed by a low-pass filter), as shown in Figure 2.9, a DC control signal is obtained that automatically corrects for local phase errors in the *voltage-controlled oscillator*.

It is apparent that phase control in the Costas receiver ceases with modulation and that phase-lock has to be reestablished with the reappearance of modulation. This is not a serious problem when receiving voice transmission, because the lock-up process normally occurs so rapidly that no distortion is perceptible.

▣ **QUADRATURE-CARRIER MULTIPLEXING**

The quadrature null effect of the coherent detector may also be put to good use in the construction of the so-called *quadrature-carrier multiplexing* or *quadrature-amplitude*

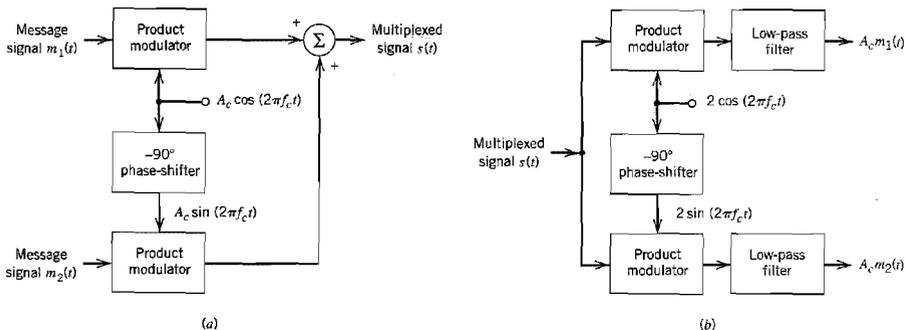


FIGURE 2.10 Quadrature-carrier multiplexing system. (a) Transmitter. (b) Receiver.

modulation (QAM). This scheme enables two DSB-SC modulated waves (resulting from the application of two physically *independent* message signals) to occupy the same channel bandwidth, and yet it allows for the separation of the two message signals at the receiver output. It is therefore a *bandwidth-conservation scheme*.

A block diagram of the quadrature-carrier multiplexing system is shown in Figure 2.10. The transmitter part of the system, shown in Figure 2.10a, involves the use of two separate product modulators that are supplied with two carrier waves of the same frequency but differing in phase by -90 degrees. The transmitted signal $s(t)$ consists of the sum of these two product modulator outputs, as shown by

$$s(t) = A_c m_1(t) \cos(2\pi f_c t) + A_c m_2(t) \sin(2\pi f_c t) \quad (2.12)$$

where $m_1(t)$ and $m_2(t)$ denote the two different message signals applied to the product modulators. Thus $s(t)$ occupies a channel bandwidth of $2W$ centered at the carrier frequency f_c , where W is the message bandwidth of $m_1(t)$ or $m_2(t)$. According to Equation (2.12), we may view $A_c m_1(t)$ as the in-phase component of the multiplexed band-pass signal $s(t)$ and $-A_c m_2(t)$ as its quadrature component.

The receiver part of the system is shown in Figure 2.10b. The multiplexed signal $s(t)$ is applied simultaneously to two separate coherent detectors that are supplied with two local carriers of the same frequency but differing in phase by -90 degrees. The output of the top detector is $A_c m_1(t)$, whereas the output of the bottom detector is $A_c m_2(t)$. For the system to operate satisfactorily, it is important to maintain the correct phase and frequency relationships between the local oscillators used in the transmitter and receiver parts of the system.

To maintain this synchronization, we may send a *pilot signal* outside the passband of the modulated signal. In this method, the pilot signal typically consists of a low-power sinusoidal tone whose frequency and phase are related to the carrier wave $c(t)$; at the receiver, the pilot signal is extracted by means of a suitably tuned circuit and then translated to the correct frequency for use in the coherent detector.

■ SINGLE-SIDEBAND MODULATION

In single-sideband modulation, only the upper or lower sideband is transmitted. We may generate such a modulated wave by using the *frequency-discrimination method* that consists of two stages:

- ▶ The first stage is a product modulator, which generates a DSB-SC modulated wave.
- ▶ The second stage is a band-pass filter, which is designed to pass one of the sidebands of this modulated wave and suppress the other.

From a practical viewpoint the most severe requirement of SSB generation using the frequency discrimination method arises from the unwanted sideband. The nearest frequency component of the unwanted sideband is separated from the desired sideband by twice the lowest frequency component of the message (modulating) signal. The implication here is that for the generation of an SSB modulated signal to be possible, the message spectrum must have an energy gap centered at the origin, as illustrated in Figure 2.11a. This requirement is naturally satisfied by voice signals, whose energy gap is about 600 Hz wide (i.e., it extends from -300 to $+300$ Hz). Thus, assuming that the upper sideband is retained, the spectrum of the SSB modulated signal is as shown in Figure 2.11b.

In designing the band-pass filter used in the frequency-discriminator for generating a SSB-modulated wave, we must meet the three basic requirements:

- ▶ The desired sideband lies inside the passband of the filter.
- ▶ The unwanted sideband lies inside the stopband of the filter.
- ▶ The filter's transition band, which separates the passband from the stopband, is twice the lowest frequency component of the message signal.

This kind of frequency discrimination usually requires the use of highly selective filters, which can only be realized in practice by means of crystal resonators.

To demodulate a SSB modulated signal $s(t)$, we may use a coherent detector, which multiplies $s(t)$ by a locally generated carrier and then low-pass filters the product. This method of demodulation assumes perfect synchronism between the oscillator in the coherent detector and the oscillator used to supply the carrier wave in the transmitter. This requirement is usually met in one of two ways:

- ▶ A low-power *pilot carrier* is transmitted in addition to the selected sideband.
- ▶ A highly stable oscillator, tuned to the same frequency as the carrier frequency, is used in the receiver.

In the latter method, it is inevitable that there would be some phase error ϕ in the local oscillator output with respect to the carrier wave used to generate the incoming SSB modulated wave. The effect of this phase error is to introduce a *phase distortion* in the demodulated signal, where each frequency component of the original message signal undergoes a constant phase shift ϕ . This phase distortion is tolerable in voice communications,

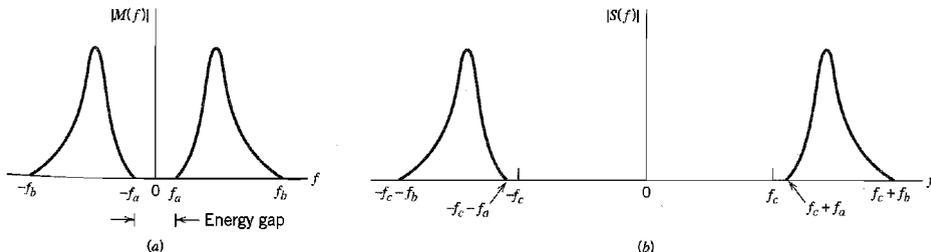


FIGURE 2.11 (a) Spectrum of a message signal $m(t)$ with an energy gap of width $2f_a$ centered on the origin. (b) Spectrum of corresponding SSB signal containing the upper sideband.

because the human ear is relatively insensitive to phase distortion. In particular, the presence of phase distortion gives rise to a *Donald Duck voice effect*. In the transmission of music and video signals, on the other hand, the presence of this form of waveform distortion is utterly unacceptable.

■ VESTIGIAL SIDEBAND MODULATION

In *vestigial sideband (VSB) modulation*, one of the sidebands is partially suppressed and a vestige of the other sideband is transmitted to compensate for that suppression. A popular method for generating a VSB-modulated wave is to use the *frequency discrimination method*. First, we generate a DSB-SC modulated wave and then pass it through a band-pass filter, as shown in Figure 2.12; it is the special design of the band-pass filter that distinguishes VSB modulation from SSB modulation. Assuming that a vestige of the lower sideband is transmitted, the frequency response $H(f)$ of the band-pass filter takes the form shown in Figure 2.13. To simplify matters, only the response for positive frequencies is shown here. This frequency response is normalized, so that at the carrier frequency f_c we have $|H(f_c)| = 1/2$. The important feature to note from Figure 2.13 is that the cutoff portion of the frequency response around the carrier frequency f_c exhibits *odd symmetry*. That is, inside the transition interval $f_c - f_v \leq |f| \leq f_c + f_v$, the following two conditions are satisfied:

1. The sum of the values of the magnitude response $|H(f)|$ at any two frequencies equally displaced above and below f_c is unity.
2. The phase response $\arg(H(f))$ is linear. That is, $H(f)$ satisfies the condition

$$H(f - f_c) + H(f + f_c) = 1 \quad \text{for } -W \leq f \leq W \quad (2.13)$$

Note also that outside the frequency band of interest (i.e., $|f| > f_c + W$), the frequency response $H(f)$ may have an arbitrary specification. Accordingly, the transmission bandwidth of VSB modulation is

$$B_T = W + f_v \quad (2.14)$$

where W is the message bandwidth, and f_v is the width of the vestigial sideband.

According to Table 2.1, the VSB modulated wave is described in the time domain as

$$s(t) = \frac{1}{2} A_c m(t) \cos(2\pi f_c t) \pm \frac{1}{2} A_c m'(t) \sin(2\pi f_c t) \quad (2.15)$$

where the plus sign corresponds to the transmission of a vestige of the upper sideband, and the minus sign corresponds to the transmission of a vestige of the lower sideband. The signal $m'(t)$ in the quadrature component of $s(t)$ is obtained by passing the message signal

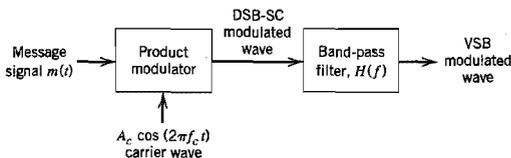


FIGURE 2.12 Filtering scheme for the generation of VSB modulated wave.

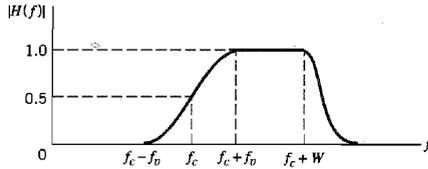


FIGURE 2.13 Magnitude response of VSB filter; only the positive-frequency portion is shown.

$m(t)$ through a filter whose frequency response $H_Q(f)$ satisfies the following requirement (see Problem 2.20):

$$H_Q(f) = j[H(f - f_c) - H(f + f_c)] \quad \text{for } -W \leq f \leq W \quad (2.16)$$

Figure 2.14 displays a plot of the frequency response $H_Q(f)$, scaled by $1/j$. The role of the quadrature component determined by $H_Q(f)$ is to interfere with the in-phase component in Equation (2.15) so as to partially reduce power in one of the sidebands of the modulated wave $s(t)$ and retain simply a vestige of the other sideband, as desired.

It is of interest to note that SSB modulation may be viewed as a special case of VSB modulation. Specifically, when the vestigial sideband is reduced to zero (i.e., we set $f_v = 0$), the modulated wave $s(t)$ of Equation (2.15) takes the limiting form of a single-sideband modulated wave.

■ TELEVISION SIGNALS

A discussion of vestigial sideband modulation would be incomplete without a mention of its role in commercial television (TV) broadcasting. The exact details of the modulation format used to transmit the video signal characterizing a TV system are influenced by two factors:

1. The video signal exhibits a large bandwidth and significant low-frequency content, which suggest the use of vestigial sideband modulation.
2. The circuitry used for demodulation in the receiver should be simple and therefore inexpensive; this suggests the use of envelope detection, which requires the addition of a carrier to the VSB-modulated wave.

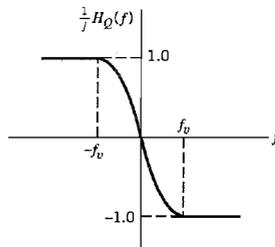


FIGURE 2.14 Frequency response of a filter for producing the quadrature component of the VSB modulated wave.

With regard to point 1, however, it should be stressed that although there is indeed a basic desire to conserve bandwidth, in commercial TV broadcasting the transmitted signal is not quite VSB modulated. The reason is that at the transmitter the power levels are high, with the result that it would be expensive to rigidly control the filtering of sidebands. Instead, a *VSB filter* is inserted in each receiver, where the power levels are low. The overall performance is the same as conventional vestigial-sideband modulation, except for some wasted power and bandwidth. These remarks are illustrated in Figure 2.15. In particular, Figure 2.15a shows the idealized spectrum of a transmitted TV signal. The upper sideband, 25 percent of the lower sideband, and the picture carrier are transmitted. The frequency response of the VSB filter used to do the required spectrum shaping in the receiver is shown in Figure 2.15b.

The channel bandwidth used for TV broadcasting in North America is 6 MHz, as indicated in Figure 2.15b. This channel bandwidth not only accommodates the bandwidth requirement of the VSB modulated video signal but also provides for the accompanying sound signal that modulates a carrier of its own. The values presented on the frequency axis in Figures 2.15a and 2.15b pertain to a specific TV channel. According to this figure, the picture carrier frequency is at 55.25 MHz, and the sound carrier frequency is at 59.75 MHz. Note, however, that the information content of the TV signal lies in a *baseband spectrum* extending from 1.25 MHz below the picture carrier to 4.5 MHz above it.

With regard to point 2, the use of envelope detection (applied to a VSB modulated

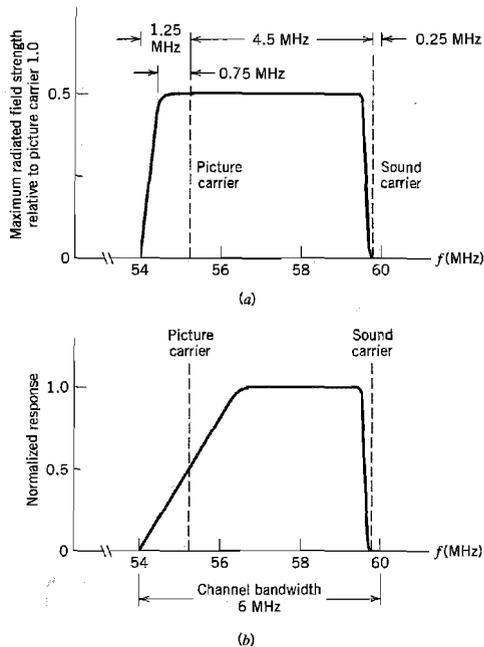


FIGURE 2.15 (a) Idealized magnitude spectrum of a transmitted TV signal. (b) Magnitude response of VSB shaping filter in the receiver.

wave plus carrier) produces *waveform distortion* in the video signal recovered at the detector output. The distortion is produced by the quadrature component of the VSB modulated wave; this issue is discussed next.

The use of the time-domain description given in Equation (2.15) enables the determination of the waveform distortion caused by the envelope detector. Specifically, adding the carrier component $A_c \cos(2\pi f_c t)$ to the VSB-modulated wave of Equation (2.15), the latter being scaled by a factor k_a , modifies the modulated signal applied to the envelope detector input as

$$s(t) = A_c \left[1 + \frac{1}{2} k_a m(t) \right] \cos(2\pi f_c t) \pm \frac{1}{2} k_a A_c m'(t) \sin(2\pi f_c t) \quad (2.17)$$

where the constant k_a determines the percentage modulation. The envelope detector output, denoted by $a(t)$, is therefore

$$\begin{aligned} a(t) &= A_c \left\{ \left[1 + \frac{1}{2} k_a m(t) \right]^2 + \left[\frac{1}{2} k_a m'(t) \right]^2 \right\}^{1/2} \\ &= A_c \left[1 + \frac{1}{2} k_a m(t) \right] \left\{ 1 + \frac{\left[\frac{1}{2} k_a m'(t) \right]^2}{\left[1 + \frac{1}{2} k_a m(t) \right]^2} \right\}^{1/2} \end{aligned} \quad (2.18)$$

Equation (2.18) indicates that the distortion is contributed by $m'(t)$, which is responsible for the quadrature component of the incoming VSB-modulated signal. This distortion can be reduced by using two methods:

- ▶ Reducing the percentage modulation to reduce the amplitude sensitivity k_a .
- ▶ Increasing the width of the vestigial sideband to reduce $m'(t)$.

Both methods are in fact used in practice. In commercial TV broadcasting, the width of the vestigial sideband (which is about 0.75 MHz, or one-sixth of a full sideband) is determined to keep the distortion due to $m'(t)$ within tolerable limits when the percentage modulation is nearly 100.

2.4 Frequency Translation

The basic operation involved in single-sideband modulation is in fact a form of *frequency translation*, which is why single-sideband modulation is sometimes referred to as *frequency changing*, *mixing*, or *heterodyning*. This operation is clearly illustrated in the spectrum of the signal shown in Figure 2.11b compared to that of the original message signal in Figure 2.11a. Specifically, we see that a message spectrum occupying the band from f_a to f_b for positive frequencies in Figure 2.11a is shifted upward by an amount equal to the carrier frequency f_c in Figure 2.11b, and the message spectrum for negative frequencies is translated downward in a symmetric fashion.

The idea of frequency translation described herein may be generalized as follows. Suppose that we have a modulated wave $s_1(t)$ whose spectrum is centered on a carrier frequency f_1 , and the requirement is to translate it upward in frequency such that its carrier frequency is changed from f_1 to a new value f_2 . This requirement may be accomplished using the *mixer* shown in Figure 2.16. Specifically, the *mixer* is a device that consists of a product modulator followed by a band-pass filter.

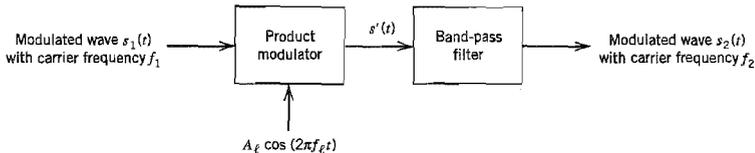


FIGURE 2.16 Block diagram of mixer.

To explain the action of the mixer, consider the situation depicted in Figure 2.17, where, for the purpose of illustration, it is assumed that the mixer input $s_1(t)$ is an AM signal with carrier frequency f_1 and bandwidth $2W$. Part (a) of Figure 2.17 displays the AM spectrum $S_1(f)$ assuming that $f_1 > W$. Part (b) of the figure displays the spectrum $S'(f)$ of the resulting signal $s'(t)$ at the product modulator output.

The signal $s'(t)$ may be viewed as the sum of two modulated components: one component represented by the shaded spectrum in Figure 2.17b, and the other component represented by the unshaded spectrum in this figure. Depending on whether the incoming carrier frequency f_1 is translated upward or downward, we may identify two different situations, as described here:

Up conversion. In this case the translated carrier frequency f_2 is greater than the incoming carrier frequency f_1 , and the required local oscillator frequency f_i is therefore defined by

$$f_2 = f_1 + f_i$$

or

$$f_i = f_2 - f_1$$

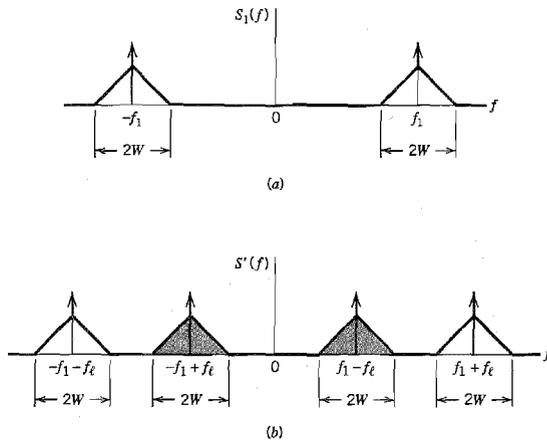


FIGURE 2.17 (a) Spectrum of modulated signal $s_1(t)$ at the mixer input. (b) Spectrum of the corresponding signal $s'(t)$ at the output of the product modulator in the mixer.

The unshaded part of the spectrum in Figure 2.17*b* defines the wanted modulated signal $s_2(t)$, and the shaded part of this spectrum defines the *image signal* associated with $s_2(t)$. For obvious reasons, the mixer in this case is referred to as a *frequency-up converter*.

Down conversion. In this second case the translated carrier frequency f_2 is smaller than the incoming carrier frequency f_1 , and the required oscillator frequency f_i is therefore defined by

$$f_2 = f_1 - f_i$$

or

$$f_i = f_1 - f_2$$

The picture we have this time is the reverse of that pertaining to up conversion. In particular, the shaded part of the spectrum in Figure 2.17*b* defines the wanted modulated signal $s_2(t)$, and the unshaded part of this spectrum defines the associated image signal. The mixer is now referred to as a *frequency-down converter*. Note that in this case the translated carrier frequency f_2 has to be larger than \mathbb{W} (i.e., one half of the bandwidth of the modulated signal) to avoid sideband overlap.

The purpose of the band-pass filter in the mixer of Figure 2.16 is to pass the wanted modulated signal $s_2(t)$ and eliminate the associated image signal. This objective is achieved by aligning the midband frequency of the filter with the translated carrier frequency f_2 and assigning it a bandwidth equal to that of the incoming modulated signal $s_1(t)$.

It is important to note that mixing is a *linear* operation. Accordingly, the relation of the sidebands of the incoming modulated wave to the carrier is completely preserved at the mixer output.

2.5 Frequency-Division Multiplexing

Another important signal processing operation is *multiplexing*, whereby a number of independent signals can be combined into a composite signal suitable for transmission over a common channel. Voice frequencies transmitted over telephone systems, for example, range from 300 to 3100 Hz. To transmit a number of these signals over the same channel, the signals must be kept apart so that they do not interfere with each other, and thus they can be separated at the receiving end. This is accomplished by separating the signals either in frequency or in time. The technique of separating the signals in frequency is referred to as *frequency-division multiplexing* (FDM), whereas the technique of separating the signals in time is called *time-division multiplexing* (TDM). In this section, we discuss FDM systems, and TDM systems are discussed in Chapter 3.

A block diagram of an FDM system is shown in Figure 2.18. The incoming message signals are assumed to be of the low-pass type, but their spectra do not necessarily have nonzero values all the way down to zero frequency. Following each signal input, we have shown a low-pass filter, which is designed to remove high-frequency components that do not contribute significantly to signal representation but are capable of disturbing other message signals that share the common channel. These low-pass filters may be omitted only if the input signals are sufficiently band limited initially. The filtered signals are applied

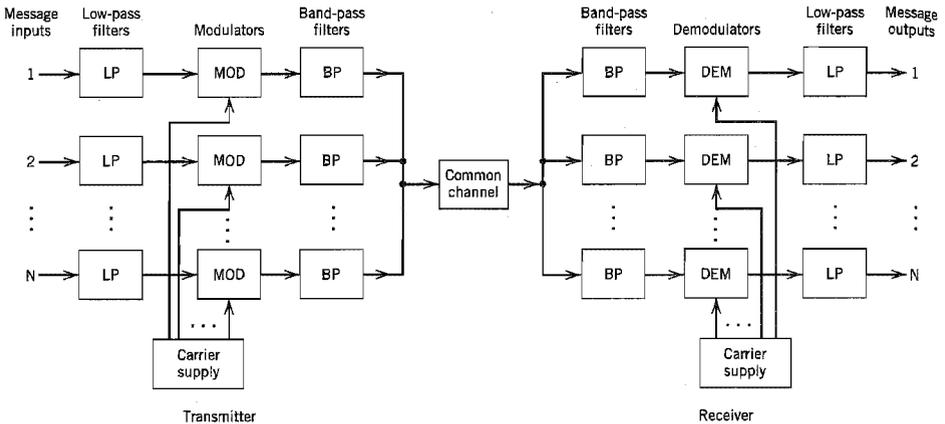


FIGURE 2.18 Block diagram of FDM system.

to modulators that shift the frequency ranges of the signals so as to occupy mutually exclusive frequency intervals. The necessary carrier frequencies needed to perform these frequency translations are obtained from a carrier supply. For the modulation, we may use any one of the methods described in previous sections of this chapter. However, the most widely used method of modulation in frequency-division multiplexing is single sideband modulation, which, in the case of voice signals, requires a bandwidth that is approximately equal to that of the original voice signal. In practice, each voice input is usually assigned a bandwidth of 4 kHz. The band-pass filters following the modulators are used to restrict the band of each modulated wave to its prescribed range. The resulting band-pass filter outputs are next combined in parallel to form the input to the common channel. At the receiving terminal, a bank of band-pass filters, with their inputs connected in parallel, is used to separate the message signals on a frequency-occupancy basis. Finally, the original message signals are recovered by individual demodulators. Note that the FDM system shown in Figure 2.18 operates in only one direction. To provide for two-way transmission, as in telephony, for example, we have to completely duplicate the multiplexing facilities, with the components connected in reverse order and with the signal waves proceeding from right to left.

EXAMPLE 2.1

The practical implementation of an FDM system usually involves many steps of modulation and demodulation, as illustrated in Figure 2.19. The first multiplexing step combines 12 voice inputs into a *basic group*, which is formed by having the n th input modulate a carrier at frequency $f_c = 60 + 4n$ kHz, where $n = 1, 2, \dots, 12$. The lower sidebands are then selected by band-pass filtering and combined to form a group of 12 lower sidebands (one for each voice input). Thus the basic group occupies the frequency band 60 to 108 kHz. The next step in the FDM hierarchy involves the combination of five basic groups into a *supergroup*. This is accomplished by using the n th group to modulate a carrier of frequency $f_c = 372 + 48n$ kHz, where $n = 1, 2, \dots, 5$. Here again the lower sidebands are selected by filtering and then

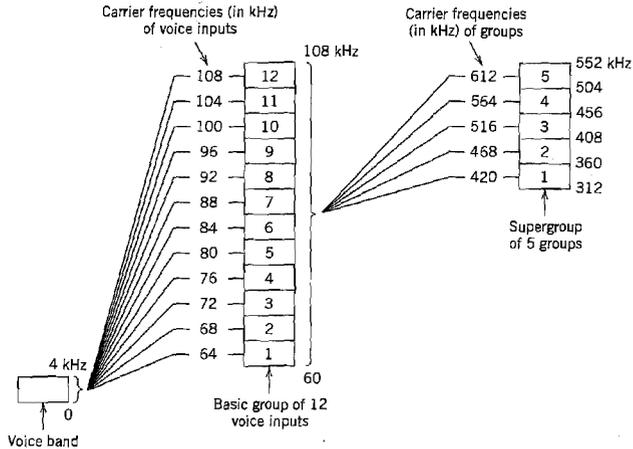


FIGURE 2.19 Illustrating the modulation steps in an FDM system.

combined to form a supergroup occupying the band 312 to 552 kHz. Thus a supergroup is designed to accommodate 60 independent voice inputs. The reason for forming the supergroup in this manner is that economical filters of the required characteristics are available only over a limited frequency range. In a similar manner, supergroups are combined into *mastergroups*, and mastergroups are combined into *very large groups*.

2.6 Angle Modulation

In the previous sections of this chapter, we investigated the effect of slowly varying the amplitude of a sinusoidal carrier wave in accordance with the baseband (information-carrying) signal. There is another way of modulating a sinusoidal carrier wave, namely, *angle modulation* in which the angle of the carrier wave is varied according to the baseband signal. In this method of modulation, the amplitude of the carrier wave is maintained constant. An important feature of angle modulation is that it can provide better discrimination against noise and interference than amplitude modulation. As will be shown later in Section 2.7, however, this improvement in performance is achieved at the expense of increased transmission bandwidth; that is, angle modulation provides us with a practical means of exchanging channel bandwidth for improved noise performance. Such a trade-off is *not* possible with amplitude modulation, regardless of its form.

■ BASIC DEFINITIONS

Let $\theta_i(t)$ denote the *angle* of a modulated sinusoidal carrier, assumed to be a function of the message signal. We express the resulting *angle-modulated wave* as

$$s(t) = A_c \cos[\theta_i(t)] \quad (2.19)$$

where A_c is the carrier amplitude. A complete oscillation occurs whenever $\theta_i(t)$ changes by 2π radians. If $\theta_i(t)$ increases monotonically with time, the average frequency in Hertz, over an interval from t to $t + \Delta t$, is given by

$$f_{\Delta t}(t) = \frac{\theta_i(t + \Delta t) - \theta_i(t)}{2\pi \Delta t} \quad (2.20)$$

We may thus define the *instantaneous frequency* of the angle-modulated signal $s(t)$ as follows:

$$\begin{aligned} f_i(t) &= \lim_{\Delta t \rightarrow 0} f_{\Delta t}(t) \\ &= \lim_{\Delta t \rightarrow 0} \left[\frac{\theta_i(t + \Delta t) - \theta_i(t)}{2\pi \Delta t} \right] \\ &= \frac{1}{2\pi} \frac{d\theta_i(t)}{dt} \end{aligned} \quad (2.21)$$

Thus, according to Equation (2.19), we may interpret the angle-modulated signal $s(t)$ as a rotating phasor of length A_c and angle $\theta_i(t)$. The angular velocity of such a phasor is $d\theta_i(t)/dt$ measured in radians per second, in accordance with Equation (2.21). In the simple case of an unmodulated carrier, the angle $\theta_i(t)$ is

$$\theta_i(t) = 2\pi f_c t + \phi_c$$

and the corresponding phasor rotates with a constant angular velocity equal to $2\pi f_c$. The constant ϕ_c is the value of $\theta_i(t)$ at $t = 0$.

There are an infinite number of ways in which the angle $\theta_i(t)$ may be varied in some manner with the message (baseband) signal. However, we shall consider only two commonly used methods, phase modulation and frequency modulation, defined as follows:

1. *Phase modulation (PM) is that form of angle modulation in which the angle $\theta_i(t)$ is varied linearly with the message signal $m(t)$, as shown by*

$$\theta_i(t) = 2\pi f_c t + k_p m(t) \quad (2.22)$$

The term $2\pi f_c t$ represents the angle of the *unmodulated* carrier; and the constant k_p represents the *phase sensitivity* of the modulator, expressed in radians per volt on the assumption that $m(t)$ is a voltage waveform. For convenience, we have assumed in Equation (2.22) that the angle of the unmodulated carrier is zero at $t = 0$. The phase-modulated signal $s(t)$ is thus described in the time domain by

$$s(t) = A_c \cos[2\pi f_c t + k_p m(t)] \quad (2.23)$$

2. *Frequency modulation (FM) is that form of angle modulation in which the instantaneous frequency $f_i(t)$ is varied linearly with the message signal $m(t)$, as shown by*

$$f_i(t) = f_c + k_f m(t) \quad (2.24)$$

The term f_c represents the frequency of the unmodulated carrier, and the constant k_f represents the *frequency sensitivity* of the modulator, expressed in Hertz per volt

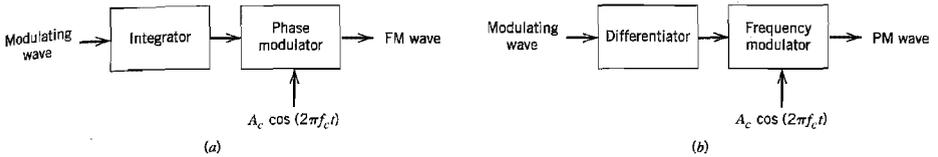


FIGURE 2.20 Illustrating the relationship between frequency modulation and phase modulation. (a) Scheme for generating an FM wave by using a phase modulator. (b) Scheme for generating a PM wave by using a frequency modulator.

on the assumption that $m(t)$ is a voltage waveform. Integrating Equation (2.24) with respect to time and multiplying the result by 2π , we get

$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \quad (2.25)$$

where, for convenience, we have assumed that the angle of the unmodulated carrier wave is zero at $t = 0$. The frequency-modulated signal is therefore described in the time domain by

$$s(t) = A_c \cos \left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right] \quad (2.26)$$

A consequence of allowing the angle $\theta_i(t)$ to become dependent on the message signal $m(t)$ as in Equation (2.22) or on its integral as in Equation (2.25) is that the *zero crossings* of a PM signal or FM signal no longer have a perfect regularity in their spacing; zero crossings refer to the instants of time at which a waveform changes from a negative to a positive value or vice versa. This is one important feature that distinguishes both PM and FM signals from an AM signal. Another important difference is that the envelope of a PM or FM signal is constant (equal to the carrier amplitude), whereas the envelope of an AM signal is dependent on the message signal.

Comparing Equation (2.23) with (2.26) reveals that an FM signal may be regarded as a PM signal in which the modulating wave is $\int_0^t m(\tau) d\tau$ in place of $m(t)$. This means that an FM signal can be generated by first integrating $m(t)$ and then using the result as the input to a phase modulator, as in Figure 2.20a. Conversely, a PM signal can be generated by first differentiating $m(t)$ and then using the result as the input to a frequency modulator, as in Figure 2.20b. We may thus deduce all the properties of PM signals from those of FM signals and vice versa. Henceforth, we concentrate our attention on FM signals.

2.7 Frequency Modulation

The FM signal $s(t)$ defined by Equation (2.26) is a nonlinear function of the modulating signal $m(t)$, which makes frequency modulation a *nonlinear modulation process*. Consequently, unlike amplitude modulation, the spectrum of an FM signal is not related in a simple manner to that of the modulating signal; rather, its analysis is much more difficult than that of an AM signal.

How then can we tackle the spectral analysis of an FM signal? We propose to provide an empirical answer to this important question by proceeding in the following manner:

- ▶ We consider the simplest case possible, namely, that of a single-tone modulation that produces a narrowband FM signal.
- ▶ We next consider the more general case also involving a single-tone modulation, but this time the FM signal is wideband.

We could, of course, go on and consider the more elaborate case of a multitone FM signal. However, we propose not to do so, because our immediate objective is to establish an empirical relationship between the transmission bandwidth of an FM signal and the message bandwidth. As we shall subsequently see, the two-stage spectral analysis described here provides us with enough insight to propose a solution to the problem.

Consider then a sinusoidal modulating signal defined by

$$m(t) = A_m \cos(2\pi f_m t) \quad (2.27)$$

The instantaneous frequency of the resulting FM signal equals

$$\begin{aligned} f_i(t) &= f_c + k_f A_m \cos(2\pi f_m t) \\ &= f_c + \Delta f \cos(2\pi f_m t) \end{aligned} \quad (2.28)$$

where

$$\Delta f = k_f A_m \quad (2.29)$$

The quantity Δf is called the *frequency deviation*, representing the maximum departure of the instantaneous frequency of the FM signal from the carrier frequency f_c . A fundamental characteristic of an FM signal is that the frequency deviation Δf is proportional to the amplitude of the modulating signal and is independent of the modulation frequency.

Using Equation (2.28), the angle $\theta_i(t)$ of the FM signal is obtained as

$$\begin{aligned} \theta_i(t) &= 2\pi \int_0^t f_i(\tau) d\tau \\ &= 2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t) \end{aligned} \quad (2.30)$$

The ratio of the frequency deviation Δf to the modulation frequency f_m is commonly called the *modulation index* of the FM signal. We denote it by β , and so write

$$\beta = \frac{\Delta f}{f_m} \quad (2.31)$$

and

$$\theta_i(t) = 2\pi f_c t + \beta \sin(2\pi f_m t) \quad (2.32)$$

From Equation (2.32) we see that, in a physical sense, the parameter β represents the phase deviation of the FM signal, that is, the maximum departure of the angle $\theta_i(t)$ from the angle $2\pi f_c t$ of the unmodulated carrier; hence, β is measured in radians.

The FM signal itself is given by

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)] \quad (2.33)$$

Depending on the value of the modulation index β , we may distinguish two cases of frequency modulation:

- ▷ *Narrowband FM*, for which β is small compared to one radian.
- ▷ *Wideband FM*, for which β is large compared to one radian.

These two cases are considered next, in that order.

■ NARROWBAND FREQUENCY MODULATION

Consider Equation (2.33), which defines an FM signal resulting from the use of a sinusoidal modulating signal. Expanding this relation, we get

$$s(t) = A_c \cos(2\pi f_c t) \cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t) \sin[\beta \sin(2\pi f_m t)] \quad (2.34)$$

Assuming that the modulation index β is small compared to one radian, we may use the following approximations:

$$\cos[\beta \sin(2\pi f_m t)] \approx 1$$

and

$$\sin[\beta \sin(2\pi f_m t)] \approx \beta \sin(2\pi f_m t)$$

Hence, Equation (2.34) simplifies to

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t) \quad (2.35)$$

Equation (2.35) defines the approximate form of a narrowband FM signal produced by a sinusoidal modulating signal $A_m \cos(2\pi f_m t)$. From this representation we deduce the modulator shown in block diagram form in Figure 2.21. This modulator involves splitting the carrier wave $A_c \cos(2\pi f_c t)$ into two paths. One path is direct; the other path contains a -90° degree phase-shifting network and a product modulator, the combination of which generates a DSB-SC modulated signal. The difference between these two signals produces a narrowband FM signal, but with some distortion.

Ideally, an FM signal has a constant envelope and, for the case of a sinusoidal modulating signal of frequency f_m , the angle $\theta_i(t)$ is also sinusoidal with the same frequency.

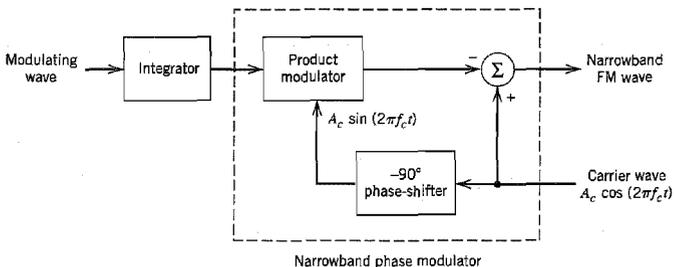


FIGURE 2.21 Block diagram of a method for generating a narrowband FM signal.

But the modulated signal produced by the narrowband modulator of Figure 2.21 differs from this ideal condition in two fundamental respects:

1. The envelope contains a *residual* amplitude modulation and, therefore, varies with time.
2. For a sinusoidal modulating wave, the angle $\theta_i(t)$ contains *harmonic distortion* in the form of third- and higher-order harmonics of the modulation frequency f_m .

However, by restricting the modulation index to $\beta \leq 0.3$ radians, the effects of residual AM and harmonic PM are limited to negligible levels.

Returning to Equation (2.35), we may expand it as follows:

$$s(t) \approx A_c \cos(2\pi f_c t) + \frac{1}{2} \beta A_c [\cos[2\pi(f_c + f_m)t] - \cos[2\pi(f_c - f_m)t]] \quad (2.36)$$

This expression is somewhat similar to the corresponding one defining an AM signal, which is as follows:

$$s_{AM}(t) = A_c \cos(2\pi f_c t) + \frac{1}{2} \mu A_c [\cos[2\pi(f_c + f_m)t] + \cos[2\pi(f_c - f_m)t]] \quad (2.37)$$

where μ is the modulation factor of the AM signal. Comparing Equations (2.36) and (2.37), we see that in the case of sinusoidal modulation, the basic difference between an AM signal and a narrowband FM signal is that the algebraic sign of the lower side frequency in the narrowband FM is reversed. Thus, a narrowband FM signal requires essentially the same transmission bandwidth (i.e., $2f_m$) as the AM signal.

We may represent the narrowband FM signal with a phasor diagram as shown in Figure 2.22a, where we have used the carrier phasor as reference. We see that the resultant

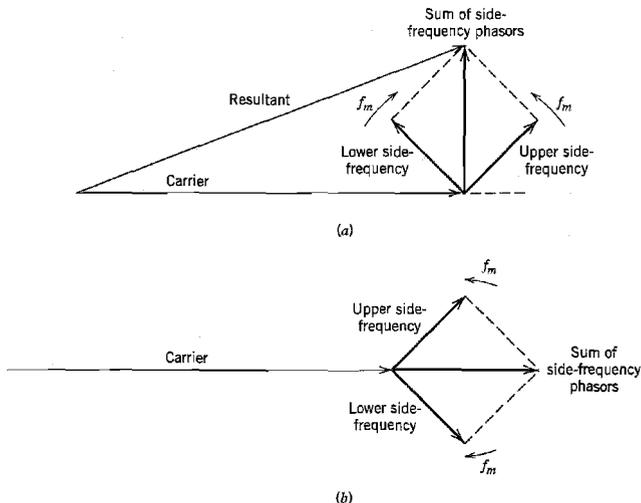


FIGURE 2.22 A phasor comparison of narrowband FM and AM waves for sinusoidal modulation. (a) Narrowband FM wave. (b) AM wave.

of the two side-frequency phasors is always at right angles to the carrier phasor. The effect of this is to produce a resultant phasor representing the narrowband FM signal that is approximately of the same amplitude as the carrier phasor, but out of phase with respect to it. This phasor diagram should be contrasted with that of Figure 2.22*b*, representing an AM signal. In this latter case we see that the resultant phasor representing the AM signal has an amplitude that is different from that of the carrier phasor but always in phase with it.

■ WIDEBAND FREQUENCY MODULATION

We next wish to determine the spectrum of the single-tone FM signal of Equation (2.33) for an arbitrary value of the modulation index β . In general, an FM signal produced by a sinusoidal modulating signal, as in Equation (2.33), is in itself nonperiodic unless the carrier frequency f_c is an integral multiple of the modulation frequency f_m . However, we may simplify matters by using the complex representation of band-pass signals described in Appendix 2. Specifically, we assume that the carrier frequency f_c is large enough (compared to the bandwidth of the FM signal) to justify rewriting this equation in the form

$$\begin{aligned} s(t) &= \text{Re}[A_c \exp(j2\pi f_c t + j\beta \sin(2\pi f_m t))] \\ &= \text{Re}[\tilde{s}(t) \exp(j2\pi f_c t)] \end{aligned} \quad (2.38)$$

where $\tilde{s}(t)$ is the *complex envelope* of the FM signal $s(t)$, defined by

$$\tilde{s}(t) = A_c \exp[j\beta \sin(2\pi f_m t)] \quad (2.39)$$

Thus, unlike the original FM signal $s(t)$, the complex envelope $\tilde{s}(t)$ is a periodic function of time with a fundamental frequency equal to the modulation frequency f_m . We may therefore expand $\tilde{s}(t)$ in the form of a complex Fourier series as follows:

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_m t) \quad (2.40)$$

where the complex Fourier coefficient c_n is defined by

$$\begin{aligned} c_n &= f_m \int_{-1/2f_m}^{1/2f_m} \tilde{s}(t) \exp(-j2\pi n f_m t) dt \\ &= f_m A_c \int_{-1/2f_m}^{1/2f_m} \exp[j\beta \sin(2\pi f_m t) - j2\pi n f_m t] dt \end{aligned} \quad (2.41)$$

Define a new variable:

$$x = 2\pi f_m t \quad (2.42)$$

Hence, we may rewrite Equation (2.41) in the new form

$$c_n = \frac{A_c}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)] dx \quad (2.43)$$

The integral on the right-hand side of Equation (2.43), except for a scaling factor, is recognized as the *n*th order Bessel function of the first kind³ and argument β . This function is commonly denoted by the symbol $J_n(\beta)$, as shown by

$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)] dx \quad (2.44)$$

Accordingly, we may reduce Equation (2.43) to

$$c_n = A_c J_n(\beta) \quad (2.45)$$

Substituting Equation (2.45) in (2.40), we get, in terms of the Bessel function $J_n(\beta)$, the following expansion for the complex envelope of the FM signal:

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j2\pi n f_m t) \quad (2.46)$$

Next, substituting Equation (2.46) in (2.38), we get

$$s(t) = A_c \cdot \text{Re} \left[\sum_{n=-\infty}^{\infty} J_n(\beta) \exp[j2\pi(f_c + n f_m)t] \right] \quad (2.47)$$

Interchanging the order of summation and evaluation of the real part in the right-hand side of Equation (2.47), we finally get

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + n f_m)t] \quad (2.48)$$

This is the desired form for the Fourier series representation of the single-tone FM signal $s(t)$ for an arbitrary value of β . The discrete spectrum of $s(t)$ is obtained by taking the Fourier transforms of both sides of Equation (2.48); we thus have

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) [\delta(f - f_c - n f_m) + \delta(f + f_c + n f_m)] \quad (2.49)$$

In Figure 2.23 we have plotted the Bessel function $J_n(\beta)$ versus the modulation index β for different positive integer values of n . We can develop further insight into the behavior

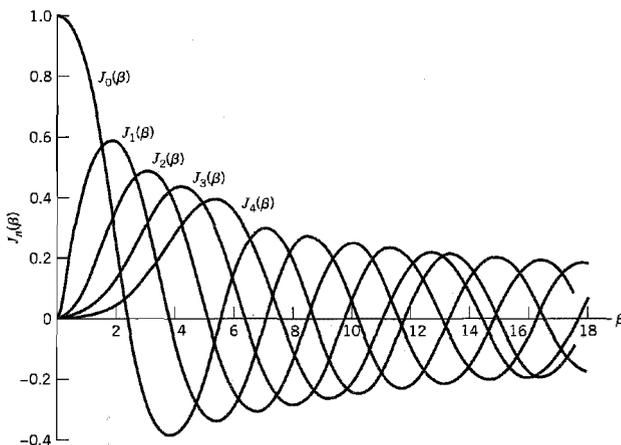


FIGURE 2.23 Plots of Bessel functions of the first kind for varying order.

of the Bessel function $J_n(\beta)$ by making use of the following properties (see Appendix 3 for more details):

1.
$$J_n(\beta) = (-1)^n J_{-n}(\beta) \quad \text{for all } n, \text{ both positive and negative} \quad (2.50)$$

2. For small values of the modulation index β , we have

$$\left. \begin{aligned} J_0(\beta) &\approx 1 \\ J_1(\beta) &\approx \frac{\beta}{2} \\ J_n(\beta) &\approx 0, \quad n > 2 \end{aligned} \right\} \quad (2.51)$$

3.
$$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1 \quad (2.52)$$

Thus, using Equations (2.49)–(2.52) and the curves of Figure 2.23, we may make the following observations:

1. The spectrum of an FM signal contains a carrier component and an infinite set of side frequencies located symmetrically on either side of the carrier at frequency separations of $f_m, 2f_m, 3f_m, \dots$. In this respect, the result is unlike that which prevails in an AM system, since in an AM system a sinusoidal modulating signal gives rise to only one pair of side frequencies.
2. For the special case of β small compared with unity, only the Bessel coefficients $J_0(\beta)$ and $J_1(\beta)$ have significant values, so that the FM signal is effectively composed of a carrier and a single pair of side frequencies at $f_c \pm f_m$. This situation corresponds to the special case of narrowband FM that was considered earlier.
3. The amplitude of the carrier component varies with β according to $J_0(\beta)$. That is, unlike an AM signal, the amplitude of the carrier component of an FM signal is dependent on the modulation index β . The physical explanation for this property is that the envelope of an FM signal is constant, so that the average power of such a signal developed across a 1-ohm resistor is also constant, as shown by

$$P = \frac{1}{2} A_c^2 \quad (2.53)$$

When the carrier is modulated to generate the FM signal, the power in the side frequencies may appear only at the expense of the power originally in the carrier, thereby making the amplitude of the carrier component dependent on β . Note that the average power of an FM signal may also be determined from Equation (2.48), obtaining

$$P = \frac{1}{2} A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta) \quad (2.54)$$

Substituting Equation (2.52) into (2.54), the expression for the average power P reduces to Equation (2.53), and so it should.

EXAMPLE 2.2

In this example, we wish to investigate the ways in which variations in the amplitude and frequency of a sinusoidal modulating signal affect the spectrum of the FM signal. Consider

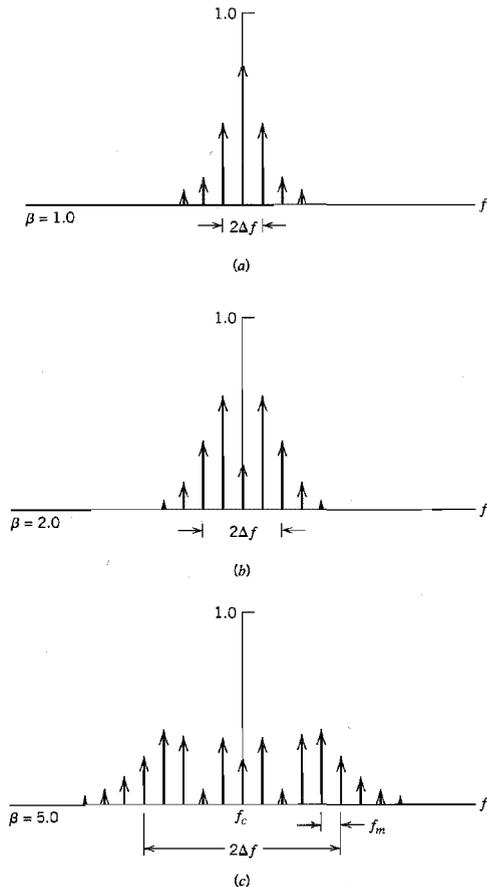


FIGURE 2.24 Discrete amplitude spectra of an FM signal, normalized with respect to the carrier amplitude, for the case of sinusoidal modulation of fixed frequency and varying amplitude. Only the spectra for positive frequencies are shown.

first the case when the frequency of the modulating signal is fixed, but its amplitude is varied, producing a corresponding variation in the frequency deviation Δf . Thus, keeping the modulation frequency f_m fixed, we find that the amplitude spectrum of the resulting FM signal is as shown plotted in Figure 2.24 for $\beta = 1, 2$, and 5. In this diagram we have normalized the spectrum with respect to the unmodulated carrier amplitude.

Consider next the case when the amplitude of the modulating signal is fixed; that is, the frequency deviation Δf is maintained constant, and the modulation frequency f_m is varied. In this case we find that the amplitude spectrum of the resulting FM signal is as shown plotted in Figure 2.25 for $\beta = 1, 2$, and 5. We see that when Δf is fixed and β is increased, we have an increasing number of spectral lines crowding into the fixed frequency interval $f_c - \Delta f < |f| < f_c + \Delta f$. That is, when β approaches infinity, the bandwidth of the FM wave

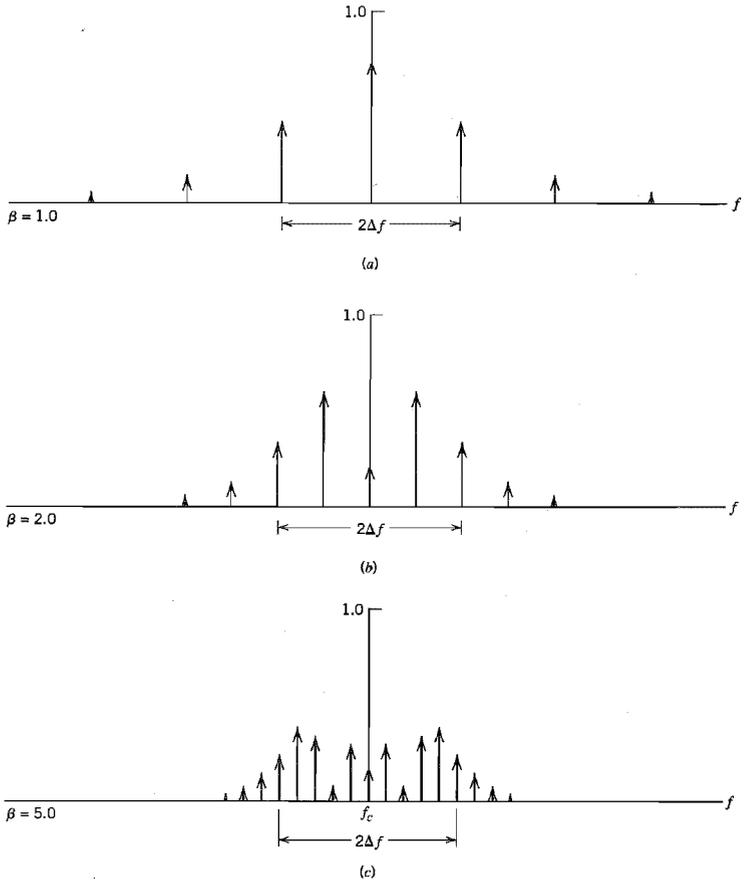


FIGURE 2.25 Discrete amplitude spectra of an FM signal, normalized with respect to the carrier amplitude, for the case of sinusoidal modulation of varying frequency and fixed amplitude. Only the spectra for positive frequencies are shown.

approaches the limiting value of $2\Delta f$, which is an important point to keep in mind for later discussion. ◀

TRANSMISSION BANDWIDTH OF FM SIGNALS

In theory, an FM signal contains an infinite number of side frequencies so that the bandwidth required to transmit such a signal is similarly infinite in extent. In practice, however, we find that the FM signal is effectively limited to a finite number of significant side frequencies compatible with a specified amount of distortion. We may therefore specify an effective bandwidth required for the transmission of an FM signal. Consider first the

case of an FM signal generated by a single-tone modulating wave of frequency f_m . In such an FM signal, the side frequencies that are separated from the carrier frequency f_c by an amount greater than the frequency deviation Δf decrease rapidly toward zero, so that the bandwidth always exceeds the total frequency excursion, but nevertheless is limited. Specifically, for large values of the modulation index β , the bandwidth approaches, and is only slightly greater than, the total frequency excursion $2\Delta f$ in accordance with the situation shown in Figure 2.25. On the other hand, for small values of the modulation index β , the spectrum of the FM signal is effectively limited to the carrier frequency f_c and one pair of side frequencies at $f_c \pm f_m$, so that the bandwidth approaches $2f_m$. We may thus define an approximate rule for the transmission bandwidth of an FM signal generated by a single-tone modulating signal of frequency f_m as follows:

$$B_T \approx 2\Delta f + 2f_m = 2\Delta f \left(1 + \frac{1}{\beta} \right) \quad (2.55)$$

This empirical relation is known as *Carson's rule*.⁴

For an alternative assessment of the bandwidth requirement of an FM signal, we may use a definition based on retaining the maximum number of significant side frequencies whose amplitudes are all greater than some selected value. A convenient choice for this value is 1 percent of the unmodulated carrier amplitude. We may thus define *the transmission bandwidth of an FM wave as the separation between the two frequencies beyond which none of the side frequencies is greater than 1 percent of the carrier amplitude obtained when the modulation is removed*. That is, we define the transmission bandwidth as $2n_{\max}f_m$, where f_m is the modulation frequency and n_{\max} is the largest value of the integer n that satisfies the requirement $|J_n(\beta)| > 0.01$. The value of n_{\max} varies with the modulation index β and can be determined readily from tabulated values of the Bessel function $J_n(\beta)$. Table 2.2 shows the total number of significant side frequencies (including both the upper and lower side frequencies) for different values of β , calculated on the 1 percent basis explained herein. The transmission bandwidth B_T calculated using this procedure can be presented in the form of a *universal curve* by normalizing it with respect to the frequency deviation Δf and then plotting it versus β . This curve is shown in Figure 2.26, which is drawn as a best fit through the set of points obtained by using Table 2.2. In Figure 2.26 we note that as the modulation index β is increased, the bandwidth occupied

TABLE 2.2 Number of significant side frequencies of a wideband FM signal for varying modulation index

Modulation Index β	Number of Significant Side Frequencies $2n_{\max}$
0.1	2
0.3	4
0.5	4
1.0	6
2.0	8
5.0	16
10.0	28
20.0	50
30.0	70

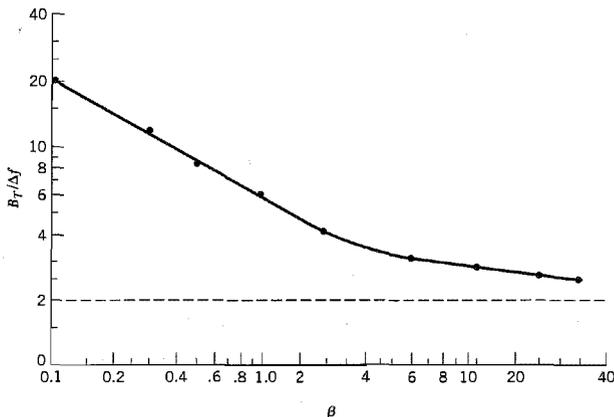


FIGURE 2.26 Universal curve for evaluating the 1 percent bandwidth of an FM wave.

by the significant side frequencies drops toward that over which the carrier frequency actually deviates. This means that small values of the modulation index β are relatively more extravagant in transmission bandwidth than are the larger values of β .

Consider next the more general case of an arbitrary modulating signal $m(t)$ with its highest frequency component denoted by W . The bandwidth required to transmit an FM signal generated by this modulating signal is estimated by using a worst-case tone-modulation analysis. Specifically, we first determine the so-called *deviation ratio* D , defined as the ratio of the frequency deviation Δf , which corresponds to the maximum possible amplitude of the modulation signal $m(t)$, to the highest modulation frequency W ; these conditions represent the extreme cases possible. *The deviation ratio D plays the same role for nonsinusoidal modulation that the modulation index β plays for the case of sinusoidal modulation.* Then, replacing β by D and replacing f_m with W , we may use Carson's rule given by Equation (2.55) or the universal curve of Figure 2.26 to obtain a value for the transmission bandwidth of the FM signal. From a practical viewpoint, Carson's rule somewhat underestimates the bandwidth requirement of an FM system, whereas using the universal curve of Figure 2.26 yields a somewhat conservative result. Thus, the choice of a transmission bandwidth that lies between the bounds provided by these two rules of thumb is acceptable for most practical purposes.

▶ EXAMPLE 2.3

In North America, the maximum value of frequency deviation Δf is fixed at 75 kHz for commercial FM broadcasting by radio. If we take the modulation frequency $W = 15$ kHz, which is typically the "maximum" audio frequency of interest in FM transmission, we find that the corresponding value of the deviation ratio is

$$D = \frac{75}{15} = 5$$

Using Carson's rule of Equation (2.55), replacing β by D , and replacing f_m by W , the approximate value of the transmission bandwidth of the FM signal is obtained as

$$B_T = 2(75 + 15) = 180 \text{ kHz}$$

On the other hand, use of the curve of Figure 2.26 gives the transmission bandwidth of the FM signal to be

$$B_T = 3.2\Delta f = 3.2 \times 75 = 240 \text{ kHz}$$

In practice, a bandwidth of 200 kHz is allocated to each FM transmitter. On this basis, Carson's rule underestimates the transmission bandwidth by 10 percent, whereas the universal curve of Figure 2.26 overestimates it by 20 percent. ◀

■ GENERATION OF FM SIGNALS

There are essentially two basic methods of generating frequency-modulated signals, namely, *direct* FM and *indirect* FM. In the direct method the carrier frequency is directly varied in accordance with the input baseband signal, which is readily accomplished using a *voltage-controlled oscillator*. In the indirect method, the modulating signal is first used to produce a narrowband FM signal, and frequency multiplication is next used to increase the frequency deviation to the desired level. The indirect method is the preferred choice for frequency modulation when the stability of carrier frequency is of major concern as in commercial radio broadcasting, as described next.

Indirect FM²

A simplified block diagram of an indirect FM system is shown in Figure 2.27. The message (baseband) signal $m(t)$ is first integrated and then used to phase-modulate a crystal-controlled oscillator; the use of crystal control provides *frequency stability*. To minimize the distortion inherent in the phase modulator, the maximum phase deviation or modulation index β is kept small, thereby resulting in a narrowband FM signal; for the implementation of the narrow-band phase modulator, we may use the arrangement described in Figure 2.21. The narrowband FM signal is next multiplied in frequency by means of a frequency multiplier so as to produce the desired wideband FM signal.

A *frequency multiplier* consists of a nonlinear device followed by a band-pass filter, as shown in Figure 2.28. The implication of the nonlinear device being memoryless is that it has no energy-storage elements. The input-output relation of such a device may be expressed in the general form

$$v(t) = a_1s(t) + a_2s^2(t) + \cdots + a_ns^n(t) \quad (2.56)$$

where a_1, a_2, \dots, a_n are coefficients determined by the operating point of the device, and n is the *highest order of nonlinearity*. In other words, the memoryless nonlinear device is an n th power-law device. The input $s(t)$ is an FM signal defined by

$$s(t) = A_c \cos \left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right]$$

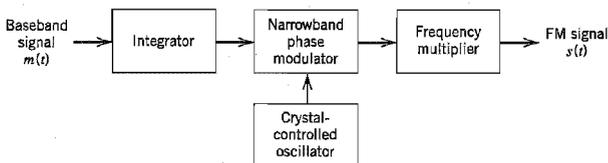


FIGURE 2.27 Block diagram of the indirect method of generating a wideband FM signal.

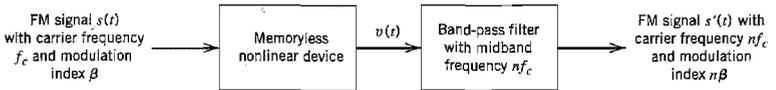


FIGURE 2.28 Block diagram of frequency multiplier.

whose instantaneous frequency is

$$f_i(t) = f_c + k_f m(t) \quad (2.57)$$

The mid-band frequency of the band-pass filter in Figure 2.28 is set equal to $n f_c$, where f_c is the carrier frequency of the incoming FM signal $s(t)$. Moreover, the band-pass filter is designed to have a bandwidth equal to n times the transmission bandwidth of $s(t)$. In Section 2.8 dealing with nonlinear effects in FM systems, we describe the spectral contributions of such nonlinear terms as the second- and third-order terms in the input-output relation of Equation (2.56). For now it suffices to say that after band-pass filtering of the nonlinear device's output $v(t)$, we have a new FM signal defined by

$$s'(t) = A'_c \cos \left[2\pi n f_c t + 2\pi n k_f \int_0^t m(\tau) d\tau \right] \quad (2.58)$$

whose instantaneous frequency is

$$f'_i(t) = n f_c + n k_f m(t) \quad (2.59)$$

Thus, comparing Equation (2.59) with (2.57), we see that the nonlinear processing circuit of Figure 2.28 acts as a frequency multiplier. The frequency multiplication ratio is determined by the highest power n in the input-output relation of Equation (2.56), characterizing the memoryless nonlinear device.

■ DEMODULATION OF FM SIGNALS

Frequency demodulation is the process that enables us to recover the original modulating signal from a frequency-modulated signal. The objective is to produce a transfer characteristic that is the inverse of that of the frequency modulator, which can be realized directly or indirectly. Here we describe a direct method of frequency demodulation involving the use of a popular device known as a frequency discriminator, whose instantaneous output amplitude is directly proportional to the instantaneous frequency of the input FM signal. In Section 2.14, we describe an indirect method of frequency demodulation that uses another popular device known as a phase-locked loop.

Basically, the *frequency discriminator* consists of a *slope circuit* followed by an *envelope detector*. An ideal slope circuit is characterized by a frequency response that is purely imaginary, varying linearly with frequency inside a prescribed frequency interval. Consider the frequency response depicted in Figure 2.29a, which is defined by

$$H_1(f) = \begin{cases} j2\pi a \left(f - f_c + \frac{B_T}{2} \right), & f_c - \frac{B_T}{2} \leq f \leq f_c + \frac{B_T}{2} \\ j2\pi a \left(f + f_c - \frac{B_T}{2} \right), & -f_c - \frac{B_T}{2} \leq f \leq -f_c + \frac{B_T}{2} \\ 0, & \text{elsewhere} \end{cases} \quad (2.60)$$

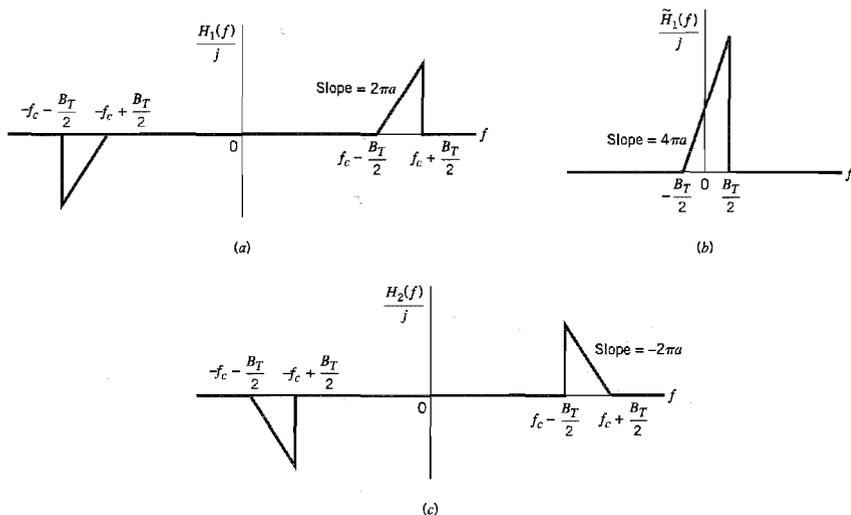


FIGURE 2.29 (a) Frequency response of ideal slope circuit. (b) Frequency response of the slope circuit's complex low-pass equivalent. (c) Frequency response of the ideal slope circuit complementary to that of part (a).

where a is a constant. We wish to evaluate the response of this slope circuit, denoted by $s_1(t)$, which is produced by an FM signal $s(t)$ of carrier frequency f_c and transmission bandwidth B_T . It is assumed that the spectrum of $s(t)$ is essentially zero outside the frequency interval $f_c - B_T/2 \leq |f| \leq f_c + B_T/2$.

We may simplify the analysis of the frequency discriminator by invoking the isomorphism between a real-valued band-pass filter and a corresponding complex-valued low-pass filter. This isomorphism is discussed in Appendix 2. According to the material presented in that appendix, we may replace the band-pass filter with frequency response $H_1(f)$ with an equivalent low-pass filter with frequency response $\tilde{H}_1(f)$ by doing two things:

1. We shift $\tilde{H}_1(f)$ to the right by f_c , where f_c is the midband frequency of the band-pass filter; this operation aligns the translated frequency response of the equivalent low-pass filter with that of the band-pass filter.
2. We set $\tilde{H}_1(f - f_c)$ equal to $2H_1(f)$ for $f > 0$.

Thus for the problem at hand we get

$$\tilde{H}_1(f - f_c) = 2H_1(f), \quad f > 0 \quad (2.61)$$

Hence, using Equations (2.60) and (2.61), we get

$$\tilde{H}_1(f) = \begin{cases} j4\pi a \left(f + \frac{B_T}{2} \right), & -\frac{B_T}{2} \leq f \leq \frac{B_T}{2} \\ 0, & \text{elsewhere} \end{cases} \quad (2.62)$$

which is plotted in Figure 2.29b.

The incoming FM signal $s(t)$ is defined by Equation (2.26), which is reproduced here for convenience:

$$s(t) = A_c \cos \left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right]$$

Given that the carrier frequency f_c is high compared to the transmission bandwidth of the FM signal $s(t)$, the complex envelope of $s(t)$ is

$$\tilde{s}(t) = A_c \exp \left[j2\pi k_f \int_0^t m(\tau) d\tau \right] \quad (2.63)$$

Let $\tilde{s}_1(t)$ denote the complex envelope of the response of the slope circuit defined by Figure 2.29a due to $\tilde{s}(t)$. Then, following the material presented in Appendix 2, we may express the Fourier transform of $\tilde{s}_1(t)$ as follows:

$$\begin{aligned} \tilde{S}_1(f) &= \frac{1}{2} \tilde{H}_1(f) \tilde{S}(f) \\ &= \begin{cases} j2\pi a \left(f + \frac{B_T}{2} \right) \tilde{S}(f), & -\frac{B_T}{2} \leq f \leq \frac{B_T}{2} \\ 0, & \text{elsewhere} \end{cases} \end{aligned} \quad (2.64)$$

where $\tilde{S}(f)$ is the Fourier transform of $\tilde{s}(t)$. From Fourier analysis we know that multiplication of the Fourier transform of a signal by $j2\pi f$ is equivalent to differentiating the signal in the time domain; see item 8 of Table A6.2. Hence, from Equation (2.64) we deduce

$$\tilde{s}_1(t) = a \left[\frac{d\tilde{s}(t)}{dt} + j\pi B_T \tilde{s}(t) \right] \quad (2.65)$$

Substituting Equation (2.63) into (2.65), we get

$$\tilde{s}_1(t) = j\pi B_T a A_c \left[1 + \frac{2k_f}{B_T} m(t) \right] \exp \left[j2\pi k_f \int_0^t m(\tau) d\tau \right] \quad (2.66)$$

The desired response of the slope circuit is therefore

$$\begin{aligned} s_1(t) &= \text{Re}[\tilde{s}_1(t) \exp(j2\pi f_c t)] \\ &= \pi B_T a A_c \left[1 + \frac{2k_f}{B_T} m(t) \right] \cos \left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau + \frac{\pi}{2} \right] \end{aligned} \quad (2.67)$$

The signal $s_1(t)$ is a hybrid-modulated signal in which both amplitude and frequency of the carrier wave vary with the message signal $m(t)$. However, provided that we choose

$$\left| \frac{2k_f}{B_T} m(t) \right| < 1 \quad \text{for all } t$$

then we may use an envelope detector to recover the amplitude variations and thus, except for a bias term, obtain the original message signal. The resulting envelope-detector output is therefore

$$|\tilde{s}_1(t)| = \pi B_T a A_c \left[1 + \frac{2k_f}{B_T} m(t) \right] \quad (2.68)$$

The bias term $\pi B_T a A_c$ in the right-hand side of Equation (2.68) is proportional to the slope a of the transfer function of the slope circuit. This suggests that the bias may be

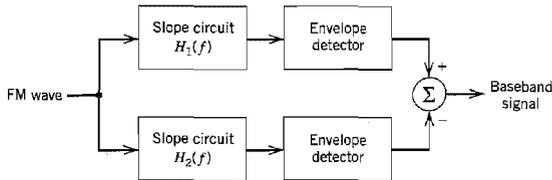


FIGURE 2.30 Block diagram of frequency discriminator.

removed by subtracting from the envelope-detector output $|\tilde{s}_1(t)|$ the output of a second envelope detector preceded by the *complementary slope circuit* with a frequency response $H_2(f)$ as described in Figure 2.29c. That is, the two slope circuits are related by

$$\tilde{H}_2(f) = \tilde{H}_1(-f) \quad (2.69)$$

Let $s_2(t)$ denote the response of the complementary slope circuit produced by the incoming FM signal $s(t)$. Then, following a procedure similar to that just described, we may write

$$|\tilde{s}_2(t)| = \pi B_T a A_c \left[1 - \frac{2k_f}{B_T} m(t) \right] \quad (2.70)$$

where $\tilde{s}_2(t)$ is the complex envelope of the signal $s_2(t)$. The difference between the two envelopes in Equations (2.68) and (2.70) is

$$\begin{aligned} s_o(t) &= |\tilde{s}_1(t)| - |\tilde{s}_2(t)| \\ &= 4\pi k_f a A_c m(t) \end{aligned} \quad (2.71)$$

which is a scaled version of the original message signal $m(t)$ and free from bias.

We may thus model the *ideal frequency discriminator* as a pair of slope circuits with their complex transfer functions related by Equation (2.69), followed by envelope detectors and finally a summer, as in Figure 2.30. This scheme is called a *balanced frequency discriminator*.

■ FM STEREO MULTIPLEXING⁶

Stereo multiplexing is a form of frequency-division multiplexing (FDM) designed to transmit two separate signals via the same carrier. It is widely used in FM radio broadcasting to send two different elements of a program (e.g., two different sections of an orchestra, a vocalist and an accompanist) so as to give a spatial dimension to its perception by a listener at the receiving end.

The specification of standards for FM stereo transmission is influenced by two factors:

1. The transmission has to operate within the allocated FM broadcast channels.
2. It has to be compatible with monophonic radio receivers.

The first requirement sets the permissible frequency parameters, including frequency deviation. The second requirement constrains the way in which the transmitted signal is configured.

Figure 2.31a shows the block diagram of the multiplexing system used in an FM stereo transmitter. Let $m_l(t)$ and $m_r(t)$ denote the signals picked up by left-hand and

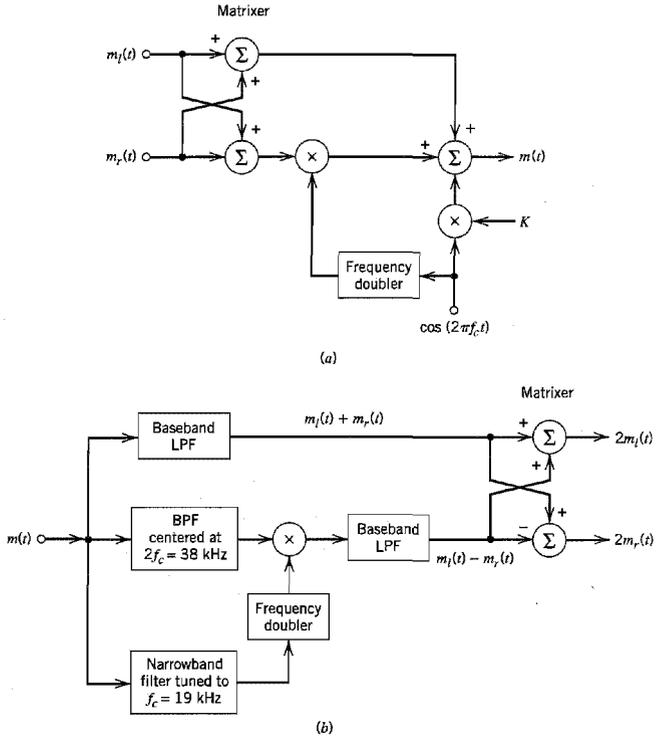


FIGURE 2.31 (a) Multiplexer in transmitter of FM stereo. (b) Demultiplexer in receiver of FM stereo.

right-hand microphones at the transmitting end of the system. They are applied to a simple *matrixer* that generates the *sum signal*, $m_l(t) + m_r(t)$, and the *difference signal*, $m_l(t) - m_r(t)$. The sum signal is left unprocessed in its baseband form; it is available for monophonic reception. The difference signal and a 38-kHz subcarrier (derived from a 19-kHz crystal oscillator by frequency doubling) are applied to a product modulator, thereby producing a DSB-SC modulated wave. In addition to the sum signal and this DSB-SC modulated wave, the multiplexed signal $m(t)$ also includes a 19-kHz pilot to provide a reference for the coherent detection of the difference signal at the stereo receiver. Thus the multiplexed signal is described by

$$m(t) = [m_l(t) + m_r(t)] + [m_l(t) - m_r(t)] \cos(4\pi f_c t) + K \cos(2\pi f_c t) \quad (2.72)$$

where $f_c = 19$ kHz, and K is the amplitude of the pilot tone. The multiplexed signal $m(t)$ then frequency-modulates the main carrier to produce the transmitted signal. The pilot is allotted between 8 and 10 percent of the peak frequency deviation; the amplitude K in Equation (2.72) is chosen to satisfy this requirement.

At a stereo receiver, the multiplexed signal $m(t)$ is recovered by frequency demodulating the incoming FM wave. Then $m(t)$ is applied to the *demultiplexing system* shown

in Figure 2.31*b*. The individual components of the multiplexed signal $m(t)$ are separated by the use of three appropriate filters. The recovered pilot (using a narrowband filter tuned to 19 kHz) is frequency doubled to produce the desired 38-kHz subcarrier. The availability of this subcarrier enables the coherent detection of the DSB-SC modulated wave, thereby recovering the difference signal, $m_i(t) - m_r(t)$. The baseband low-pass filter in the top path of Figure 2.31*b* is designed to pass the sum signal, $m_i(t) + m_r(t)$. Finally, the simple matrixer reconstructs the left-hand signal $m_i(t)$ and right-hand signal $m_r(t)$, except for scaling factors, and applies them to their respective speakers.

2.8 Nonlinear Effects in FM Systems

In the preceding two sections, we studied frequency modulation theory and methods for its generation and demodulation. We complete the discussion of frequency modulation by considering nonlinear effects in FM systems.

Nonlinearities, in one form or another, are present in all electrical networks. There are two basic forms of nonlinearity to consider:

1. The nonlinearity is said to be *strong* when it is introduced intentionally and in a controlled manner for some specific application. Examples of strong nonlinearity include square-law modulators, hard-limiters, and frequency multipliers.
2. The nonlinearity is said to be *weak* when a linear performance is desired, but nonlinearities of a parasitic nature arise due to imperfections. The effect of such weak nonlinearities is to limit the useful signal levels in a system and thereby become an important design consideration.

In this section we examine the effects of weak nonlinearities on frequency modulation.

Consider a communications channel, the transfer characteristic of which is defined by the nonlinear input–output relation

$$v_o(t) = a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t) \quad (2.73)$$

where $v_i(t)$ and $v_o(t)$ are the input and output signals, respectively, and a_1 , a_2 , and a_3 are constants; Equation (2.73) is a truncated version of Equation (2.56) used in the context of frequency multiplication. The channel described in Equation (2.73) is said to be *memoryless* in that the output signal $v_o(t)$ is an instantaneous function of the input signal $v_i(t)$ (i.e., there is no energy storage involved in the description). We wish to determine the effect of transmitting a frequency-modulated wave through such a channel. The FM signal is defined by

$$v_i(t) = A_c \cos[2\pi f_c t + \phi(t)]$$

where

$$\phi(t) = 2\pi k_f \int_0^t m(\tau) d\tau$$

For this input signal, the use of Equation (2.73) yields

$$\begin{aligned} v_o(t) = & a_1 A_c \cos[2\pi f_c t + \phi(t)] + a_2 A_c^2 \cos^2[2\pi f_c t + \phi(t)] \\ & + a_3 A_c^3 \cos^3[2\pi f_c t + \phi(t)] \end{aligned} \quad (2.74)$$

Expanding the squared and cubed cosine terms in Equation (2.74) and then collecting common terms, we get

$$\begin{aligned} v_o(t) = & \frac{1}{2} a_2 A_c^2 + \left(a_1 A_c + \frac{3}{4} a_3 A_c^3 \right) \cos[2\pi f_c t + \phi(t)] \\ & + \frac{1}{2} a_2 A_c^2 \cos[4\pi f_c t + 2\phi(t)] \\ & + \frac{1}{4} a_3 A_c^3 \cos[6\pi f_c t + 3\phi(t)] \end{aligned} \quad (2.75)$$

Thus the channel output consists of a DC component and three frequency-modulated signals with carrier frequencies f_c , $2f_c$, and $3f_c$; the latter components are contributed by the linear, second-order, and third-order terms of Equation (2.73), respectively.

To extract the desired FM signal from the channel output $v_o(t)$, that is, the particular component with carrier frequency, f_c , it is necessary to separate the FM signal with this carrier frequency from the one with the closest carrier frequency, $2f_c$. Let Δf denote the frequency deviation of the incoming FM signal $v_i(t)$, and W denote the highest frequency component of the message signal $m(t)$. Then, applying Carson's rule and noting that the frequency deviation about the second harmonic of the carrier frequency is doubled, we find that the necessary condition for separating the desired FM signal with the carrier frequency f_c from that with the carrier frequency $2f_c$ is

$$2f_c - (2\Delta f + W) > f_c + \Delta f + W$$

or

$$f_c > 3\Delta f + 2W \quad (2.76)$$

Thus, by using a band-pass filter of midband frequency f_c and bandwidth $2\Delta f + 2W$, the channel output is reduced to

$$v'_o(t) = \left(a_1 A_c + \frac{3}{4} a_3 A_c^3 \right) \cos[2\pi f_c t + \phi(t)] \quad (2.77)$$

We see therefore that the only effect of passing an FM signal through a channel with amplitude nonlinearities, followed by appropriate filtering, is simply to modify its amplitude. That is, unlike amplitude modulation, frequency modulation is not affected by distortion produced by transmission through a channel with amplitude nonlinearities. It is for this reason that we find frequency modulation used in microwave radio systems: It permits the use of highly nonlinear amplifiers and power transmitters, which are particularly important to producing a maximum power output at radio frequencies.

An FM system is extremely sensitive to *phase nonlinearities*, however, as we would intuitively expect. A common type of phase nonlinearity that is encountered in microwave radio systems is known as *AM-to-PM conversion*. This is the result of the phase characteristic of repeaters or amplifiers used in the system being dependent on the instantaneous amplitude of the input signal. In practice, AM-to-PM conversion is characterized by a constant K , which is measured in degrees per dB and may be interpreted as the peak phase change at the output for a 1-dB change in envelope at the input. When an FM wave is transmitted through a microwave radio link, it picks up spurious amplitude variations due to noise and interference during the course of transmission, and when such an FM wave is passed through a repeater with AM-to-PM conversion, the output will contain unwanted

phase modulation and resultant distortion. It is therefore important to keep the AM-to-PM conversion at a low level. For example, for a good microwave repeater, the AM-to-PM conversion constant K is less than 2 degrees per dB.

2.9 Superheterodyne Receiver⁷

In a *broadcasting* system, irrespective of whether it is based on amplitude modulation or frequency modulation, the receiver not only has the task of demodulating the incoming modulated signal, but it is also required to perform some other system functions:

- ▶ *Carrier-frequency tuning*, the purpose of which is to select the desired signal (i.e., desired radio or TV station).
- ▶ *Filtering*, which is required to separate the desired signal from other modulated signals that may be picked up along the way.
- ▶ *Amplification*, which is intended to compensate for the loss of signal power incurred in the course of transmission.

The *superheterodyne receiver*, or *superhet* as it is often referred to, is a special type of receiver that fulfills all three functions, particularly the first two, in an elegant and practical fashion. Specifically, it overcomes the difficulty of having to build a tunable highly selective and variable filter. Indeed, practically all radio and TV receivers now being made are of the superheterodyne type.

Basically, the receiver consists of a radio-frequency (RF) section, a mixer and local oscillator, an intermediate-frequency (IF) section, demodulator, and power amplifier. Typical frequency parameters of commercial AM and FM radio receivers are listed in Table 2.3. Figure 2.32 shows the block diagram of a superheterodyne receiver for amplitude modulation using an envelope detector for demodulation.

The incoming amplitude-modulated wave is picked up by the receiving antenna and amplified in the RF section that is tuned to the carrier frequency of the incoming wave. The combination of mixer and local oscillator (of adjustable frequency) provides a *heterodyning* function, whereby the incoming signal is converted to a predetermined fixed *intermediate frequency*, usually lower than the incoming carrier frequency. This frequency translation is achieved without disturbing the relation of the sidebands to the carrier; see Section 2.4. The result of the heterodyning is to produce an intermediate-frequency carrier defined by

$$f_{\text{IF}} = f_{\text{LO}} - f_{\text{RF}} \quad (2.78)$$

where f_{LO} is the frequency of the local oscillator and f_{RF} is the carrier frequency of the incoming RF signal. We refer to f_{IF} as the intermediate frequency (IF), because the signal

TABLE 2.3 Typical frequency parameters of AM and FM radio receivers

	AM Radio	FM Radio
RF carrier range	0.535–1.605 MHz	88–108 MHz
Midband frequency of IF section	0.455 MHz	10.7 MHz
IF bandwidth	10 kHz	200 kHz

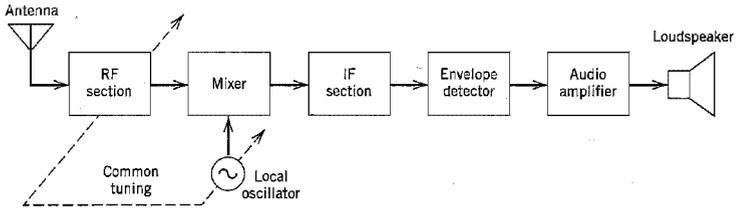


FIGURE 2.32 Basic elements of an AM radio receiver of the superheterodyne type.

is neither at the original input frequency nor at the final baseband frequency. The mixer–local oscillator combination is sometimes referred to as the *first detector*, in which case the demodulator is called the *second detector*.

The IF section consists of one or more stages of tuned amplification, with a bandwidth corresponding to that required for the particular type of modulation that the receiver is intended to handle. The IF section provides most of the amplification and selectivity in the receiver. The output of the IF section is applied to a demodulator, the purpose of which is to recover the baseband signal. If coherent detection is used, then a coherent signal source must be provided in the receiver. The final operation in the receiver is the power amplification of the recovered message signal.

In a superheterodyne receiver the mixer will develop an intermediate frequency output when the input signal frequency is greater or less than the local oscillator frequency by an amount equal to the intermediate frequency. That is, there are two input frequencies, namely, $|f_{LO} \pm f_{IF}|$, which will result in f_{IF} at the mixer output. This introduces the possibility of simultaneous reception of two signals differing in frequency by twice the intermediate frequency. For example, a receiver tuned to 0.65 MHz and having an IF of 0.455 MHz is subject to an *image interference* at 1.56 MHz; indeed, any receiver with this value of IF, when tuned to any station, is subject to image interference at a frequency of 0.910 MHz higher than the desired station. Since the function of the mixer is to produce the difference between two applied frequencies, it is incapable of distinguishing between the desired signal and its image in that it produces an IF output from either one of them. The only practical cure for image interference is to employ highly selective stages in the RF section (i.e., between the antenna and the mixer) in order to favor the desired signal and discriminate against the undesired or *image signal*. The effectiveness of suppressing unwanted image signals increases as the number of selective stages in the RF section increases, and as the ratio of intermediate to signal frequency increases.

The basic difference between AM and FM superheterodyne receivers lies in the use of an FM demodulator such as limiter–frequency discriminator. In an FM system, the message information is transmitted by variations of the instantaneous frequency of a sinusoidal carrier wave, and its amplitude is maintained constant. Therefore, any variations of the carrier amplitude at the receiver input must result from noise or interference. An *amplitude limiter*, following the IF section, is used to remove amplitude variations by clipping the modulated wave at the IF section output almost to the zero axis. The resulting rectangular wave is rounded off by a band-pass filter that suppresses harmonics of the carrier frequency. Thus the filter output is again sinusoidal, with an amplitude that is practically independent of the carrier amplitude at the receiver input (see Problem 2.42).

2.10 Noise in CW Modulation Systems

Up to this point in our discussion we have focused attention on the characterization of continuous-wave (CW) modulation techniques, entirely from a deterministic perspective. In the remainder of the chapter, we study the effects of *channel noise* on the reception of CW modulated signals and thereby develop a deeper understanding of the behavior of analog communications.

To undertake such a study we follow the customary practice by formulating two models:

1. *Channel model*, which assumes a communication channel that is distortionless but perturbed by *additive white Gaussian noise* (AWGN).
2. *Receiver model*, which assumes a receiver consisting of an ideal band-pass filter f_{0l} followed by an ideal demodulator appropriate for the application at hand; the band-pass filter is used to minimize the effect of channel noise.

These simplifying assumptions are made in order to obtain a basic understanding of the way in which noise affects the performance of the receiver. Moreover, they provide a framework for the comparison of different CW modulation-demodulation schemes.

Figure 2.33 shows the *noisy receiver model* that combines the above two assumptions. In this figure, $s(t)$ denotes the incoming modulated signal and $w(t)$ denotes the channel noise. The *received signal* is therefore made up of the sum of $s(t)$ and $w(t)$; this is the signal that the receiver has to work on. The *band-pass filter* in the model of Figure 2.33 represents the combined filtering action of the tuned amplifiers used in the actual receiver for the purpose of signal amplification prior to demodulation. The bandwidth of this band-pass filter is just wide enough to pass the modulated signal $s(t)$ without distortion. As for the *demodulator* in the model of Figure 2.33, its details naturally depend on the type of modulation used.

■ SIGNAL-TO-NOISE RATIOS: BASIC DEFINITIONS

Let the power spectral density of the noise $w(t)$ be denoted by $N_0/2$, defined for both positive and negative frequencies; that is, N_0 is the *average noise power per unit bandwidth measured at the front end of the receiver*. We also assume that the band-pass filter in the receiver model of Figure 2.33 is ideal, having a bandwidth equal to the transmission bandwidth B_T of the modulated signal $s(t)$ and a midband frequency equal to the carrier frequency f_c . The latter assumption is justified for double sideband-suppressed carrier (DSB-SC) modulation, full amplitude modulation (AM), and frequency modulation (FM); the cases of single sideband (SSB) modulation and vestigial sideband (VSB) modulation require special considerations. Taking the midband frequency of the band-pass filter to be the same as the carrier frequency f_c , we may model the power spectral density $S_N(f)$ of the noise $n(t)$, resulting from the passage of the white noise $w(t)$ through the filter, as shown

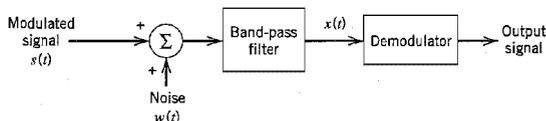


FIGURE 2.33 Receiver model.

in Figure 2.34. Typically, the carrier frequency f_c is large compared to the transmission bandwidth B_T . We may therefore treat the *filtered noise* $n(t)$ as a narrowband noise represented in the canonical form

$$n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \quad (2.79)$$

where $n_I(t)$ is the *in-phase noise component* and $n_Q(t)$ is the *quadrature noise component*, both measured with respect to the carrier wave $A_c \cos(2\pi f_c t)$. The filtered signal $x(t)$ available for demodulation is defined by

$$x(t) = s(t) + n(t) \quad (2.80)$$

The details of $s(t)$ depend on the type of modulation used. In any event, the average noise power at the demodulator input is equal to the total area under the curve of the power spectral density $S_N(f)$. From Figure 2.34 we readily see that this average noise power is equal to $N_0 B_T$. Given the format of $s(t)$, we may also determine the average signal power at the demodulator input. With the demodulated signal $s(t)$ and the filtered noise $n(t)$ appearing additively at the demodulator input in accordance with Equation (2.80), we may go on to define an *input signal-to-noise ratio*, $(\text{SNR})_I$, as the *ratio of the average power of the modulated signal $s(t)$ to the average power of the filtered noise $n(t)$* .

A more useful measure of noise performance, however, is the *output signal-to-noise ratio*, $(\text{SNR})_O$, defined as the *ratio of the average power of the demodulated message signal to the average power of the noise, both measured at the receiver output*. The output signal-to-noise ratio provides an intuitive measure for describing the fidelity with which the demodulation process in the receiver recovers the message signal from the modulated signal in the presence of additive noise. For such a criterion to be well defined, the recovered message signal and the corruptive noise component must appear *additively* at the demodulator output. This condition is perfectly valid in the case of a receiver using coherent detection. On the other hand, when the receiver uses envelope detection as in full AM or frequency discrimination as in FM, we have to assume that the average power of the *filtered noise* $n(t)$ is relatively low to justify the use of output signal-to-noise ratio as a measure of receiver performance.

The output signal-to-noise ratio depends, among other factors, on the type of modulation used in the transmitter and the type of demodulation used in the receiver. Thus it is informative to compare the output signal-to-noise ratios for different modulation-demodulation systems. However, for this comparison to be of meaningful value, it must be made on an equal basis as described here:

- ▶ The modulated signal $s(t)$ transmitted by each system has the same average power.
- ▶ The channel noise $w(t)$ has the same average power measured in the message bandwidth W .

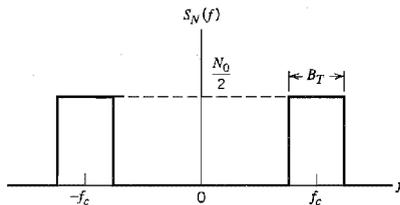


FIGURE 2.34 Idealized characteristic of band-pass filtered noise.

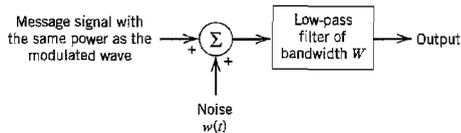


FIGURE 2.35 The baseband transmission model, assuming a message signal of bandwidth W , used for calculating the channel signal-to-noise ratio.

Accordingly, as a frame of reference we define the *channel signal-to-noise ratio*, $(\text{SNR})_C$, as *the ratio of the average power of the modulated signal to the average power of channel noise in the message bandwidth, both measured at the receiver input*. This definition is illustrated in Figure 2.35.

For the purpose of comparing different continuous-wave (CW) modulation systems, we *normalize* the receiver performance by dividing the output signal-to-noise ratio by the channel signal-to-noise ratio. We thus define a *figure of merit* for the receiver as follows:

$$\text{Figure of merit} = \frac{(\text{SNR})_O}{(\text{SNR})_C} \quad (2.81)$$

Clearly, the higher the value of the figure of merit, the better will the noise performance of the receiver be. The figure of merit may equal one, be less than one, or be greater than one, depending on the type of modulation used, which will become apparent from the discussion that follows.

2.11 Noise in Linear Receivers Using Coherent Detection

From Sections 2.2 and 2.3 we recall that the demodulation of an amplitude-modulated wave depends on whether the carrier is suppressed or not. When the carrier is suppressed we usually require the use of coherent detection, in which case the receiver is *linear*. On the other hand, when the amplitude modulation includes transmission of the carrier, demodulation is accomplished simply by using an envelope detector, in which case the receiver is *nonlinear*. In this section we study the effect of noise on the performance of a linear receiver. The more difficult case of a nonlinear receiver is deferred to Section 2.12.

Consider the case of DSB-SC modulation. Figure 2.36 shows the model of a DSB-SC receiver using a coherent detector. The use of coherent detection requires multiplication of the filtered signal $x(t)$ by a locally generated sinusoidal wave $\cos(2\pi f_c t)$ and then low-pass filtering the product. To simplify the analysis, we assume that the amplitude of the locally generated sinusoidal wave is unity. For this demodulation scheme to operate satisfactorily, however, it is necessary that the local oscillator be synchronized both in phase and in frequency with the oscillator generating the carrier wave in the transmitter. We assume that this synchronization has been achieved.

The DSB-SC component of the filtered signal $x(t)$ is expressed as

$$s(t) = CA_c \cos(2\pi f_c t)m(t) \quad (2.82)$$

where $A_c \cos(2\pi f_c t)$ is the sinusoidal carrier wave and $m(t)$ is the message signal. In the expression for $s(t)$ in Equation (2.82) we have included a *system-dependent scaling factor* C , the purpose of which is to ensure that the signal component $s(t)$ is measured in the same units as the additive noise component $n(t)$. We assume that $m(t)$ is the sample function of

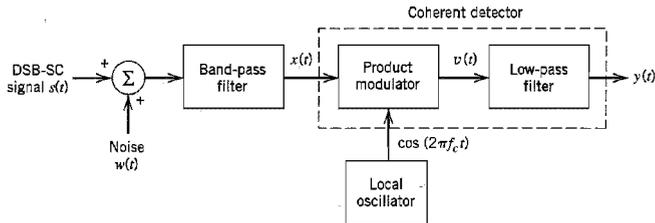


FIGURE 2.36 Model of DSB-SC receiver using coherent detection.

a stationary process of zero mean, whose power spectral density $S_M(f)$ is limited to a maximum frequency W ; that is, W is the *message bandwidth*. The average power P of the message signal is the total area under the curve of power spectral density, as shown by

$$P = \int_{-W}^W S_M(f) df \quad (2.83)$$

The carrier wave is statistically independent of the message signal. To emphasize this independence, the carrier should include a random phase that is uniformly distributed over 2π radians. In the defining equation for $s(t)$ this random phase angle has been omitted for convenience of presentation. Using the result of Example 1.7 of Chapter 1 on a modulated random process, we may express the average power of the DSB-SC modulated signal component $s(t)$ as $C^2 A_c^2 P/2$. With a noise spectral density of $N_0/2$, the average noise power in the message bandwidth W is equal to WN_0 . The channel signal-to-noise ratio of the DSB-SC modulation system is therefore

$$(\text{SNR})_{\text{C,DSB}} = \frac{C^2 A_c^2 P}{2WN_0} \quad (2.84)$$

where the constant C^2 in the numerator ensures that this ratio is dimensionless.

Next, we wish to determine the output signal-to-noise ratio of the system. Using the narrowband representation of the filtered noise $n(t)$, the total signal at the coherent detector input may be expressed as

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= CA_c \cos(2\pi f_c t)m(t) + n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \end{aligned} \quad (2.85)$$

where $n_I(t)$ and $n_Q(t)$ are the in-phase and quadrature components of $n(t)$ with respect to the carrier. The output of the product-modulator component of the coherent detector is therefore

$$\begin{aligned} v(t) &= x(t) \cos(2\pi f_c t) \\ &= \frac{1}{2}CA_c m(t) + \frac{1}{2}n_I(t) \\ &\quad + \frac{1}{2}[CA_c m(t) + n_I(t)] \cos(4\pi f_c t) - \frac{1}{2}n_Q(t) \sin(4\pi f_c t) \end{aligned}$$

The low-pass filter in the coherent detector in Figure 2.36 removes the high-frequency components of $v(t)$, yielding the receiver output

$$y(t) = \frac{1}{2}CA_c m(t) + \frac{1}{2}n_I(t) \quad (2.86)$$

Equation (2.86) indicates the following:

1. The message signal $m(t)$ and in-phase noise component $n_I(t)$ of the filtered noise $n(t)$ appear additively at the receiver output.

2. The quadrature component $n_o(t)$ of the noise $n(t)$ is completely rejected by the coherent detector.

These two results are independent of the input signal-to-noise ratio. Thus, coherent detection distinguishes itself from other demodulation techniques in an important property: The output message component is unmutated and the noise component always appears additively with the message, irrespective of the input signal-to-noise ratio.

The message signal component at the receiver output is $CA_r m(t)/2$. Therefore, the average power of this component may be expressed as $C^2 A_r^2 P/4$, where P is the average power of the original message signal $m(t)$ and C is the system-dependent scaling factor referred to earlier.

In the case of DSB-SC modulation, the band-pass filter in Figure 2.36 has a bandwidth B_T equal to $2W$ in order to accommodate the upper and lower sidebands of the modulated signal $s(t)$. It follows therefore that the average power of the filtered noise $n(t)$ is $2WN_0$. From the discussion of narrowband noise presented in Section 1.11, we know that the average power of the (low-pass) in-phase noise component $n_i(t)$ is the same as that of the (band-pass) filtered noise $n(t)$. Since from Equation (2.86) the noise component at the receiver output is $n_i(t)/2$, it follows that the average power of the noise at the receiver output is

$$\left(\frac{1}{2}\right)^2 2WN_0 = \frac{1}{2}WN_0$$

The output signal-to-noise for a DSB-SC receiver using coherent detection is therefore

$$\begin{aligned} (\text{SNR})_{\text{O, DSB-SC}} &= \frac{C^2 A_r^2 P/4}{WN_0/2} \\ &= \frac{C^2 A_r^2 P}{2WN_0} \end{aligned} \quad (2.87)$$

Using Equations (2.84) and (2.87), we obtain the figure of merit

$$\frac{(\text{SNR})_{\text{O}}}{(\text{SNR})_{\text{C}}} \Big|_{\text{DSB-SC}} = 1 \quad (2.88)$$

Note that the factor C^2 is common to both the output and channel signal-to-noise ratios, and therefore cancels out in evaluating the figure of merit.

Following through the noise analysis of a coherent detector for SSB, we find that, despite the fundamental differences between it and the coherent detector for DSB-SC modulation, the figure of merit is exactly the same for both of them; see Problem 2.49.

The important conclusions to be drawn from the discussions presented in this section and Problem 2.49 are two-fold:

1. For the same average transmitted or modulated signal power and the same average noise power in the message bandwidth, a coherent SSB receiver will have exactly the same output signal-to-noise ratio as a coherent DSB-SC receiver.
2. In both cases, the noise performance of the receiver is exactly the same as that obtained by simply transmitting the message signal in the presence of the same channel noise. The only effect of the modulation process is to translate the message signal to a different frequency band to facilitate its transmission over a band-pass channel.

Simply put, neither DSB-SC modulation nor SSB modulation offers the means for a trade-off between improved noise performance and increased channel bandwidth. This is a serious problem when high quality of reception is a requirement.

2.12 Noise in AM Receivers Using Envelope Detection

The next noise analysis we perform is for an amplitude modulation (AM) system using an envelope detector in the receiver, as shown in the model of Figure 2.37. In a full AM signal, both sidebands and the carrier wave are transmitted, as shown by

$$s(t) = A_c[1 + k_a m(t)] \cos(2\pi f_c t) \quad (2.89)$$

where $A_c \cos(2\pi f_c t)$ is the carrier wave, $m(t)$ is the message signal, and k_a is a constant that determines the percentage modulation. In the expression for the amplitude-modulated signal component $s(t)$ given in Equation (2.89), we see no need for the use of a scaling factor, because it is reasonable to assume that the carrier amplitude A_c has the same units as the additive noise component.

The average power of the carrier component in the AM signal $s(t)$ is $A_c^2/2$. The average power of the information-bearing component $A_c k_a m(t) \cos(2\pi f_c t)$ is $A_c^2 k_a^2 P/2$, where P is the average power of the message signal $m(t)$. The average power of the full AM signal $s(t)$ is therefore equal to $A_c^2(1 + k_a^2 P)/2$. As for the DSB-SC system, the average power of noise in the message bandwidth is WN_0 . The channel signal-to-noise ratio for AM is therefore

$$(\text{SNR})_{C,AM} = \frac{A_c^2(1 + k_a^2 P)}{2WN_0} \quad (2.90)$$

To evaluate the output signal-to-noise ratio, we first represent the filtered noise $n(t)$ in terms of its in-phase and quadrature components. We may therefore define the filtered signal $x(t)$ applied to the envelope detector in the receiver model of Figure 2.37 as follows:

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= [A_c + A_c k_a m(t) + n_I(t)] \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \end{aligned} \quad (2.91)$$

It is informative to represent the components that comprise the signal $x(t)$ by means of phasors, as in Figure 2.38a. From this phasor diagram, the receiver output is readily obtained as

$$\begin{aligned} y(t) &= \text{envelope of } x(t) \\ &= \{[A_c + A_c k_a m(t) + n_I(t)]^2 + n_Q^2(t)\}^{1/2} \end{aligned} \quad (2.92)$$

The signal $y(t)$ defines the output of an ideal envelope detector. The phase of $x(t)$ is of no interest to us, because an ideal envelope detector is totally insensitive to variations in the phase of $x(t)$.

The expression defining $y(t)$ is somewhat complex and needs to be simplified in some manner to permit the derivation of insightful results. Specifically, we would like to approximate the output $y(t)$ as the sum of a message term plus a term due to noise. In general, this is quite difficult to achieve. However, when the average carrier power is large com-

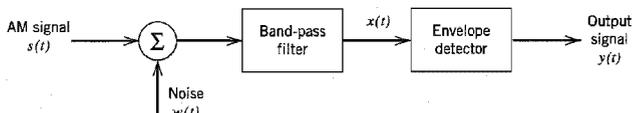


FIGURE 2.37 Model of AM receiver.

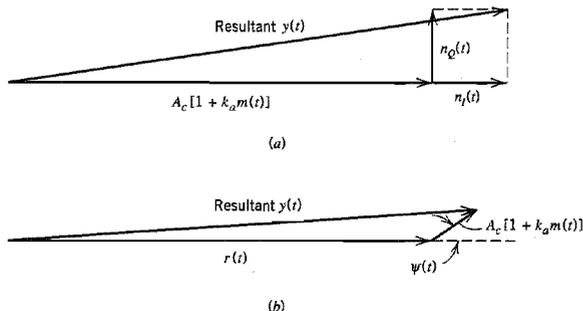


FIGURE 2.38 (a) Phasor diagram for AM wave plus narrowband noise for the case of high carrier-to-noise ratio. (b) Phasor diagram for AM wave plus narrowband noise for the case of low carrier-to-noise ratio.

pared with the average noise power, so that the receiver is operating satisfactorily, then the signal term $A_c[1 + k_a m(t)]$ will be large compared with the noise terms $n_1(t)$ and $n_0(t)$, at least most of the time. Then we may approximate the output $y(t)$ as (see Problem 2.51):

$$y(t) \approx A_c + A_c k_a m(t) + n_1(t) \quad (2.93)$$

The presence of the DC or constant term A_c in the envelope detector output $y(t)$ of Equation (2.93) is due to demodulation of the transmitted carrier wave. We may ignore this term, however, because it bears no relation whatsoever to the message signal $m(t)$. In any case, it may be removed simply by means of a blocking capacitor. Thus if we neglect the DC term A_c in Equation (2.93), we find that the remainder has, except for scaling factors, a form similar to the output of a DSB-SC receiver using coherent detection. Accordingly, the output signal-to-noise ratio of an AM receiver using an envelope detector is approximately

$$(\text{SNR})_{O,AM} \approx \frac{A_c^2 k_a^2 P}{2WN_0} \quad (2.94)$$

Equation (2.94) is, however, valid only if the following two conditions are satisfied:

1. The average noise power is small compared to the average carrier power at the envelope detector input.
2. The amplitude sensitivity k_a is adjusted for a percentage modulation less than or equal to 100 percent.

Using Equations (2.90) and (2.94), we obtain the following figure of merit for amplitude modulation:

$$\frac{(\text{SNR})_O}{(\text{SNR})_C} \Big|_{AM} \approx \frac{k_a^2 P}{1 + k_a^2 P} \quad (2.95)$$

Thus, whereas the figure of merit of a DSB-SC receiver or that of an SSB receiver using coherent detection is always unity, the corresponding figure of merit of an AM receiver using envelope detection is always less than unity. In other words, *the noise performance of a full AM receiver is always inferior to that of a DSB-SC receiver*. This is due to the

wastage of transmitter power, which results from transmitting the carrier as a component of the AM wave.

► EXAMPLE 2.4 Single-Tone Modulation

Consider the special case of a sinusoidal wave of frequency f_m and amplitude A_m as the modulating wave, as shown by

$$m(t) = A_m \cos(2\pi f_m t)$$

The corresponding AM wave is

$$s(t) = A_c [1 + \mu \cos(2\pi f_m t)] \cos(2\pi f_c t)$$

where $\mu = k_a A_m$ is the modulation factor. The average power of the modulating wave $m(t)$ is (assuming a load resistor of 1 ohm)

$$P = \frac{1}{2} A_m^2$$

Therefore, using Equation (2.95), we get

$$\begin{aligned} \frac{(\text{SNR})_O}{(\text{SNR})_C} \Big|_{\text{AM}} &= \frac{\frac{1}{2} k_a^2 A_m^2}{1 + \frac{1}{2} k_a^2 A_m^2} \\ &= \frac{\mu^2}{2 + \mu^2} \end{aligned} \quad (2.96)$$

When $\mu = 1$, which corresponds to 100 percent modulation, we get a figure of merit equal to 1/3. This means that, other factors being equal, an AM system (using envelope detection) must transmit three times as much average power as a suppressed-carrier system (using coherent detection) to achieve the same quality of noise performance. ◀

■ THRESHOLD EFFECT

When the carrier-to-noise ratio is small compared with unity, the noise term dominates and the performance of the envelope detector changes completely from that just described. In this case it is more convenient to represent the narrowband noise $n(t)$ in terms of its envelope $r(t)$ and phase $\psi(t)$, as shown by

$$n(t) = r(t) \cos[2\pi f_c t + \psi(t)] \quad (2.97)$$

The corresponding phasor diagram for the detector input $x(t) = s(t) + n(t)$ is shown in Figure 2.38*b*, where we have used the noise envelope as reference, because it is now the dominant term. To the noise phasor $r(t)$ we have added a phasor representing the signal term $A_c[1 + k_a m(t)]$, with the angle between them being equal to the phase $\psi(t)$ of the noise $n(t)$. In Figure 2.38*b* it is assumed that the carrier-to-noise ratio is so low that the carrier amplitude A_c is small compared with the noise envelope $r(t)$, at least most of the time. Then we may neglect the quadrature component of the signal with respect to the noise, and thus find from Figure 2.38*b* that the envelope detector output is

$$y(t) \approx r(t) + A_c \cos[\psi(t)] + A_c k_a m(t) \cos[\psi(t)] \quad (2.98)$$

This relation reveals that when the carrier-to-noise ratio is low, the detector output has no component strictly proportional to the message signal $m(t)$. The last term of the expression defining $y(t)$ contains the message signal $m(t)$ multiplied by noise in the form of

$\cos[\psi(t)]$. From Section 1.11 we recall that the phase $\psi(t)$ of the narrowband noise $n(t)$ is uniformly distributed over 2π radians. It follows therefore that we have a complete loss of information in that the detector output does not contain the message signal $m(t)$ at all. The loss of a message in an envelope detector that operates at a low carrier-to-noise ratio is referred to as the *threshold effect*. By *threshold* we mean a *value of the carrier-to-noise ratio below which the noise performance of a detector deteriorates much more rapidly than proportionately to the carrier-to-noise ratio*. It is important to recognize that every nonlinear detector (e.g., envelope detector) exhibits a threshold effect. On the other hand, such an effect does *not* arise in a coherent detector.

A rigorous mathematical analysis of the threshold effect for the general case of an AM wave is beyond the scope of this book. In the next subsection we simplify matters by considering the case of an unmodulated carrier. Despite this simplification, we can still develop a great deal of insight into the threshold effect experienced in an envelope detector.

General Formula for $(\text{SNR})_O$ in Envelope Detection*

Consider an envelope detector whose input signal is defined by

$$x(t) = A_c \cos(2\pi f_c t) + n(t) \quad (2.99)$$

where $A_c \cos(2\pi f_c t)$ is the unmodulated carrier and $n(t)$ is the sample function of band-limited, zero-mean, white Gaussian noise $N(t)$. The power spectral density of $N(t)$ is

$$S_N(f) = \begin{cases} \frac{N_0}{2} & \text{for } |f - f_c| \leq W \\ 0 & \text{otherwise} \end{cases} \quad (2.100)$$

Representing the narrowband noise $n(t)$ in terms of its in-phase component $n_I(t)$ and quadrature component $n_Q(t)$, we may express the noisy signal at the detector input as

$$x(t) = (A_c + n_I(t)) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \quad (2.101)$$

The noise components $n_I(t)$ and $n_Q(t)$ are zero-mean, jointly Gaussian, mutually independent low-pass random processes with identical power spectral densities (see Equation 1.101):

$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c) & \text{for } |f| \leq W \\ 0 & \text{otherwise} \end{cases} \quad (2.102)$$

For the problem at hand, the input signal consists of an unmodulated carrier with average power equal to $A_c^2/2$. The average noise power at the detector input is

$$\sigma_N^2 = 2WN_0 \quad (2.103)$$

The *carrier-to-noiser ratio* is therefore defined by

$$\begin{aligned} \rho &= \frac{A_c^2/2}{\sigma_N^2} \\ &= \frac{A_c^2}{4WN_0} \end{aligned} \quad (2.104)$$

We may think of ρ as an input signal-to-noise ratio for the problem described herein.

*A reader who is not interested in the mathematical details of how noise affects the envelope detection of an AM signal may skip the material up to Eq. (2.124) and read the two limiting cases of the formula in that equation.

However, determination of the output signal-to-noise ratio is a more difficult undertaking because the envelope detector output

$$y(t) = \sqrt{(A_c + n_r(t))^2 + n_Q^2(t)} \quad (2.105)$$

is a nonlinear combination of signal and noise terms. With no clear-cut separation between signal and noise at the detector output $y(t)$, how then do we isolate the contribution of the signal $s(t)$ to $y(t)$ from the contribution due to the noise $n(t)$? To resolve this issue, we adopt a *heuristic* approach based on signal averaging. Specifically, we introduce the following two definitions:

1. The *mean output signal*, s_o , is the difference between the expectation of $y(t)$ in the combined presence of signal and noise and the expectation of $y(t)$ in the presence of noise alone, as shown by

$$s_o = E[y(t)] - E[y_o(t)] \quad (2.106)$$

where $y(t)$ is itself defined by Equation (2.105) and $y_o(t)$ is defined by

$$y_o(t) = \sqrt{n_r^2(t) + n_Q^2(t)} \quad (2.107)$$

2. The *mean output noise power* is the difference between the mean-square value of the detector output $y(t)$ and the square of the mean value of $y(t)$, as shown by

$$\text{var}[y(t)] = E[y^2(t)] - (E[y(t)])^2 \quad (2.108)$$

On this basis, we define the *output signal-to-noise ratio* as

$$(\text{SNR})_O = \frac{s_o^2}{\text{var}[y(t)]} \quad (2.109)$$

From Section 1.12, we recall that the envelope detector output due to noise alone is Rayleigh distributed; that is

$$f_{Y_o}(y) = \begin{cases} \frac{y}{\sigma_N^2} \exp\left(-\frac{y^2}{2\sigma_N^2}\right), & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.110)$$

The expectation of $y_o(t)$ is therefore

$$\begin{aligned} E[y_o(t)] &= \int_{-\infty}^{\infty} y f_{Y_o}(y) dy \\ &= \int_0^{\infty} \frac{y^2}{\sigma_N^2} \exp\left(-\frac{y^2}{2\sigma_N^2}\right) dy \end{aligned} \quad (2.111)$$

From the definition of the *gamma function* for real positive values of the argument x , we have

$$\Gamma(x) = \int_0^{\infty} z^{x-1} \exp(-z) dz \quad (2.112)$$

We may therefore rewrite Equation (2.111) as

$$\begin{aligned} E[y_o(t)] &= \sqrt{2}\sigma_N \Gamma\left(\frac{3}{2}\right) \\ &= \sqrt{\frac{\pi}{2}} \sigma_N \end{aligned} \quad (2.113)$$

where we have used the value $\Gamma(3/2) = \sqrt{\pi}/2$. To calculate the mean signal s_o at the detector output, we also need the expectation of $y(t)$. Due to the combined presence of signal and noise, we recall from Section 1.13 that $y(t)$ is Rician distributed, as shown by

$$f_Y(y) = \begin{cases} \frac{y}{\sigma_N^2} \exp\left(-\frac{y^2 + A_c^2}{2\sigma_N^2}\right) I_0\left(\frac{A_c y}{\sigma_N^2}\right) & \text{for } y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.114)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of zero order (see Appendix 3). Hence,

$$E[y(t)] = \int_0^\infty \frac{y^2}{\sigma_N^2} \exp\left(-\frac{y^2 + A_c^2}{2\sigma_N^2}\right) I_0\left(\frac{A_c y}{\sigma_N^2}\right) dy \quad (2.115)$$

Putting $A_c y/\sigma_N^2 = u$ and recognizing that $\rho = A_c^2/2\sigma_N^2$, we may recast this expectation in the form

$$E[y(t)] = \frac{\sigma_N}{(2\rho)^{3/2}} \exp(-\rho) \int_0^\infty u^2 \exp\left(-\frac{u^2}{4\rho}\right) I_0(u) du \quad (2.116)$$

The integral in Equation (2.116) can be written in a concise form by using *confluent hypergeometric functions*; see Appendix 4. In particular, using the integral representation

$$\int_0^\infty u^{m-1} \exp(-b^2 u^2) I_0(u) du = \frac{\Gamma(m/2)}{2b^m} \left({}_1F_1\left(\frac{m}{2}; 1; \frac{1}{4b^2}\right) \right) \quad (2.117)$$

with $m = 3$, $\Gamma(m/2) = \sqrt{\pi}/2$ and $b^2 = 1/4\rho$, we may express the expectation of $y(t)$ in terms of the confluent hypergeometric function ${}_1F_1(3/2; 1; \rho)$ as

$$E[y(t)] = \sqrt{\frac{\pi}{2}} \sigma_N \exp(-\rho) \left({}_1F_1\left(\frac{3}{2}; 1; \rho\right) \right) \quad (2.118)$$

We may further simplify matters by using the following identity:

$$\exp(-u) ({}_1F_1(\alpha; \beta; u)) = {}_1F_1(\beta - \alpha; \beta; -u) \quad (2.119)$$

and so finally express the expectation of $y(t)$ in the concise form

$$E[y(t)] = \sqrt{\frac{\pi}{2}} \sigma_N \left({}_1F_1\left(-\frac{1}{2}; 1; -\rho\right) \right) \quad (2.120)$$

Thus using Equations (2.113) and (2.120) in Equation (2.106) yields the mean output signal as

$$s_o = \sqrt{\frac{\pi}{2}} \sigma_N \left({}_1F_1\left(-\frac{1}{2}; 1; -\rho\right) - 1 \right) \quad (2.121)$$

whose dependence on the standard deviation σ_N of the noise $n(t)$ is testimony to the intermingling of signal and noise at the detector output.

Following a similar procedure, we may express the mean-square value of the detector output $y(t)$ as

$$\begin{aligned} E[y^2(t)] &= \int_0^\infty \frac{y^3}{\sigma_N^2} \exp\left(-\frac{y^2 + A_c^2}{2\sigma_N^2}\right) I_0\left(\frac{A_c y}{\sigma_N^2}\right) dy \\ &= 2\sigma_N^2 ({}_1F_1(-1; 1; -\rho)) \end{aligned} \quad (2.122)$$

Hence using Equations (2.120) and (2.122) in Equation (2.108) yields the mean output noise power as

$$\text{var}[y(t)] = 2\sigma_N^2 \left({}_1F_1(-1; 1; -\rho) - \frac{\pi}{4} \left({}_1F_1\left(-\frac{1}{2}; 1; -\rho\right) \right)^2 \right) \quad (2.123)$$

Finally, using Equations (2.121) and (2.123) in Equation (2.109) yields the output signal-to-noise ratio for the envelope detection problem at hand as

$$(\text{SNR})_O = \frac{\left({}_1F_1\left(-\frac{1}{2}; 1; -\rho\right) - 1 \right)^2}{(4/\pi) \left({}_1F_1(-1; 1; -\rho) \right) - \left({}_1F_1\left(-\frac{1}{2}; 1; -\rho\right) \right)^2} \quad (2.124)$$

Equation (2.124) is the general formula for the output signal-to-noise of an envelope detector whose input consists of an unmodulated carrier and band-limited, white Gaussian noise. Two limiting cases of this general formula are of particular interest:

1. *Large carrier-to-noise ratio.* For large ρ , we may use the following asymptotic formula (see Appendix 4)

$${}_1F_1\left(-\frac{1}{2}; 1; -\rho\right) \approx 2\sqrt{\frac{\rho}{\pi}} \quad \text{for } \rho \rightarrow \infty \quad (2.125)$$

Moreover, the following identity

$${}_1F_1(-1; 1; -\rho) = 1 + \rho \quad (2.126)$$

holds exactly for all ρ . Accordingly, the use of Equations (2.125) and (2.126) in Equation (2.124) yields the following approximate formula for the output signal-to-noise ratio:

$$(\text{SNR})_O \approx \rho \quad \text{for } \rho \rightarrow \infty \quad (2.127)$$

where we have ignored contributions due to $\rho^{1/2}$ and ρ^0 in the numerator of Equation (2.124) as being subdominant compared to ρ for large ρ . Equation (2.127) shows that for large carrier-to-noise ρ the envelope detector behaves like a coherent detector, in that the output signal-to-noise ratio is proportional to the input signal-to-noise ratio.

2. *Small carrier-to-noise ratio.* For small ρ , we have (see Appendix 4)

$${}_1F_1(a; c; -\rho) \approx 1 - \frac{a}{c} \rho \quad \text{for } \rho \rightarrow 0 \quad (2.128)$$

Hence, using this asymptotic formula, we may approximate the output signal-to-noise ratio for small ρ as

$$\begin{aligned} (\text{SNR})_O &\approx \frac{\pi\rho^2}{16 - 4\pi} \\ &\approx 0.91\rho^2 \quad \text{for } \rho \rightarrow 0 \end{aligned} \quad (2.129)$$

where, in the denominator, we have ignored contributions due to ρ and ρ^2 as being subdominant compared to ρ^0 for small ρ . Equation (2.129) shows that for a small carrier-to-noise ratio, the output signal-to-noise ratio of the envelope detector is proportional to the squared input signal-to-noise ratio.

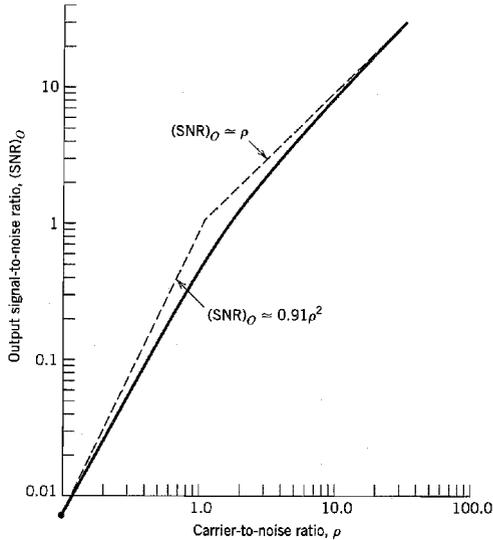


FIGURE 2.39 Output signal-to-noise ratio of an envelope detector for varying carrier-to-noise ratio.

The conclusions drawn from the two limiting cases considered herein are that an envelope detector favors strong signals and penalizes weak signals. The phenomenon of weak signals being penalized by the detector is referred to as *weak signal suppression*, which is a manifestation of the threshold effect.

Using the formula of Equation (2.124), in Figure 2.39 we have plotted the output signal-to-noise ratio $(\text{SNR})_O$ of the envelope detector versus the carrier-to-noise ratio ρ using tabulated values of confluent hypergeometric functions. This figure also includes the two asymptotes for large ρ and small ρ . From Figure 2.39 we see that the output signal-to-noise ratio deviates from a linear behavior around a carrier-to-noise ratio of 10 dB (i.e., $\rho \approx 10$).

2.13 Noise in FM Receivers

Finally, we turn our attention to the noise analysis of a frequency modulation (FM) system, for which we use the receiver model shown in Figure 2.40. As before, the noise $w(t)$ is modeled as white Gaussian noise of zero mean and power spectral density $N_0/2$. The

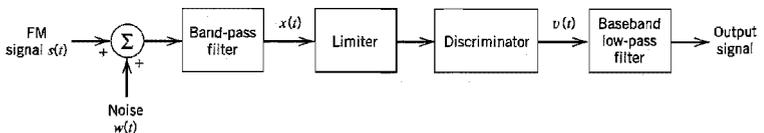


FIGURE 2.40 Model of an FM receiver.

received FM signal $s(t)$ has a carrier frequency f_c and transmission bandwidth B_T , such that only a negligible amount of power lies outside the frequency band $f_c \pm B_T/2$ for positive frequencies, and similarly for negative frequencies.

As in the AM case, the band-pass filter has a midband frequency f_c and bandwidth B_T and therefore passes the FM signal essentially without distortion. Ordinarily, B_T is small compared with the midband frequency f_c , so that we may use the narrowband representation for $n(t)$, the filtered version of channel noise $w(t)$, in terms of its in-phase and quadrature components.

In an FM system, the message signal is transmitted by variations of the instantaneous frequency of a sinusoidal carrier wave, and its amplitude is maintained constant. Therefore, any variations of the carrier amplitude at the receiver input must result from noise or interference. The amplitude *limiter*, following the band-pass filter in the receiver model of Figure 2.40, is used to remove amplitude variations by clipping the modulated wave at the filter output almost to the zero axis. The resulting rectangular wave is rounded off by another band-pass filter that is an integral part of the limiter, thereby suppressing harmonics of the carrier frequency. Thus, the filter output is again sinusoidal, with an amplitude that is practically independent of the carrier amplitude at the receiver input.

The discriminator in the model of Figure 2.40 consists of two components:

1. A *slope network* or *differentiator* with a purely imaginary frequency response that varies linearly with frequency. It produces a hybrid-modulated wave in which both amplitude and frequency vary in accordance with the message signal.
2. An envelope detector that recovers the amplitude variation and thus reproduces the message signal.

The slope network and envelope detector are usually implemented as integral parts of a single physical unit.

The *postdetection filter*, labeled “baseband low-pass filter” in Figure 2.40, has a bandwidth that is just large enough to accommodate the highest frequency component of the message signal. This filter removes the out-of-band components of the noise at the discriminator output and thereby keeps the effect of the output noise to a minimum.

The filtered noise $n(t)$ at the band-pass filter output in Figure 2.40 is defined in terms of its in-phase and quadrature components by

$$n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t)$$

Equivalently, we may express $n(t)$ in terms of its envelope and phase as

$$n(t) = r(t) \cos[(2\pi f_c t) + \psi(t)] \quad (2.130)$$

where the envelope is

$$r(t) = [n_I^2(t) + n_Q^2(t)]^{1/2} \quad (2.131)$$

and the phase is

$$\psi(t) = \tan^{-1} \left[\frac{n_Q(t)}{n_I(t)} \right] \quad (2.132)$$

The envelope $r(t)$ is Rayleigh distributed, and the phase $\psi(t)$ is uniformly distributed over 2π radians (see Section 1.12).

The incoming FM signal $s(t)$ is defined by

$$s(t) = A_c \cos \left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right] \quad (2.133)$$

where A_c is the carrier amplitude, f_c is the carrier frequency, k_f is the frequency sensitivity, and $m(t)$ is the message signal. Note that, as with the standard AM, in FM there is no need to introduce a scaling factor in the definition of the modulated signal $s(t)$, since it is reasonable to assume that its amplitude A_c has the same units as the additive noise component $n(t)$. To proceed, we define

$$\phi(t) = 2\pi k_f \int_0^t m(\tau) d\tau \quad (2.134)$$

We may thus express $s(t)$ in the simple form

$$s(t) = A_c \cos[2\pi f_c t + \phi(t)] \quad (2.135)$$

The noisy signal at the band-pass filter output is therefore

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A_c \cos[2\pi f_c t + \phi(t)] + r(t) \cos[2\pi f_c t + \psi(t)] \end{aligned} \quad (2.136)$$

It is informative to represent $x(t)$ by means of a phasor diagram, as in Figure 2.41. In this diagram we have used the signal term as reference. The phase $\theta(t)$ of the resultant phasor representing $x(t)$ is obtained directly from Figure 2.41 as

$$\theta(t) = \phi(t) + \tan^{-1} \left\{ \frac{r(t) \sin[\psi(t) - \phi(t)]}{A_c + r(t) \cos[\psi(t) - \phi(t)]} \right\} \quad (2.137)$$

The envelope of $x(t)$ is of no interest to us, because any envelope variations at the band-pass filter output are removed by the limiter.

Our motivation is to determine the error in the instantaneous frequency of the carrier wave caused by the presence of the filtered noise $n(t)$. With the discriminator assumed ideal, its output is proportional to $\theta'(t)/2\pi$ where $\theta'(t)$ is the derivative of $\theta(t)$ with respect to time. In view of the complexity of the expression defining $\theta(t)$, however, we need to make certain simplifying approximations, so that our analysis may yield useful results.

We assume that the carrier-to-noise ratio measured at the discriminator input is large compared with unity. Let R denote the random variable obtained by observing (at some fixed time) the envelope process with sample function $r(t)$ [due to the noise $n(t)$]. Then, at least most of the time, the random variable R is small compared with the carrier amplitude A_c , and so the expression for the phase $\theta(t)$ simplifies considerably as follows:

$$\theta(t) \approx \phi(t) + \frac{r(t)}{A_c} \sin[\psi(t) - \phi(t)] \quad (2.138)$$

or, using the expression for $\phi(t)$ given in Equation (2.134),

$$\theta(t) \approx 2\pi k_f \int_0^t m(\tau) d\tau + \frac{r(t)}{A_c} \sin[\psi(t) - \phi(t)] \quad (2.139)$$

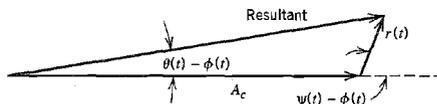


FIGURE 2.41 Phasor diagram for FM wave plus narrowband noise for the case of high carrier-to-noise ratio.

The discriminator output is therefore

$$\begin{aligned} v(t) &= \frac{1}{2\pi} \frac{d\theta(t)}{dt} \\ &= k_f m(t) + n_d(t) \end{aligned} \quad (2.140)$$

where the noise term $n_d(t)$ is defined by

$$n_d(t) = \frac{1}{2\pi A_c} \frac{d}{dt} \{r(t) \sin[\psi(t) - \phi(t)]\} \quad (2.141)$$

We thus see that provided the carrier-to-noise ratio is high, the discriminator output $v(t)$ consists of the original message signal $m(t)$ multiplied by the constant factor k_f , plus an additive noise component $n_d(t)$. Accordingly, we may use the output signal-to-noise ratio as previously defined to assess the quality of performance of the FM receiver. Before doing this, however, it is instructive to see if we can simplify the expression defining the noise $n_d(t)$.

From the phasor diagram of Figure 2.41, we note that the effect of variations in the phase $\psi(t)$ of the narrowband noise appear referred to the signal term $\phi(t)$. We know that the phase $\psi(t)$ is uniformly distributed over 2π radians. It would therefore be tempting to assume that the phase difference $\psi(t) - \phi(t)$ is also uniformly distributed over 2π radians. If such an assumption were true, then the noise $n_d(t)$ at the discriminator output would be independent of the modulating signal and would depend only on the characteristics of the carrier and narrowband noise. Theoretical considerations show that this assumption is justified provided that the carrier-to-noise ratio is high.⁹ Then we may simplify Equation (2.141) as:

$$n_d(t) = \frac{1}{2\pi A_c} \frac{d}{dt} \{r(t) \sin[\psi(t)]\} \quad (2.142)$$

However, from the defining equations for $r(t)$ and $\psi(t)$, we note that the quadrature component $n_Q(t)$ of the filtered noise $n(t)$ is

$$n_Q(t) = r(t) \sin[\psi(t)] \quad (2.143)$$

Therefore, we may rewrite Equation (2.142) as

$$n_d(t) = \frac{1}{2\pi A_c} \frac{dn_Q(t)}{dt} \quad (2.144)$$

This means that *the additive noise $n_d(t)$ appearing at the discriminator output is determined effectively by the carrier amplitude A_c and the quadrature component $n_Q(t)$ of the narrowband noise $n(t)$.*

The output signal-to-noise ratio is defined as the ratio of the average output signal power to the average output noise power. From Equation (2.140), we see that the message component in the discriminator output, and therefore the low-pass filter output, is $k_f m(t)$. Hence, the average output signal power is equal to $k_f^2 P$, where P is the average power of the message signal $m(t)$.

To determine the average output noise power, we note that the noise $n_d(t)$ at the discriminator output is proportional to the time derivative of the quadrature noise component $n_Q(t)$. Since the differentiation of a function with respect to time corresponds to multiplication of its Fourier transform by $j2\pi f$, it follows that we may obtain the noise process $n_d(t)$ by passing $n_Q(t)$ through a linear filter with a frequency response equal to

$$\frac{j2\pi f}{2\pi A_c} = \frac{jf}{A_c}$$

This means that the power spectral density $S_{N_d}(f)$ of the noise $n_d(t)$ is related to the power spectral density $S_{N_Q}(f)$ of the quadrature noise component $n_Q(t)$ as follows:

$$S_{N_d}(f) = \frac{f^2}{A_c^2} S_{N_Q}(f) \quad (2.145)$$

With the band-pass filter in the receiver model of Figure 2.40 having an ideal frequency response characterized by bandwidth B_T and midband frequency f_c , it follows that the narrowband noise $n(t)$ will have a power spectral density characteristic that is similarly shaped. This means that the quadrature component $n_Q(t)$ of the narrowband noise $n(t)$ will have the ideal low-pass characteristic shown in Figure 2.42a. The corresponding power spectral density of the noise $n_d(t)$ is shown in Figure 2.42b; that is,

$$S_{N_d}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2}, & |f| \leq \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases} \quad (2.146)$$

In the receiver model of Figure 2.40, the discriminator output is followed by a low-pass filter with a bandwidth equal to the message bandwidth W . For wideband FM, we usually find that W is smaller than $B_T/2$, where B_T is the transmission bandwidth of the FM signal. This means that the out-of-band components of noise $n_d(t)$ will be rejected. Therefore, the power spectral density $S_{N_o}(f)$ of the noise $n_o(t)$ appearing at the receiver output is defined by

$$S_{N_o}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2}, & |f| \leq W \\ 0, & \text{otherwise} \end{cases} \quad (2.147)$$

as shown in Figure 2.42c. The average output noise power is determined by integrating the power spectral density $S_{N_o}(f)$ from $-W$ to W . We thus get the following result:

$$\begin{aligned} \text{Average power of output noise} &= \frac{N_0}{A_c^2} \int_{-W}^W f^2 df \\ &= \frac{2N_0 W^3}{3A_c^2} \end{aligned} \quad (2.148)$$

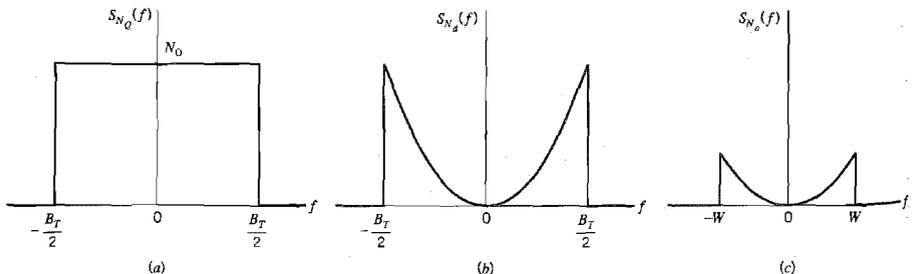


FIGURE 2.42 Noise analysis of FM receiver. (a) Power spectral density of quadrature component $n_Q(t)$ of narrowband noise $n(t)$. (b) Power spectral density of noise $n_d(t)$ at the discriminator output. (c) Power spectral density of noise $n_o(t)$ at the receiver output.

Note that the average output noise power is inversely proportional to the average carrier power $A_c^2/2$. Accordingly, in an FM system, increasing the carrier power has a *noise-quieting effect*.

Earlier we determined the average output signal power as $k_f^2 P$. Therefore, provided the carrier-to-noise ratio is high, we may divide this average output signal power by the average output noise power of Equation (2.148) to obtain the output signal-to-noise ratio

$$(\text{SNR})_{O,\text{FM}} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3} \quad (2.149)$$

The average power in the modulated signal $s(t)$ is $A_c^2/2$, and the average noise power in the message bandwidth is WN_0 . Thus the channel signal-to-noise ratio is

$$(\text{SNR})_{C,\text{FM}} = \frac{A_c^2}{2WN_0} \quad (2.150)$$

Dividing the output signal-to-noise ratio by the channel signal-to-noise ratio, we get the following figure of merit for frequency modulation:

$$\frac{(\text{SNR})_O}{(\text{SNR})_C} \Big|_{\text{FM}} = \frac{3k_f^2 P}{W^2} \quad (2.151)$$

From Section 2.7 we recall that the frequency deviation Δf is proportional to the frequency sensitivity k_f of the modulator. Also, by definition, the deviation ratio D is equal to the frequency deviation Δf divided by the message bandwidth W . In other words, the deviation ratio D is proportional to the ratio $k_f P^{1/2}/W$. It follows therefore from Equation (2.151) that the figure of merit of a wideband FM system is a quadratic function of the deviation ratio. Now, in wideband FM, the transmission bandwidth B_T is approximately proportional to the deviation ratio D . Accordingly, we may state that *when the carrier-to-noise ratio is high, an increase in the transmission bandwidth B_T provides a corresponding quadratic increase in the output signal-to-noise ratio or figure of merit of the FM system*. The important point to note from this statement is that, unlike amplitude modulation, the use of frequency modulation does provide a practical mechanism for the exchange of increased transmission bandwidth for improved noise performance.

► EXAMPLE 2.5 Single-Tone Modulation

Consider the case of a sinusoidal wave of frequency f_m as the modulating signal, and assume a peak frequency deviation Δf . The modulated FM signal is thus defined by

$$s(t) = A_c \cos \left[2\pi f_c t + \frac{\Delta f}{f_m} \sin(2\pi f_m t) \right]$$

Therefore, we may write

$$2\pi k_f \int_0^t m(\tau) d\tau = \frac{\Delta f}{f_m} \sin(2\pi f_m t)$$

Differentiating both sides with respect to time and solving for $m(t)$, we get

$$m(t) = \frac{\Delta f}{k_f} \cos(2\pi f_m t)$$

Hence, the average power of the message signal $m(t)$, developed across a 1-ohm load, is

$$P = \frac{(\Delta f)^2}{2k_f^2}$$

Substituting this result into the formula for the output signal-to-noise ratio given in Equation (2.149), we get

$$\begin{aligned} (\text{SNR})_{O,FM} &= \frac{3A_c^2(\Delta f)^2}{4N_0W^3} \\ &= \frac{3A_c^2\beta^2}{4N_0W} \end{aligned}$$

where $\beta = \Delta f/W$ is the modulation index. Using Equation (2.151) to evaluate the corresponding figure of merit, we get

$$\begin{aligned} \frac{(\text{SNR})_O}{(\text{SNR})_C} \Big|_{FM} &= \frac{3}{2} \left(\frac{\Delta f}{W} \right)^2 \\ &= \frac{3}{2} \beta^2 \end{aligned} \quad (2.152)$$

It is important to note that the modulation index $\beta = \Delta f/W$ is determined by the bandwidth W of the postdetection low-pass filter and is not related to the sinusoidal message frequency f_m , except insofar as this filter is usually chosen so as to pass the spectrum of the desired message; this is merely a matter of consistent design. For a specified system bandwidth W , the sinusoidal message frequency f_m may lie anywhere between 0 and W and would yield the same output signal-to-noise ratio.

It is of particular interest to compare the noise performance of AM and FM systems. An insightful way of making this comparison is to consider the figures of merit of the two systems based on a sinusoidal modulating signal. For an AM system operating with a sinusoidal modulating signal and 100 percent modulation, we have (from Example 2.4):

$$\frac{(\text{SNR})_O}{(\text{SNR})_C} \Big|_{AM} = \frac{1}{3}$$

Comparing this figure of merit with the corresponding result described in Equation (2.152) for an FM system, we see that the use of frequency modulation offers the possibility of improved noise performance over amplitude modulation when

$$\frac{3}{2}\beta^2 > \frac{1}{3}$$

that is,

$$\beta > \frac{\sqrt{2}}{3} = 0.471$$

We may therefore consider $\beta = 0.5$ as defining roughly *the transition between narrowband FM and wideband FM*. This statement, based on noise considerations, further confirms a similar observation that was made in Section 2.7 when considering the bandwidth of FM waves. ◀

■ CAPTURE EFFECT

The inherent ability of an FM system to minimize the effects of unwanted signals (e.g., noise, as just discussed) also applies to *interference* produced by another frequency-modulated signal whose frequency content is close to the carrier frequency of the desired FM wave. However, interference suppression in an FM receiver works well only when the interference is weaker than the desired FM input. When the interference is the stronger one of the two, the receiver locks onto the stronger signal and thereby suppresses the

desired FM input. When they are of nearly equal strength, the receiver fluctuates back and forth between them. This phenomenon is known as the *capture effect*, which describes another distinctive characteristic of frequency modulation.

■ FM THRESHOLD EFFECT

The formula of Equation (2.149), defining the output signal-to-noise ratio of an FM receiver, is valid only if the carrier-to-noise ratio, measured at the discriminator input, is high compared with unity. It is found experimentally that as the input noise power is increased so that the carrier-to-noise ratio is decreased, the FM receiver *breaks*. At first, individual clicks are heard in the receiver output, and as the carrier-to-noise ratio decreases still further, the clicks rapidly merge into a *crackling* or *sputtering sound*. Near the breaking point, Equation (2.149) begins to fail by predicting values of output signal-to-noise ratio larger than the actual ones. This phenomenon is known as the *threshold effect*.¹⁰ The threshold is defined as the minimum carrier-to-noise ratio yielding an FM improvement that is not significantly deteriorated from the value predicted by the usual signal-to-noise formula assuming a small noise power.

For a qualitative discussion of the FM threshold effect, consider first the case when there is a no signal present, so that the carrier wave is unmodulated. Then the composite signal at the frequency discriminator input is

$$x(t) = [A_c + n_I(t)] \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t) \quad (2.153)$$

where $n_I(t)$ and $n_Q(t)$ are the in-phase and quadrature components of the narrowband noise $n(t)$ with respect to the carrier wave. The phasor diagram of Figure 2.43 displays the phase relations between the various components of $x(t)$ in Equation (2.153). As the amplitudes and phases of $n_I(t)$ and $n_Q(t)$ change with time in a random manner, the point P_1 [the tip of the phasor representing $x(t)$] wanders around the point P_2 (the tip of the phasor representing the carrier). When the carrier-to-noise ratio is large, $n_I(t)$ and $n_Q(t)$ are usually much smaller than the carrier amplitude A_c , and so the wandering point P_1 in Figure 2.43 spends most of its time near point P_2 . Thus the angle $\theta(t)$ is approximately $n_Q(t)/A_c$ to within a multiple of 2π . When the carrier-to-noise ratio is low, on the other hand, the wandering point P_1 occasionally sweeps around the origin and $\theta(t)$ increases or decreases by 2π radians. Figure 2.44 illustrates how in a rough way the excursions in $\theta(t)$, depicted in Figure 2.44a, produce impulselike components in $\theta'(t) = d\theta/dt$. The discriminator output $v(t)$ is equal to $\theta'(t)/2\pi$. These impulselike components have different heights depending on how close the wandering point P_1 comes to the origin O, but all have areas nearly equal to $\pm 2\pi$ radians, as illustrated in Figure 2.44b. When the signal shown in Figure 2.44b is passed through the postdetection low-pass filter, corresponding but wider impulselike components are excited in the receiver output and are heard as clicks. The clicks are produced only when $\theta(t)$ changes by $\pm 2\pi$ radians.

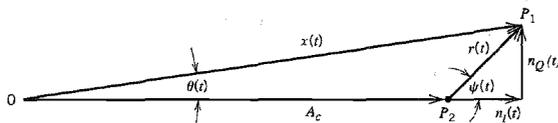


FIGURE 2.43 Phasor diagram interpretation of Equation (2.153).

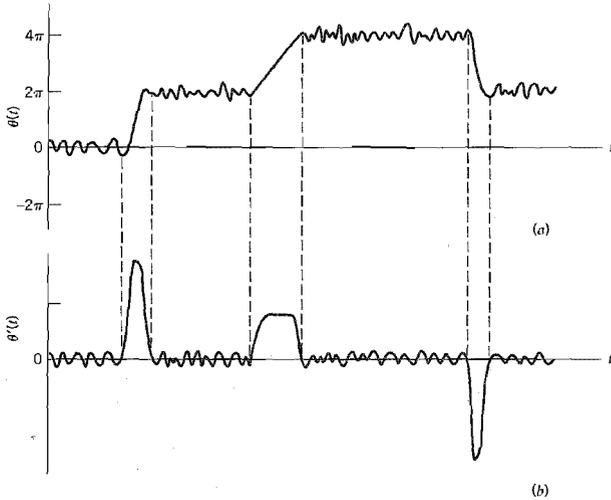


FIGURE 2.44 Illustrating impulselike components in $\theta'(t) = d\theta(t)/dt$ produced by changes of 2π in $\theta(t)$; (a) and (b) are graphs of $\theta(t)$ and $\theta'(t)$, respectively.

From the phasor diagram of Figure 2.43, we may deduce the conditions required for clicks to occur. A positive-going click occurs when the envelope $r(t)$ and phase $\psi(t)$ of the narrowband noise $n(t)$ satisfy the following conditions:

$$\begin{aligned} r(t) &> A_c \\ \psi(t) &< \pi \leq \psi(t) + d\psi(t) \\ \frac{d\psi(t)}{dt} &> 0 \end{aligned}$$

These conditions ensure that the phase $\theta(t)$ of the resultant phasor $x(t)$ changes by 2π radians in the time increment dt , during which the phase of the narrowband noise increases by incremental amount $d\psi(t)$. Similarly, the conditions for a negative-going click to occur are as follows:

$$\begin{aligned} r(t) &> A_c \\ \psi(t) &> -\pi > \psi(t) + d\psi(t) \\ \frac{d\psi(t)}{dt} &< 0 \end{aligned}$$

These conditions ensure that $\theta(t)$ changes by -2π radians during the time increment dt .

The *carrier-to-noise ratio* is defined by

$$\rho = \frac{A_c^2}{2B_T N_0} \quad (2.154)$$

As ρ is decreased, the average number of clicks per unit time increases. When this number becomes appreciably large, threshold is said to occur.

The output signal-to-noise ratio is calculated as follows:

1. The output signal is taken as the receiver output measured in the absence of noise. The average output signal power is calculated assuming a sinusoidal modulation that produces a frequency deviation Δf equal to $B_T/2$, so that the carrier swings back and forth across the entire input frequency band.
2. The average output noise power is calculated when there is no signal present; that is, the carrier is unmodulated, with no restriction imposed on the value of the carrier-to-noise ratio ρ .

On this heuristic basis, theory¹¹ yields Curve I of Figure 2.45 presenting a plot of the output signal-to-noise ratio versus the carrier-to-noise ratio when the ratio $B_T/2W$ is equal to 5. This curve shows that the output signal-to-noise ratio deviates appreciably from a linear function of the carrier-to-noise ratio ρ when ρ is less than about 10 dB. Curve II of Figure 2.45 shows the effect of modulation on the output signal-to-noise ratio when the modulating signal (assumed sinusoidal) and the noise are present at the same time. The average output signal power pertaining to curve II may be taken to be effectively the same as for curve I. The average output noise power, however, is strongly dependent on the presence of the modulating signal, which accounts for the noticeable deviation of curve II from curve I. In particular, we find that as ρ decreases from infinity, the output signal-to-

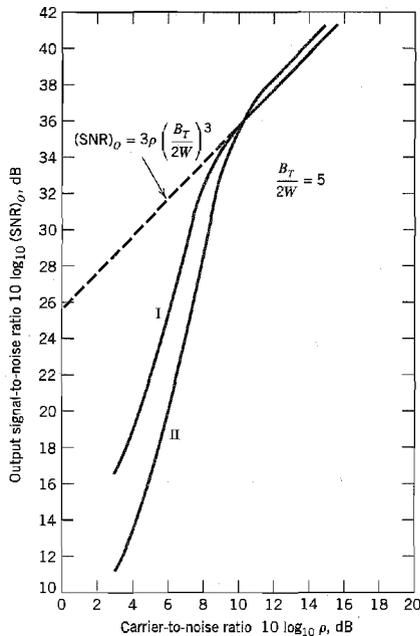


FIGURE 2.45 Dependence of output signal-to-noise ratio on input carrier-to-noise ratio for FM receiver. In curve I, the average output noise power is calculated assuming an unmodulated carrier. In curve II, the average output noise power is calculated assuming a sinusoidally modulated carrier. Both curves I and II are calculated from theory.

noise deviates appreciably from a linear function of ρ when ρ is about 11 dB. Also when the signal is present, the resulting modulation of the carrier tends to increase the average number of clicks per second. Experimentally, it is found that occasional clicks are heard in the receiver output at a carrier-to-noise ratio of about 13 dB, which appears to be only slightly higher than what theory indicates. Also it is of interest to note that the increase in the average number of clicks per second tends to cause the output signal-to-noise ratio to fall off somewhat more sharply just below the threshold level in the presence of modulation.

From the foregoing discussion we may conclude that threshold effects in FM receivers may be avoided in most practical cases of interest if the carrier-to-noise ratio ρ is equal to or greater than 20 or, equivalently, 13 dB. Thus using Equation (2.154) we find that the loss of message at the discriminator output is negligible if

$$\frac{A_c^2}{2B_T N_0} \geq 20$$

or, equivalently, if the average transmitted power $A_c^2/2$ satisfies the condition

$$\frac{A_c^2}{2} \geq 20B_T N_0 \quad (2.155)$$

To use this formula, we may proceed as follows:

1. For a specified modulation index β and message bandwidth W , we determine the transmission bandwidth of the FM wave, B_T , using the universal curve of Figure 2.26 or Carson's rule.
2. For a specified average noise power per unit bandwidth, N_0 , we use Equation (2.155) to determine the minimum value of the average transmitted power $A_c^2/2$ that is necessary to operate above threshold.

■ FM THRESHOLD REDUCTION

In communication systems using frequency modulation, there is particular interest in reducing the noise threshold in an FM receiver so as to satisfactorily operate the receiver with the minimum signal power possible. Threshold reduction in FM receivers may be achieved by using an FM demodulator with negative feedback¹² (commonly referred to as an *FMFB demodulator*), or by using a *phase-locked loop demodulator*. Such devices are referred to as *extended-threshold demodulators*, the idea of which is illustrated in Figure 2.46. The threshold extension shown in this figure is measured with respect to the standard frequency discriminator (i.e., one without feedback).

The block diagram of an FMFB demodulator¹³ is shown in Figure 2.47. We see that the local oscillator of the conventional FM receiver has been replaced by a voltage-controlled oscillator (VCO) whose instantaneous output frequency is controlled by the demodulated signal. In order to understand the operation of this receiver, suppose for the moment that the VCO is removed from the circuit and the feedback loop is left open. Assume that a wideband FM signal is applied to the receiver input, and a second FM signal, from the same source but whose modulation index is a fraction smaller, is applied to the VCO terminal of the mixer. The output of the mixer would consist of the difference frequency component, because the sum frequency component is removed by the band-pass filter. The frequency deviation of the mixer output would be small, although the frequency deviation of both input FM waves is large, since the difference between their instantaneous deviations is small. Hence, the modulation indices would subtract and the resulting FM wave at the mixer output would have a smaller modulation index. The FM wave with

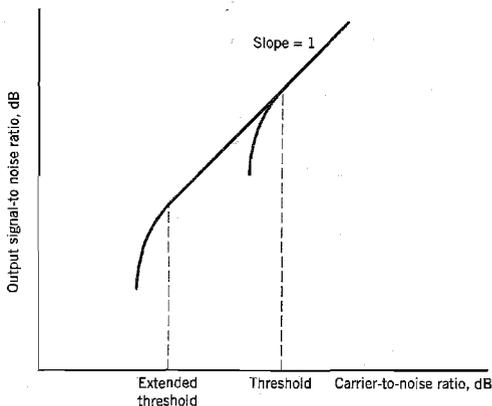


FIGURE 2.46 FM threshold extension.

reduced modulation index may be passed through a band-pass filter, whose bandwidth need only be a fraction of that required for either wideband FM, and then frequency demodulated. It is now apparent that the second wideband FM signal applied to the mixer may be obtained by feeding the output of the frequency discriminator back to the VCO.

It will now be argued that the signal-to-noise ratio of an FMFB receiver is the same as that of a conventional FM receiver with the same input signal and noise power if the carrier-to-noise ratio is sufficiently large. Assume for the moment that there is no feedback around the demodulator. In the combined presence of an unmodulated carrier $A_c \cos(2\pi f_c t)$ and narrowband noise

$$n(t) = n_I(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t)$$

the phase of the composite signal $x(t)$ at the limiter-discriminator input is approximately equal to $n_Q(t)/A_c$, assuming that the carrier-to-noise ratio is high. The envelope of $x(t)$ is of no interest to us, because the limiter removes all variations in the envelope. Thus the composite signal at the frequency discriminator input consists of a small index phase-modulated wave with the modulation derived from the component $n_Q(t)$ of noise that is in phase quadrature with the carrier. When feedback is applied, the VCO generates a frequency-modulated signal that reduces the phase-modulation index of the wave in the band-pass filter output, that is, the quadrature component $n_Q(t)$ of noise. Thus we see that as long as the carrier-to-noise ratio is sufficiently large, the FMFB receiver does not respond to the in-phase noise component $n_I(t)$, but that it would demodulate the quadrature noise component $n_Q(t)$ in exactly the same fashion as it would demodulate signal modulation.

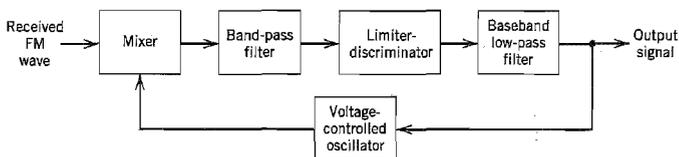


FIGURE 2.47 FM demodulator with negative feedback.

Signal and quadrature noise are reduced in the same proportion by the applied feedback, with the result that the baseband signal-to-noise ratio is independent of feedback. For large carrier-to-noise ratios, the baseband signal-to-noise ratio of an FMFB receiver is then the same as that of a conventional FM receiver.

The reason that an FMFB receiver is able to extend the threshold is that, unlike a conventional FM receiver, it uses a very important piece of *a priori* information, namely, that even though the carrier frequency of the incoming FM wave will usually have large frequency deviations, its rate of change will be at the baseband rate. An FMFB demodulator is essentially a *tracking filter* that can track only the slowly varying frequency of a wideband FM signal, and consequently it responds only to a narrowband of noise centered about the instantaneous carrier frequency. The bandwidth of noise to which the FMFB receiver responds is precisely the band of noise that the VCO tracks. The end result is that an FMFB receiver is capable of realizing a threshold extension on the order of 5–7 dB, which represents a significant improvement in the design of minimum power FM systems.

Like the FMFB demodulator, the phase-locked loop (discussed later in Section 2.14) is also a tracking filter and, as such, the noise bandwidth to which it responds is precisely the band of noise tracked by the VCO. Indeed, the phase-locked loop demodulator offers a threshold extension capability with a relatively simple circuit. Unfortunately, the amount of threshold extension is not predictable by any existing theory, and it depends on signal parameters. Roughly speaking, improvement by a few (on the order of 2 to 3) decibels is achieved in typical applications, which is not as good as an FMFB demodulator.

■ PRE-EMPHASIS AND DE-EMPHASIS IN FM

Equation (2.147) shows that the power spectral density of the noise at the output of an FM receiver has a square-law dependence on the operating frequency; this is illustrated in Figure 2.48a. In Figure 2.48b, we have included the power spectral density of a typical message source; audio and video signals typically have spectra of this form. In particular, we see that the power spectral density of the message usually falls off appreciably at higher frequencies. On the other hand, the power spectral density of the output noise increases rapidly with frequency. Thus around $f = \pm W$, the relative spectral density of the message is quite low, whereas that of the output noise is quite high in comparison. Clearly, the message is not using the frequency band allotted to it in an efficient manner. It may appear that one way of improving the noise performance of the system is to slightly reduce the bandwidth of the postdetection low-pass filter so as to reject a large amount of noise power while losing only a small amount of message power. Such an approach, however, is usually not satisfactory because the distortion of the message caused by the reduced filter bandwidth, even though slight, may not be tolerable. For example, in the case of music, we find that although the high-frequency notes contribute only a very small fraction of the total power, nonetheless, they contribute a great deal from an esthetic viewpoint.

A more satisfactory approach to the efficient use of the allowed frequency band is based on the use of *pre-emphasis* in the transmitter and *de-emphasis* in the receiver, as

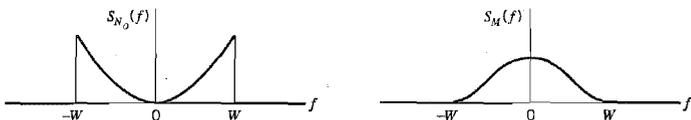


FIGURE 2.48 (a) Power spectral density of noise at FM receiver output. (b) Power spectral density of a typical message signal.

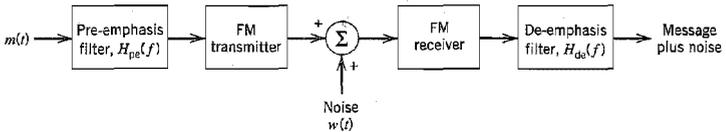


FIGURE 2.49 Use of pre-emphasis and de-emphasis in an FM system.

illustrated in Figure 2.49. In this method, we artificially emphasize the high-frequency components of the message signal prior to modulation in the transmitter, and therefore before the noise is introduced in the receiver. In effect, the low-frequency and high-frequency portions of the power spectral density of the message are equalized in such a way that the message fully occupies the frequency band allotted to it. Then, at the discriminator output in the receiver, we perform the inverse operation by de-emphasizing the high-frequency components, so as to restore the original signal-power distribution of the message. In such a process, the high-frequency components of the noise at the discriminator output are also reduced, thereby effectively increasing the output signal-to-noise ratio of the system. Such a pre-emphasis and de-emphasis process is widely used in commercial FM radio transmission and reception.

In order to produce an undistorted version of the original message at the receiver output, the pre-emphasis filter in the transmitter and the de-emphasis filter in the receiver must ideally have frequency responses that are the inverse of each other. That is, if $H_{pe}(f)$ designates the frequency response of the pre-emphasis filter, then the frequency response $H_{de}(f)$ of the de-emphasis filter must ideally be (ignoring transmission delay)

$$H_{de}(f) = \frac{1}{H_{pe}(f)}, \quad -W \leq f \leq W \quad (2.156)$$

This choice of frequency responses makes the average message power at the receiver output independent of the pre-emphasis and de-emphasis procedure.

From our previous noise analysis in FM systems, assuming a high carrier-to-noise ratio, the power spectral density of the noise $n_d(t)$ at the discriminator output is given by Equation (2.146). The modified power spectral density of the noise at the de-emphasis filter output is therefore

$$|H_{de}(f)|^2 S_{N_d}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2} |H_{de}(f)|^2, & |f| \leq \frac{B_T}{2} \\ 0, & \text{otherwise} \end{cases} \quad (2.157)$$

Recognizing, as before, that the postdetection low-pass filter has a bandwidth W that is, in general, less than $B_T/2$, we find that the average power of the modified noise at the receiver output is as follows:

$$\left(\begin{array}{l} \text{Average output noise} \\ \text{power with de-emphasis} \end{array} \right) = \frac{N_0}{A_c^2} \int_{-W}^W f^2 |H_{de}(f)|^2 df \quad (2.158)$$

Because the average message power at the receiver output is ideally unaffected by the combined pre-emphasis and de-emphasis procedure, it follows that the improvement in output signal-to-noise ratio produced by the use of pre-emphasis in the transmitter and de-emphasis in the receiver is defined by

$$I = \frac{\text{average output noise power without pre-emphasis and de-emphasis}}{\text{average output noise power with pre-emphasis and de-emphasis}} \quad (2.159)$$

Earlier we showed that the average output noise power without pre-emphasis and de-emphasis is equal to $(2N_0W^3/3A_c^2)$; see Equation (2.148). Therefore, after cancellation of common terms, we may express the improvement factor I as

$$I = \frac{2W^3}{3 \int_{-W}^W f^2 |H_{de}(f)|^2 df} \quad (2.160)$$

It must be emphasized that this improvement factor assumes the use of a high carrier-to-noise ratio at the discriminator input in the receiver.

▶ EXAMPLE 2.6

A simple pre-emphasis filter that emphasizes high frequencies and is commonly used in practice is defined by the frequency response

$$H_{pe}(f) = 1 + \frac{jf}{f_0}$$

which is closely realized by the RC-amplifier network shown in Figure 2.50a, provided that $R \ll r$ and $2\pi fCr \ll 1$ inside the frequency band of interest. The amplifier in Figure 2.50a is intended to make up for the attenuation introduced by the RC network at low frequencies. The frequency parameter f_0 is $1/(2\pi Cr)$.

The corresponding de-emphasis filter in the receiver is defined by the frequency response

$$H_{de}(f) = \frac{1}{1 + jf/f_0}$$

which can be realized using the simple RC network of Figure 2.50b.

The improvement in output signal-to-noise ratio of the FM receiver, resulting from the combined use of the pre-emphasis and de-emphasis filters of Figure 2.50, is therefore

$$\begin{aligned} I &= \frac{2W^3}{3 \int_{-W}^W \frac{f^2 df}{1 + (f/f_0)^2}} \\ &= \frac{(W/f_0)^3}{3[(W/f_0) - \tan^{-1}(W/f_0)]} \end{aligned} \quad (2.161)$$

In commercial FM broadcasting, we typically have $f_0 = 2.1$ kHz, and we may reasonably assume $W = 15$ kHz. This set of values yields $I = 22$, which corresponds to an improvement of 13 dB in the output signal-to-noise ratio of the receiver. The output signal-to-noise

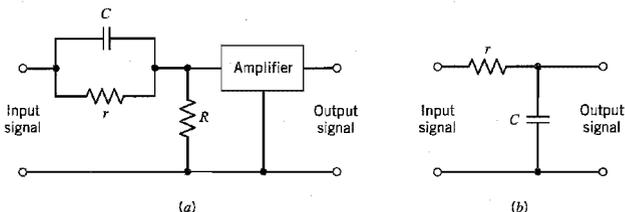


FIGURE 2.50 (a) Pre-emphasis filter. (b) De-emphasis filter.

ratio of an FM receiver without pre-emphasis and de-emphasis is typically 40–50 dB. We see, therefore, that by using the simple pre-emphasis and de-emphasis filters shown in Figure 2.50, we can realize a significant improvement in the noise performance of the receiver. ◀

The use of the simple *linear* pre-emphasis and de-emphasis filters just described is an example of how the performance of an FM system may be improved by using the differences between characteristics of signals and noise in the system. These simple filters also find application in audio tape-recording. Specifically, *nonlinear* pre-emphasis and de-emphasis techniques have been applied successfully to tape recording. These techniques¹⁴ (known as *Dolby-A*, *Dolby-B*, and *DBX* systems) use a combination of filtering and dynamic range compression to reduce the effects of noise, particularly when the signal level is low.

2.14 Computer Experiments: Phase-Locked Loop

The experimental study presented in this section focuses on the use of a phase-locked loop for the demodulation of a frequency modulated signal. Before proceeding with the experiments, however, we first present a brief exposition of phase-locked loop theory.

Basically, the *phase-locked loop* consists of three major components: a *multiplier*, a *loop filter*, and a *voltage-controlled oscillator* (VCO) connected together in the form of a feedback system, as shown in Figure 2.51. The VCO is a sinusoidal generator whose frequency is determined by a voltage applied to it from an external source. In effect, any frequency modulator may serve as a VCO. We assume that initially we have adjusted the VCO so that when the control voltage is zero, two conditions are satisfied:

1. The frequency of the VCO is precisely set at the unmodulated carrier frequency f_c .
2. The VCO output has a 90 degree phase-shift with respect to the unmodulated carrier wave.

Suppose then that the input signal applied to the phase-locked loop is an FM signal defined by

$$s(t) = A_c \sin[2\pi f_c t + \phi_1(t)]$$

where A_c is the carrier amplitude. With a modulating signal $m(t)$, the angle $\phi_1(t)$ is related to $m(t)$ by the integral

$$\phi_1(t) = 2\pi k_f \int_0^t m(\tau) d\tau$$

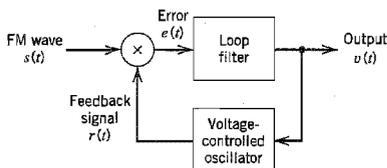


FIGURE 2.51 Phase-locked loop.

where k_f is the frequency sensitivity of the frequency modulator. Let the VCO output in the phase-locked loop be defined by

$$r(t) = A_v \cos[2\pi f_c t + \phi_2(t)]$$

where A_v is the amplitude. With a control voltage $v(t)$ applied to the VCO input, the angle $\phi_2(t)$ is related to $v(t)$ by the integral

$$\phi_2(t) = 2\pi k_v \int_0^t v(\tau) d\tau \quad (2.162)$$

where k_v is the frequency sensitivity of the VCO, measured in Hertz per volt. The object of the phase-locked loop is to generate a VCO output $r(t)$ that has the same phase angle (except for the fixed difference of 90 degrees) as the input FM signal $s(t)$. The time-varying phase angle $\phi_1(t)$ characterizing $s(t)$ may be due to modulation by a message signal $m(t)$, in which case we wish to recover $\phi_1(t)$ and thereby produce an estimate of $m(t)$. In other applications of the phase-locked loop, the time-varying phase angle $\phi_1(t)$ of the incoming signal $s(t)$ may be an unwanted phase shift caused by fluctuations in the communication channel; in this latter case, we wish to *track* $\phi_1(t)$ so as to produce a signal with the same phase angle for the purpose of coherent detection (synchronous demodulation).

■ MODEL OF THE PHASE-LOCKED LOOP¹⁵

To develop an understanding of the phase-locked loop, it is desirable to have a *model* of the loop. We start by developing a nonlinear model, which is subsequently linearized to simplify the analysis. According to Figure 2.51, the incoming FM signal $s(t)$ and the VCO output $r(t)$ are applied to the multiplier, producing two components:

1. A high-frequency component, represented by the *double-frequency* term

$$k_m A_c A_v \sin[4\pi f_c t + \phi_1(t) + \phi_2(t)]$$

2. A low-frequency component represented by the *difference-frequency* term

$$k_m A_c A_v \sin[\phi_1(t) - \phi_2(t)]$$

where k_m is the *multiplier gain*, measured in volt^{-1} .

The loop filter in the phase-locked loop is a low-pass filter, and its response to the high-frequency component will be negligible. The VCO also contributes to the attenuation of this component. Therefore, discarding the high-frequency component (i.e., the double-frequency term), the input to the loop filter is reduced to

$$e(t) = k_m A_c A_v \sin[\phi_e(t)] \quad (2.163)$$

where $\phi_e(t)$ is the *phase error* defined by

$$\begin{aligned} \phi_e(t) &= \phi_1(t) - \phi_2(t) \\ &= \phi_1(t) - 2\pi k_v \int_0^t v(\tau) d\tau \end{aligned} \quad (2.164)$$

The loop filter operates on the error $e(t)$ to produce an output $v(t)$ defined by the convolution integral:

$$v(t) = \int_{-\infty}^{\infty} e(\tau)h(t - \tau) d\tau \quad (2.165)$$

where $h(t)$ is the impulse response of the loop filter. Using Equations (2.164) and (2.165) to relate $\phi_e(t)$ and $\phi_1(t)$, we obtain the following nonlinear integro-differential equation as the descriptor of the dynamic behavior of the phase-locked loop:

$$\frac{d\phi_e(t)}{dt} = \frac{d\phi_1(t)}{dt} - 2\pi K_0 \int_{-\infty}^{\infty} \sin[\phi_e(\tau)] h(t - \tau) d\tau \quad (2.166)$$

where K_0 is as *loop-gain parameter* defined by

$$K_0 = k_m k_v A_c A_v \quad (2.167)$$

The amplitudes A_c and A_v are both measured in volts, the multiplier gain k_m in volt⁻¹ and the frequency sensitivity k_v in Hertz per volt. Hence, it follows from Equation (2.167) that K_0 has the dimensions of frequency. Equation (2.166) suggests the model shown in Figure 2.52 for a phase-locked loop. In this model we have also included the relationship between $v(t)$ and $e(t)$ as represented by Equations (2.163) and (2.165). We see that the model of Figure 2.52 resembles the actual block diagram of Figure 2.51. The multiplier at the input of the phase-locked loop is replaced by a subtracter and a sinusoidal nonlinearity, and the VCO by an integrator.

The sinusoidal nonlinearity in the model of Figure 2.52 complicates the task of analyzing the behavior of the phase-locked loop. It would be helpful to *linearize* this model to simplify the analysis and yet give a good approximate description of the loop's behavior in certain modes of operation. When the phase error $\phi_e(t)$ is zero, the phase-locked loop is said to be in *phase-lock*. When $\phi_e(t)$ is at all times small compared with one radian, we may use the approximation

$$\sin[\phi_e(t)] \approx \phi_e(t)$$

which is accurate to within 4 percent for $\phi_e(t)$ less than 0.5 radians. In this case, the loop is said to be *near phase-lock*, and the sinusoidal nonlinearity of Figure 2.52 may be disregarded. Under this condition, $v(t)$ is approximately equal to $m(t)$, except for the scaling factor k_f/k_v .

The complexity of the phase-locked loop is determined by the frequency response $H(f)$ of the loop filter. The simplest form of a phase-locked loop is obtained when $H(f) = 1$; that is, there is no loop filter, and the resulting phase-locked loop is referred to as a *first-order phase-locked loop*. A major limitation of a first-order phase-locked loop is that the loop gain parameter K_0 controls both the loop bandwidth as well as the hold-in frequency range of the loop; the *hold-in frequency range* refers to the range of frequencies

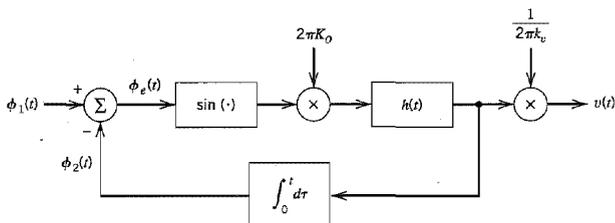


FIGURE 2.52 Nonlinear model of the phase-locked loop.

for which the loop remains phase-locked to the input signal. We may overcome this limitation by using a loop filter with the frequency response

$$H(f) = 1 + \frac{a}{jf} \quad (2.168)$$

where a is a constant. Then with this loop filter in place and the phase-locked loop operating in its linear mode, we find from Equation (2.166) that the phase-locked loop behaves as a *second-order feedback system*, as shown by the standard frequency response

$$\frac{\Phi_o(f)}{\Phi_i(f)} = \frac{(jff_n)^2}{1 + 2\zeta(jff_n) + (jff_n)^2} \quad (2.169)$$

where $\Phi_o(f)$ and $\Phi_i(f)$ are the Fourier transforms of $\phi_o(t)$ and $\phi_i(t)$, respectively. The system is parameterized by the *natural frequency*, f_n , and *damping factor*, ζ , which are respectively defined by

$$f_n = \sqrt{aK_0} \quad (2.170)$$

and

$$\zeta = \sqrt{\frac{K_0}{4a}} \quad (2.171)$$

The second-order phase-locked loop so described is the subject of the computer experiments presented next.

Experiment 1: Acquisition Mode

When a phase-locked loop is used for coherent detection (synchronous demodulation), the loop must first lock onto the input signal and then follow the variations of its phase angle with time. The process of bringing a loop into phase-lock is called *acquisition*, and the ensuing process of following angular variations in the input signal is called *tracking*. In the acquisition mode and quite possibly the tracking mode, the phase error $\phi_e(t)$ between the input signal $s(t)$ and the VCO output $r(t)$ will certainly be large, thereby mandating the use of the nonlinear model of Figure 2.52. However, a nonlinear analysis of the acquisition process based on this latter model is beyond the scope of this book. In this experiment, we use computer simulations to study the acquisition process and thereby develop insight into some of its features.

Consider a second-order phase-locked loop using the loop filter of Equation (2.168) and having the following parameters:

$$\text{Natural frequency } f_n = \frac{1}{2\pi} \text{ Hz}$$

$$\text{Damping factor } \zeta = 0.3, 0.707, 1.0$$

To accommodate variation in ζ , the filter parameter a is varied in accordance with the formula

$$a = \frac{f_n}{2\zeta}$$

which follows from Equations (2.170) and (2.171). Figure 2.53 presents the variation in the phase error $\phi_e(t)$ with time for each of the three specified values of damping factor ζ .

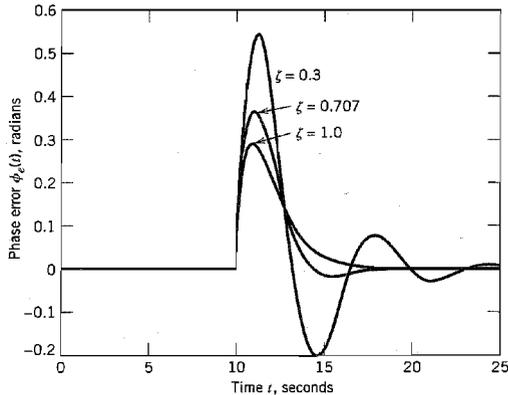


FIGURE 2.53 Variation of the phase error for three different values of damping factor.

assuming a frequency step of 0.125 Hz. These results show that the damping factor $\zeta = 0.707$ gives the best compromise between a fast response time and an underdamped oscillatory behavior.

Experiment 2: Phase-Plane Portrait

A *phase-plane portrait* is a family of trajectories, with each trajectory representing a single solution of Equation (2.166). For the second experiment we plot the phase-plane portrait of a second-order phase-locked loop for the case of sinusoidal modulation. The system parameters of the loop are as follows:

$$\text{Loop-gain parameter } K_0 = \frac{50}{2\pi} \text{ Hz}$$

$$\text{Loop-natural frequency } f_n = \frac{50}{2\sqrt{2}} \text{ Hz}$$

$$\text{Sinusoidal modulation frequency } f_m = \frac{50}{2\pi\sqrt{2}\pi} \text{ Hz}$$

Figure 2.54 presents the phase-plane portrait of the phase-locked loop adjusted for critical damping, where the trajectories (frequency error versus phase error) are plotted for different starting points. From this portrait we make the following observations:

1. For a sinusoidal nonlinearity, the phase-plane portrait is itself periodic with period 2π in the phase error ϕ_e , but it is aperiodic in $d\phi_e/dt$.
2. For an initial frequency error

$$\frac{1}{K} \frac{d\phi_e}{dt}$$

with an absolute value less than or equal to 1, the phase-locked loop is assured of attaining a *stable (equilibrium) point* at $(0, 0)$ or $(0, 2\pi)$; the multiplicity of equilibrium points is a manifestation of periodicity of the phase-plane portrait.

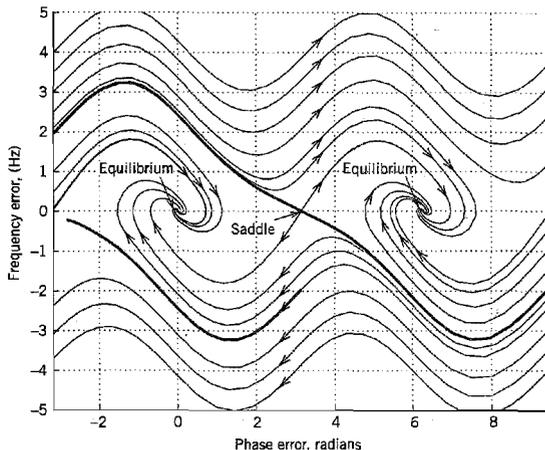


FIGURE 2.54 Phase-plane portrait for critical damping and sinusoidal modulation.

3. For an initial frequency error

$$\frac{1}{K} \frac{d\phi_e}{dt}$$

with an absolute value equal to 2, we have a *saddle point* at $(0, \pi)$ where the slightest perturbation applied to the phase-locked loop causes it to shift to the equilibrium point $(0, 0)$ or $(0, 2\pi)$.

2.15 Summary and Discussion

In this chapter we studied the principles of continuous-wave (CW) modulation. This analog form of modulation uses a sinusoidal carrier whose amplitude or angle is varied in accordance with a message signal. We may thus distinguish two families of CW modulation: amplitude modulation and angle modulation.

■ AMPLITUDE MODULATION

Amplitude modulation may itself be classified into four types, depending on the spectral content of the modulated signal. The four types of amplitude modulation and their practical merits are as follows:

1. *Full amplitude modulation (AM)*, in which the upper and lower sidebands are transmitted in full, accompanied by the carrier wave.

Accordingly, demodulation of an AM signal is accomplished rather simply in the receiver by using an envelope detector, for example. It is for this reason we find that full AM is commonly used in commercial AM radio broadcasting, which involves a single powerful transmitter and numerous receivers that are relatively inexpensive to build.

2. *Double sideband-suppressed carrier (DSB-SC) modulation, in which only the upper and lower sidebands are transmitted.*

The suppression of the carrier wave means that DSB-SC modulation requires much less power than full AM to transmit the same message signal; this advantage of DSB-SC modulation over full AM is, however, attained at the expense of increased receiver complexity. DSB-SC modulation is therefore well suited for *point-to-point communication* involving one transmitter and one receiver; in this form of communication, transmitted power is at a premium and the use of a complex receiver is therefore justifiable.

3. *Single sideband (SSB) modulation, in which only the upper sideband or lower sideband is transmitted.*

SSB modulation is the optimum form of CW modulation in the sense that it requires the minimum transmitted power and the minimum channel bandwidth for conveying a message signal from one point to another. However, its use is limited to message signals with an energy gap centered on zero frequency.

4. *Vestigial sideband modulation, in which almost all of one sideband and a vestige of the other sideband are transmitted in a prescribed complementary fashion.*

VSF modulation requires a channel bandwidth that is between that required for SSB and DSB-SC systems, and the saving in bandwidth can be significant if modulating signals with large bandwidths are being handled, as in the case of television signals and high-speed data.

DSB-SC, SSB, and VSB are examples of linear modulation, whereas, strictly speaking, full AM is nonlinear. However, the deviation of full AM from linearity is of a mild sort. Accordingly, all four forms of amplitude modulation lend themselves readily to spectral analysis using the Fourier transform.

■ ANGLE MODULATION

Angle modulation may be classified into frequency modulation (FM) and phase modulation (PM). *In FM, the instantaneous frequency of a sinusoidal carrier is varied in proportion to the message signal. In PM, on the other hand, it is the phase of the carrier that is varied in proportion to the message signal.* The instantaneous frequency is defined as the derivative of the phase with respect to time, except for the scaling factor $1/(2\pi)$. Accordingly, FM and PM are closely related to each other; if we know the properties of the one, we can determine those of the other. For this reason, and because FM is commonly used in broadcasting, much of the material on angle modulation in the chapter was devoted to FM.

Unlike amplitude modulation, FM is a nonlinear modulation process. Accordingly, spectral analysis of FM is more difficult than for AM. Nevertheless, by studying single-tone FM, we were able to develop a great deal of insight into the spectral properties of FM. In particular, we derived an empirical rule known as Carson's rule for an approximate evaluation of the transmission bandwidth B_T of FM. According to this rule, B_T is controlled by a single parameter: the modulation index β for sinusoidal FM, or the deviation ratio D for nonsinusoidal FM.

■ NOISE ANALYSIS

We conclude the chapter on CW modulation systems by presenting a comparison of their noise performances. For this comparison, we assume that the modulation is produced by

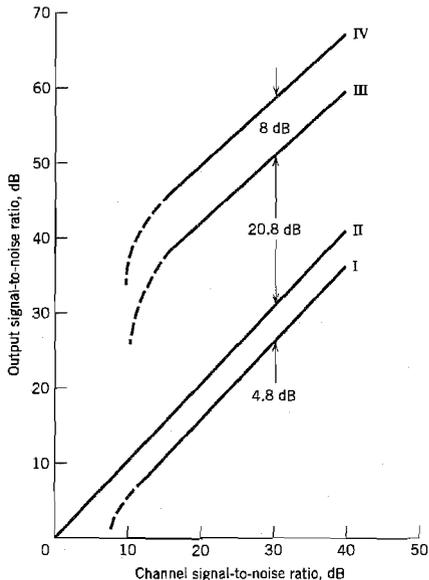


FIGURE 2.55 Comparison of the noise performance of various CW modulation systems. Curve I: Full AM, $\mu = 1$. Curve II: DSB-SC, SSB. Curve III: FM, $\beta = 2$. Curve IV: FM, $\beta = 5$. (Curves III and IV include 13-dB pre-emphasis, de-emphasis improvement.)

a sinusoidal wave. For the comparison to be meaningful, we also assume that the modulation systems operate with exactly the same channel signal-to-noise ratio. We may thus plot the output signal-to-noise ratio versus the channel signal-to-noise ratio as in Figure 2.55 for the following modulation schemes:

- ▶ Full AM with 100 percent modulation
- ▶ Coherent DSB-SC, SSB
- ▶ FM with $\beta = 2$ and $\beta = 5$

Figure 2.55 also includes the AM and FM threshold effects. In making the comparison, it is informative to keep in mind the transmission bandwidth requirement of the modulation system in question. In this regard, we use a *normalized transmission bandwidth* defined by

$$B_n = \frac{B_T}{W}$$

where B_T is the transmission bandwidth of the modulated signal, and W is the message bandwidth. Table 2.4 presents the values of B_n for the different CW modulation schemes.

From Figure 2.55 and Table 2.4 we make the following observations:

- ▶ Among the family of AM systems, SSB modulation is optimum with regard to noise performance as well as bandwidth conservation.
- ▶ The use of FM improves noise performance but at the expense of an excessive transmission bandwidth. This assumes that the FM system operates above threshold for the noise improvement to be realized.

TABLE 2.4 Values of B_n for various CW modulation schemes

	AM, DSB-SC	SSB	FM	
			$\beta = 2$	$\beta = 5$
B_n	2	1	8	16

On an important point to conclude the discussion on CW modulation, only frequency modulation offers the capability to trade off transmission bandwidth for improved noise performance. The trade-off follows a square law, which is the best that we can do with CW modulation (i.e., analog communications). In Chapter 3 we describe pulse-code modulation, which is basic to the transmission of analog information-bearing signals by a digital communication system, and which can indeed do much better.

NOTES AND REFERENCES

1. It appears that the terms *continuous wave* and *heterodyning* were first used by Reginald Fessenden in the early 1900s.
2. The Costas receiver is named in honor of its inventor; see the paper by Costas (1956).
3. Bessel functions play an important role in the study of both analog and digital communication systems. They can be of the so-called first kind or second kind. Appendix 3 discusses mathematical details and properties of both kinds of Bessel functions. A table of Bessel functions of the first kind is presented in Table A6.5.
4. Carson's rule for the bandwidth of FM signals is named in honor of its originator; Carson and Fry (1937) wrote one of the early classic papers on frequency modulation theory.
5. The indirect method of generating a wideband FM wave was first proposed by Armstrong (1936). Armstrong was also the first to recognize the noise-robustness properties of frequency modulation.
6. Stereo multiplexing usually involves the use of frequency modulation for radio transmission. However, it may also be transmitted using amplitude modulation as discussed in Problem 2.14; for more details, see the paper by Mennie (1978).
7. For detailed description of the superheterodyne receiver, see the *Radio Engineering Handbook* edited by Henney (1958, pp. 19-34-19-41).
8. The qualitative study of threshold in envelope detection presented here follows Downing (1964, p. 71).
9. For a justification of the critical assumption on which the simplification presented in Equation (2.142) rests, see Rice (1963).
10. For a detailed discussion of the threshold effect in FM receivers, see the paper by Rice (1963) and the book by Schwartz, Bennett, and Stein (1966, pp. 129-163).
11. Figure 2.45 is adapted from Rice (1963). The validity of the theoretical curve II in this figure has been confirmed experimentally; see Schwartz, Bennett, and Stein (1966, p. 153). For some earlier experimental work on the threshold phenomenon in FM, see the paper by Crosby (1937).
12. The idea of using feedback around an FM demodulator was originally proposed by Chaffee (1939).

13. The treatment of the FMFB demodulator presented in Section 2.13 is based on the paper by Enloe (1962); see also Roberts (1977, pp. 166–181).
14. For a detailed discussion of Dolby systems mentioned in the latter part of Section 2.13, see Stremler (1990, pp. 732–734).
15. For a full treatment of the nonlinear analysis of a phase-locked loop, see Gardner (1979), Lindsey (1972), and Viterbi (1966).

PROBLEMS

Amplitude Modulation

- 2.1 Suppose that nonlinear devices are available for which the output current i_o and input voltage v_i are related by

$$i_o = a_1 v_i + a_3 v_i^3$$

where a_1 and a_3 are constants. Explain how these devices may be used to provide: (a) a product modulator and (b) an amplitude modulator.

- 2.2 Figure P2.2 shows the circuit diagram of a *square-law modulator*. The signal applied to the nonlinear device is relatively weak, such that it can be represented by a square law:

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$$

where a_1 and a_2 are constants, $v_1(t)$ is the input voltage, and $v_2(t)$ is the output voltage. The input voltage is defined by

$$v_1(t) = A_c \cos(2\pi f_c t) + m(t)$$

where $m(t)$ is a message signal and $A_c \cos(2\pi f_c t)$ is the carrier wave.

- (a) Evaluate the output voltage $v_2(t)$.
- (b) Specify the frequency response that the tuned circuit in Figure P2.2 must satisfy in order to generate an AM signal with f_c as the carrier frequency.
- (c) What is the amplitude sensitivity of this AM signal?

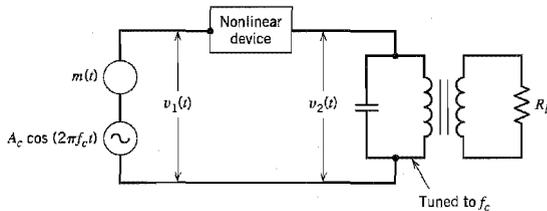


FIGURE P2.2

- 2.3 Figure P2.3a shows the circuit diagram of a *switching modulator*. Assume that the carrier wave $c(t)$ applied to the diode is large in amplitude, so that the diode acts like an ideal switch: it presents zero impedance when forward biased (i.e., $c(t) > 0$). We may thus

approximate the transfer characteristic of the diode-load resistor combination by a piecewise-linear characteristic defined as (see Figure P2.3b)

$$v_2(t) = \begin{cases} v_1(t), & c(t) > 0 \\ 0, & c(t) < 0 \end{cases}$$

That is, the load voltage $v_2(t)$ varies periodically between the values $v_1(t)$ and zero at a rate equal to the carrier frequency f_c . Hence, we may write

$$v_2(t) \approx [A_c \cos(2\pi f_c t) + m(t)]g_{T_0}(t)$$

where $g_{T_0}(t)$ is a periodic pulse train defined by

$$g_{T_0}(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos[2\pi f_c t(2n-1)]$$

- (a) Find the AM wave component contained in the output voltage $v_2(t)$.
- (b) Specify the unwanted components in $v_2(t)$ that need to be removed by a band-pass filter of suitable design.

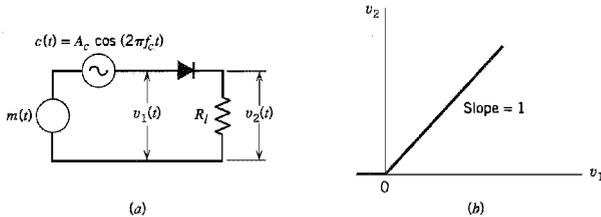


FIGURE P2.3

2.4 Consider the AM signal

$$s(t) = A_c[1 + \mu \cos(2\pi f_m t)] \cos(2\pi f_c t)$$

produced by a sinusoidal modulating signal of frequency f_m . Assume that the modulation factor is $\mu = 2$, and the carrier frequency f_c is much greater than f_m . The AM signal $s(t)$ is applied to an ideal envelope detector, producing the output $v(t)$.

- (a) Determine the Fourier series representation of $v(t)$.
 - (b) What is the ratio of second-harmonic amplitude to fundamental amplitude in $v(t)$?
- 2.5 Figure P2.5 shows the circuit diagram of an *envelope detector*. It consists simply of a diode and resistor-capacitor (RC) filter. On a positive half-cycle of the input signal, the diode is forward-biased and the capacitor C charges up rapidly to the peak value of the input signal. When the input signal falls below this value, the diode becomes reverse-biased and the capacitor C discharges slowly through the load resistor R_l . The discharging process continues until the next positive half-cycle. Thereafter, the charging-discharging routine is continued.
- (a) Specify the condition that must be satisfied by the capacitor C for it to charge rapidly and thereby follow the input voltage up to the positive peak when the diode is conducting.
 - (b) Specify the condition which the load resistor R_l must satisfy so that the capacitor C discharges slowly between positive peaks of the carrier wave, but not so long that the

capacitor voltage will not discharge at the maximum rate of change of the modulating wave.

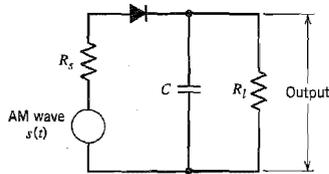


FIGURE P2.5

- 2.6 Consider a *square-law detector*, using a nonlinear device whose transfer characteristic is defined by

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$$

where a_1 and a_2 are constants, $v_1(t)$ is the input, and $v_2(t)$ is the output. The input consists of the AM wave

$$v_1(t) = A_c[1 + k_a m(t)] \cos(2\pi f_c t)$$

- (a) Evaluate the output $v_2(t)$.
 (b) Find the conditions for which the message signal $m(t)$ may be recovered from $v_2(t)$.
- 2.7 The AM signal

$$s(t) = A_c[1 + k_a m(t)] \cos(2\pi f_c t)$$

is applied to the system shown in Figure P2.7. Assuming that $|k_a m(t)| < 1$ for all t and the message signal $m(t)$ is limited to the interval $-W \leq f \leq W$ and that the carrier frequency $f_c > 2W$ show that $m(t)$ can be obtained from the square-rooter output $v_3(t)$.



FIGURE P2.7

- 2.8 Consider a message signal $m(t)$ with the spectrum shown in Figure P2.8. The message bandwidth $W = 1$ kHz. This signal is applied to a product modulator, together with a carrier wave $A_c \cos(2\pi f_c t)$, producing the DSB-SC modulated signal $s(t)$. The modulated signal is next applied to a coherent detector. Assuming perfect synchronism between the carrier waves in the modulator and detector, determine the spectrum of the detector output when: (a) the carrier frequency $f_c = 1.25$ kHz and (b) the carrier frequency $f_c = 0.75$ kHz. What is the lowest carrier frequency for which each component of the modulated signal $s(t)$ is uniquely determined by $m(t)$?

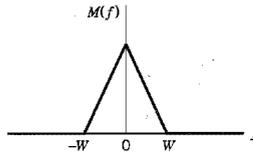


FIGURE P2.8

2.9 Figure P2.9 shows the circuit diagram of a *balanced modulator*. The input applied to the top AM modulator is $m(t)$, whereas that applied to the lower AM modulator is $-m(t)$; these two modulators have the same amplitude sensitivity. Show that the output $s(t)$ of the balanced modulator consists of a DSB-SC modulated signal.

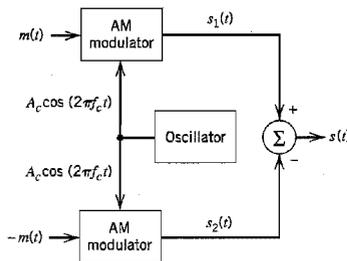


FIGURE P2.9

- 2.10 A DSB-SC modulated signal is demodulated by applying it to a coherent detector.
- Evaluate the effect of a frequency error Δf in the local carrier frequency of the detector, measured with respect to the carrier frequency of the incoming DSB-SC signal.
 - For the case of a sinusoidal modulating wave, show that because of this frequency error, the demodulated signal exhibits *beats* at the error frequency. Illustrate your answer with a sketch of this demodulated signal.
- 2.11 Consider the DSB-SC signal

$$s(t) = A_c \cos(2\pi f_c t) m(t)$$

where $A_c \cos(2\pi f_c t)$ is the carrier wave and $m(t)$ is the message signal. This modulated signal is applied to a square-law device characterized by

$$y(t) = s^2(t)$$

The output $y(t)$ is next applied to a narrowband filter with a passband magnitude response of one, midband frequency $2f_c$, and bandwidth Δf . Assume that Δf is small enough to treat the spectrum of $y(t)$ as essentially constant inside the passband of the filter.

- Determine the spectrum of the square-law device output $y(t)$.
- Show that the filter output $v(t)$ is approximately sinusoidal, given by

$$v(t) \approx \frac{A_c^2}{2} E \Delta f \cos(4\pi f_c t)$$

where E is the energy of the message signal $m(t)$.

- 2.12 Consider the quadrature-carrier multiplex system of Figure 2.10. The multiplexed signal $s(t)$ produced at the transmitter output in Figure 2.10a is applied to a communication channel of frequency response $H(f)$. The output of this channel is, in turn, applied to the receiver input in Figure 2.10b. Prove that the condition

$$H(f_c + f) = H^*(f_c - f), \quad 0 \leq f \leq W$$

is necessary for recovery of the message signals $m_1(t)$ and $m_2(t)$ at the receiver outputs; f_c is the carrier frequency, and W is the message bandwidth. *Hint:* Evaluate the spectra of the two receiver outputs.

- 2.13 Suppose that in the receiver of the quadrature-carrier multiplex system of Figure 2.10b the local carrier available for demodulation has a phase error ϕ with respect to the carrier source used in the transmitter. Assuming a distortionless communication channel between transmitter and receiver, show that this phase error will cause *cross-talk* to arise between the two demodulated signals at the receiver outputs. By cross-talk we mean that a portion of one message signal appears at the receiver output belonging to the other message signal, and vice versa.
- 2.14 A particular version of *AM stereo* uses quadrature multiplexing. Specifically, the carrier $A_c \cos(2\pi f_c t)$ is used to modulate the sum signal

$$m_1(t) = V_0 + m_e(t) + m_r(t)$$

where V_0 is a DC offset included for the purpose of transmitting the carrier component, $m_e(t)$ is the left-hand audio signal, and $m_r(t)$ is the right-hand audio signal. The quadrature carrier $A_c \sin(2\pi f_c t)$ is used to modulate the difference signal

$$m_2(t) = m_e(t) - m_r(t)$$

- (a) Show that an envelope detector may be used to recover the sum $m_e(t) + m_r(t)$ from the quadrature-multiplexed signal. How would you minimize the signal distortion produced by the envelope detector?
- (b) Show that a coherent detector can recover the difference $m_e(t) - m_r(t)$.
- (c) How are the desired $m_e(t)$ and $m_r(t)$ finally obtained?
- 2.15 Using the message signal

$$m(t) = \frac{1}{1 + t^2}$$

determine and sketch the modulated waves for the following methods of modulation:

- (a) Amplitude modulation with 50 percent modulation.
 (b) Double sideband-suppressed carrier modulation.
 (c) Single sideband modulation with only the upper sideband transmitted.
 (d) Single sideband modulation with only the lower sideband transmitted.
- 2.16 The *Hilbert transform* of a Fourier transformable signal $m(t)$, denoted by $\hat{m}(t)$, is defined by (see Appendix 2)

$$\hat{m}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{m(\tau)}{t - \tau} d\tau$$

In the frequency domain, we have

$$\hat{M}(f) = -j \operatorname{sgn}(f)M(f)$$

where $m(t) \Leftrightarrow M(f)$, $\hat{m}(t) \Leftrightarrow \hat{M}(f)$, and $\operatorname{sgn}(f)$ is the signum function.

Using the definition of the Hilbert transform, show that a single-sideband modulated signal resulting from the message signal $m(t)$ and carrier $\cos(2\pi f_c t)$ of unit amplitude is given by (see Table 2.1)

$$s(t) = \frac{1}{2} m(t) \cos(2\pi f_c t) \pm \frac{1}{2} \hat{m}(t) \sin(2\pi f_c t)$$

where the minus sign corresponds to the transmission of the upper sideband and the plus sign corresponds to the transmission of the lower sideband.

- 2.17 The local oscillator used for the demodulation of an SSB signal $s(t)$ has a frequency error Δf measured with respect to the carrier frequency f_c used to generate $s(t)$. Otherwise, there is perfect synchronism between this oscillator in the receiver and the oscillator supplying the carrier wave in the transmitter. Evaluate the demodulated signal for the following two situations:
- The SSB signal $s(t)$ consists of the upper sideband only.
 - The SSB signal $s(t)$ consists of the lower sideband only.
- 2.18 Figure P2.18 shows the block diagram of *Weaver's method* for generating SSB modulated waves. The message (modulating) signal $m(t)$ is limited to the band $f_a \leq |f| \leq f_b$. The auxiliary carrier applied to the first pair of product modulators has a frequency f_0 , which lies at the center of this band, as shown by

$$f_0 = \frac{f_a + f_b}{2}$$

The low-pass filters in the in-phase and quadrature channels are identical, each with a cutoff frequency equal to $(f_b - f_a)/2$. The carrier applied to the second pair of product modulators has a frequency f_c that is greater than $(f_b - f_a)/2$. Sketch the spectra at the various points in the modulator of Figure P2.18, and hence show that:

- For the lower sideband, the contributions of the in-phase and quadrature channels are of opposite polarity, and by adding them at the modulator output, the lower sideband is suppressed.
- For the upper sideband, the contributions of the in-phase and quadrature channels are of the same polarity, and by adding them, the upper sideband is transmitted.
- How would you modify the modulator of Figure P2.18 so that only the lower sideband is transmitted?

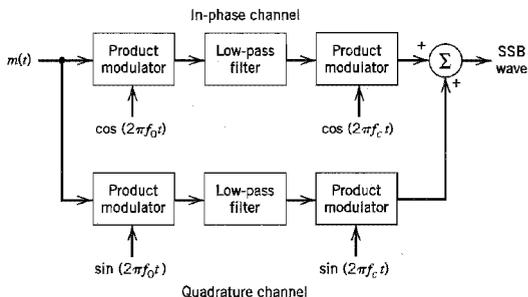


FIGURE P2.18

- 2.19 The spectrum of a voice signal $m(t)$ is zero outside the interval $f_a \leq |f| \leq f_b$. To ensure communication privacy, this signal is applied to a *scrambler* that consists of the following cascade of components: product modulator, high-pass filter, second product modulator, and low-pass filter. The carrier wave applied to the first product modulator has a frequency equal to f_c , whereas that applied to the second product modulator has a frequency equal to $f_b + f_c$; both of them have unit amplitude. The high-pass and low-pass filters have the same cutoff frequency at f_c . Assume that $f_c > f_b$.
- Derive an expression for the scrambler output $s(t)$, and sketch its spectrum.
 - Show that the original voice signal $m(t)$ may be recovered from $s(t)$ by using an *unscrambler* that is identical to the unit just described.
- 2.20 In this problem we derive Equation (2.16) that defines the frequency response $H_Q(f)$ of the filter that operates on the message signal $m(t)$ to produce $m'(t)$ for VSB modulation. The signal $m'(t)$, except for a scaling factor, constitutes the quadrature component of $s(t)$. To do the derivation, we apply $s(t)$ to the coherent detector of Figure P2.20 so as to recover a scaled version of the original message signal $m(t)$.
- Starting with the block diagram of Figure 2.12 for the generation of a VSB modulated wave, determine the Fourier transform $V(f)$ of the product modulator output $v(t)$ in Figure P2.20 in terms of the Fourier transform of the message signal $m(t)$ and the frequency response $H(f)$ of the band-pass filter in Figure 2.12.
 - Hence, by evaluating the Fourier transform of the low-pass filter output in Figure P2.20, determine the conditions that must be satisfied by $H_Q(f)$ in terms of $H(f)$ to assure perfect recovery of the original message signal $m(t)$, except for a scaling factor.

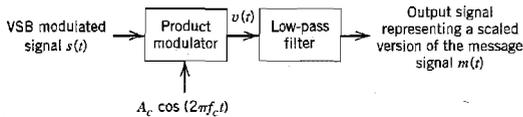


FIGURE P2.20

- 2.21 The single-tone modulating signal $m(t) = A_m \cos(2\pi f_m t)$ is used to generate the VSB signal

$$s(t) = \frac{1}{2} a A_m A_c \cos[2\pi(f_c + f_m)t] + \frac{1}{2} A_m A_c (1 - a) \cos[2\pi(f_c - f_m)t]$$

where a is a constant, less than unity, representing the attenuation of the upper side frequency.

- Find the quadrature component of the VSB signal $s(t)$.
 - The VSB signal, plus the carrier $A_c \cos(2\pi f_c t)$, is passed through an envelope detector. Determine the distortion produced by the quadrature component.
 - What is the value of constant a for which this distortion reaches its worst possible condition?
- 2.22 In this problem we study the idea of mixing in a superheterodyne receiver. To be specific, consider the block diagram of the *mixer* shown in Figure P2.22 that consists of a product modulator with a local oscillator of *variable frequency* f_l , followed by a band-pass filter. The input signal is an AM wave of bandwidth 10 kHz and carrier frequency that may lie anywhere in the range of 0.535 to 1.605 MHz; these parameters are typical of AM radio broadcasting. It is required to translate this signal to a frequency band centered at a fixed

intermediate frequency (IF) of 0.455 MHz. Find the range of tuning that must be provided in the local oscillator to achieve this requirement.

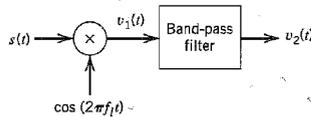


FIGURE P2.22

2.23 Figure P2.23 shows the block diagram of a *heterodyne spectrum analyzer*. It consists of a variable-frequency oscillator, multiplier, band-pass filter, and root mean square (RMS) meter. The oscillator has an amplitude A and operates over the range f_0 to $f_0 + W$, where f_0 is the midband frequency of the filter and W is the signal bandwidth. Assume that $f_0 = 2W$, the filter bandwidth Δf is small compared with f_0 and that the passband magnitude response of the filter is one. Determine the value of the RMS meter output for a low-pass input signal $g(t)$.

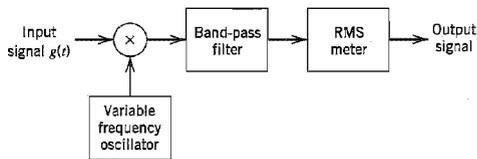


FIGURE P2.23

Angle Modulation

2.24 Sketch the PM and FM waves produced by the sawtooth wave shown in Figure P2.24.

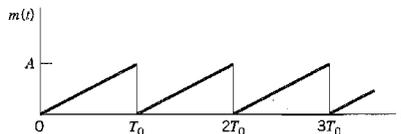


FIGURE P2.24

2.25 In a *frequency-modulated radar*, the instantaneous frequency of the transmitted carrier is varied as in Figure P2.25, which is obtained by using a triangular modulating signal. The instantaneous frequency of the received echo signal is shown dashed in Figure P2.25, where τ is the round-trip delay time. The transmitted and received echo signals are applied to a mixer, and the difference frequency component is retained. Assuming that $f_0\tau \ll 1$, determine the number of beat cycles at the mixer output, averaged over one second, in

terms of the peak deviation Δf of the carrier frequency, the delay τ , and the repetition frequency f_0 of the transmitted signal.

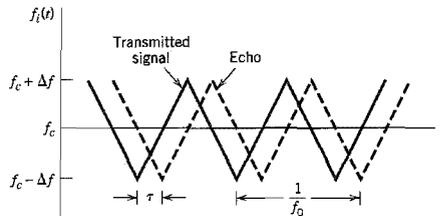


FIGURE P2.25

- 2.26 The instantaneous frequency of a sine wave is equal to $f_c - \Delta f$ for $|t| \leq T/2$, and f_c for $|t| > T/2$. Determine the spectrum of this frequency-modulated wave. *Hint:* Divide up the time interval of interest into three regions: $-\infty < t < -T/2$, $-T/2 \leq t \leq T/2$, and $T/2 < t < \infty$.
- 2.27 Single-sideband modulation may be viewed as a hybrid form of amplitude modulation and frequency modulation. Evaluate the envelope and instantaneous frequency of an SSB wave for the following two cases:
- When only the upper sideband is transmitted.
 - When only the lower sideband is transmitted.
- 2.28 Consider a narrowband FM signal approximately defined by

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t) \sin(2\pi f_m t)$$

- Determine the envelope of this modulated signal. What is the ratio of the maximum to the minimum value of this envelope? Plot this ratio versus β , assuming that β is restricted to the interval $0 \leq \beta \leq 0.3$.
- Determine the average power of the narrowband FM signal, expressed as a percentage of the average power of the unmodulated carrier wave. Plot this result versus β , assuming that β is restricted to the interval $0 \leq \beta \leq 0.3$.
- By expanding the angle $\theta_i(t)$ of the narrow-band FM signal $s(t)$ in the form of a power series, and restricting the modulation index β to a maximum value of 0.3 radians, show that

$$\theta_i(t) \approx 2\pi f_c t + \beta \sin(2\pi f_m t) - \frac{\beta^3}{3} \sin^3(2\pi f_m t)$$

What is the power ratio of third harmonic to fundamental component for $\beta = 0.3$?

- 2.29 The sinusoidal modulating wave

$$m(t) = A_m \cos(2\pi f_m t)$$

is applied to a phase modulator with phase sensitivity k_p . The unmodulated carrier wave has frequency f_c and amplitude A_c .

- Determine the spectrum of the resulting phase-modulated signal, assuming that the maximum phase deviation $\beta_p = k_p A_m$ does not exceed 0.3 radians.
 - Construct a phasor diagram for this modulated signal, and compare it with that of the corresponding narrowband FM signal.
- 2.30 Suppose that the phase-modulated signal of Problem 2.29 has an arbitrary value for the maximum phase deviation β_p . This modulated signal is applied to an ideal band-pass filter with midband frequency f_c and a passband extending from $f_c - 1.5f_m$ to $f_c + 1.5f_m$.

Determine the envelope, phase, and instantaneous frequency of the modulated signal at the filter output as functions of time.

- 2.31 A carrier wave is frequency-modulated using a sinusoidal signal of frequency f_m and amplitude A_m .
- Determine the values of the modulation index β for which the carrier component of the FM signal is reduced to zero. For this calculation you may use the values of $J_0(\beta)$ given in Table A6.5.
 - In a certain experiment conducted with $f_m = 1$ kHz and increasing A_m (starting from 0 volts), it is found that the carrier component of the FM signal is reduced to zero for the first time when $A_m = 2$ volts. What is the frequency sensitivity of the modulator? What is the value of A_m for which the carrier components is reduced to zero for the second time?
- 2.32 An FM signal with modulation index $\beta = 1$ is transmitted through an ideal band-pass filter with midband frequency f_c and bandwidth $5f_m$, where f_c is the carrier frequency and f_m is the frequency of the sinusoidal modulating wave. Determine the magnitude spectrum of the filter output.
- 2.33 A carrier wave of frequency 100 MHz is frequency-modulated by a sinusoidal wave of amplitude 20 volts and frequency 100 kHz. The frequency sensitivity of the modulator is 25 kHz per volt.
- Determine the approximate bandwidth of the FM signal, using Carson's rule.
 - Determine the bandwidth by transmitting only those side frequencies whose amplitudes exceed 1 percent of the unmodulated carrier amplitude. Use the universal curve of Figure 2.26 for this calculation.
 - Repeat your calculations, assuming that the amplitude of the modulating signal is doubled.
 - Repeat your calculations, assuming that the modulation frequency is doubled.
- 2.34 Consider a wideband PM signal produced by a sinusoidal modulating wave $A_m \cos(2\pi f_m t)$, using a modulator with a phase sensitivity equal to k_p radians per volt.
- Show that if the maximum phase deviation of the PM signal is large compared with one radian, the bandwidth of the PM signal varies linearly with the modulation frequency f_m .
 - Compare this characteristic of a wideband PM signal with that of a wideband FM signal.
- 2.35 Figure P2.35 shows the block diagram of a real-time *spectrum analyzer* working on the principle of frequency modulation. The given signal $g(t)$ and a frequency-modulated signal $s(t)$ are applied to a multiplier and the output $g(t)s(t)$ is fed into a filter of impulse response $h(t)$. The $s(t)$ and $h(t)$ are *linear FM signals* whose instantaneous frequencies vary linearly with time at opposite rates, as shown by

$$s(t) = \cos(2\pi f_c t - \pi k t^2)$$

$$h(t) = \cos(2\pi f_c t + \pi k t^2)$$

where k is a constant. Show that the envelope of the filter output is proportional to the magnitude spectrum of the input signal $g(t)$ with kt playing the role of frequency f . *Hint:* Use the complex notations described in Appendix 2 for the analysis of band-pass signals and band-pass filters.

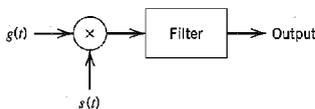


FIGURE P2.35

- 2.36 An FM signal with a frequency deviation of 10 kHz at a modulation frequency of 5 kHz is applied to two frequency multipliers connected in cascade. The first multiplier doubles the frequency and the second multiplier triples the frequency. Determine the frequency deviation and the modulation index of the FM signal obtained at the second multiplier output. What is the frequency separation of the adjacent side frequencies of this FM signal?
- 2.37 Figure P2.37 shows the block diagram of a wideband frequency modulator using the indirect method. This modulator is used to transmit audio signals containing frequencies in the range of 100 Hz to 15 kHz. The narrowband phase modulator is supplied with a carrier of frequency $f_1 = 0.1$ MHz by a crystal-controlled oscillator. A second crystal-controlled oscillator supplies a sinusoidal wave of frequency 9.5 MHz to the mixer. The system specifications are as follows:
 Carrier frequency at the transmitter output, $f_c = 100$ MHz
 Minimum frequency deviation, $\Delta f = 75$ kHz
 Maximum modulation index in the phase modulator = 0.3 radians
- Calculate the frequency multiplication ratios n_1 and n_2 (preceding and following the mixer), which will satisfy these specifications.
 - Specify the values of the carrier frequency and frequency deviation at the various points in the modulator of Figure P2.37.

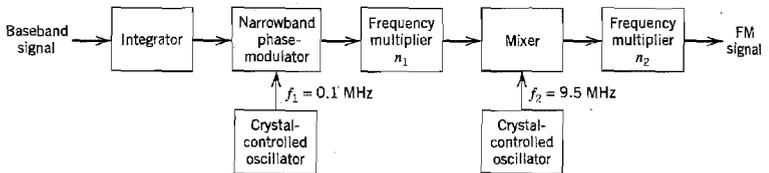


FIGURE P2.37

- 2.38 Figure P2.38 shows the frequency-determining network of a voltage-controlled oscillator. Frequency modulation is produced by applying the modulating signal $A_m \sin(2\pi f_m t)$ plus a bias V_b to a pair of varactor diodes connected across the parallel combination of a 200- μH inductor and 100-pF capacitor. The capacitor of each varactor diode is related to the voltage V (in volts) applied across its electrodes by

$$C = 100V^{-1/2} \text{ pF}$$

The unmodulated frequency of oscillation is 1 MHz. The VCO output is applied to a frequency multiplier to produce an FM signal with a carrier frequency of 64 MHz and a modulation index of 5. Determine (a) the magnitude of the bias voltage V_b and (b) the amplitude A_m of the modulating wave, given that $f_m = 10$ kHz.

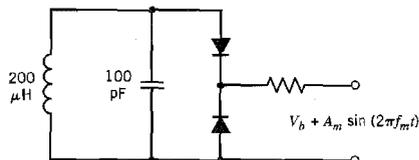


FIGURE P2.38

2.39 The FM signal

$$s(t) = A_c \cos \left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau \right]$$

is applied to the system shown in Figure P2.39 consisting of a high-pass RC filter and an envelope detector. Assume that (a) the resistance R is small compared with the reactance of the capacitor C for all significant frequency components of $s(t)$, and (b) the envelope detector does not load the filter. Determine the resulting signal at the envelope detector output, assuming that $k_f |m(t)| < f_c$ for all t .

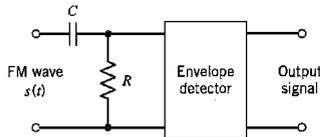


FIGURE P2.39

- 2.40 Consider the frequency demodulation scheme shown in Figure P2.40 in which the incoming FM signal $s(t)$ is passed through a delay line that produces a phase-shift of $\pi/2$ radians at the carrier frequency f_c . The delay-line output is subtracted from the incoming FM signal, and the resulting composite signal is then envelope-detected. This demodulator finds application in demodulating microwave FM signals. Assuming that

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

analyze the operation of this demodulator when the modulation index β is less than unity and the delay T produced by the delay line is sufficiently small to justify making the approximations

$$\cos(2\pi f_m T) \approx 1$$

and

$$\sin(2\pi f_m T) \approx 2\pi f_m T$$

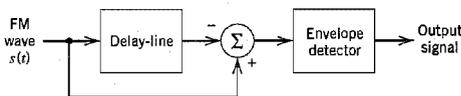


FIGURE P2.40

- 2.41 Figure 2.41 shows the block diagram of a *zero-crossing detector* for demodulating an FM signal. It consists of a limiter, a pulse generator for producing a short pulse at each zero-crossing of the input, and a low-pass filter for extracting the modulating wave.

(a) Show that the instantaneous frequency of the input FM signal is proportional to the number of zero crossings in the time interval $t - (T_1/2)$ to $t + (T_1/2)$, divided by T_1 . Assume that the modulating signal is essentially constant during this time interval.

- (b) Illustrate the operation of this demodulator, using the sawtooth wave of Figure P2.24 as the modulating wave.

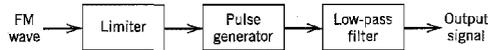


FIGURE P2.41

- 2.42 Suppose that the received signal in an FM system contains some residual amplitude modulation of positive amplitude $a(t)$, as shown by

$$s(t) = a(t) \cos[2\pi f_c t + \phi(t)]$$

where f_c is the carrier frequency. The phase $\phi(t)$ is related to the modulating signal $m(t)$ by

$$\phi(t) = 2\pi k_f \int_0^t m(\tau) d\tau$$

where k_f is a constant. Assume that the signal $s(t)$ is restricted to a frequency band of width B_T , centered at f_c , where B_T is the transmission bandwidth of the FM signal in the absence of amplitude modulation, and that the amplitude modulation is slowly varying compared with $\phi(t)$. Show that the output of an ideal frequency discriminator produced by $s(t)$ is proportional to $a(t)m(t)$. *Hint:* Use the complex notation described in Appendix 2 to represent the modulated wave $s(t)$.

- 2.43 (a) Let the modulated wave $s(t)$ in Problem 2.42 be applied to a *hard limiter*, whose output $z(t)$ is defined by

$$z(t) = \text{sgn}[s(t)] = \begin{cases} +1, & s(t) > 0 \\ -1, & s(t) < 0 \end{cases}$$

Show that the limiter output may be expressed in the form of a Fourier series as follows:

$$z(t) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos[2\pi f_c t(2n+1) + (2n+1)\phi(t)]$$

- (b) Suppose that the limiter output is applied to a band-pass filter with a passband magnitude response of one and bandwidth B_T centered about the carrier frequency f_c , where B_T is the transmission bandwidth of the FM signal in the absence of amplitude modulation. Assuming that f_c is much greater than B_T , show that the resulting filter output equals

$$y(t) = \frac{4}{\pi} \cos[2\pi f_c t + \phi(t)]$$

By comparing this output with the original modulated signal $s(t)$ defined in Problem 2.42, comment on the practical usefulness of the result.

- 2.44 (a) Consider an FM signal of carrier frequency f_c , which is produced by a modulating signal $m(t)$. Assume that f_c is large enough to justify treating this FM signal as a narrowband signal. Find an approximate expression for its Hilbert transform.
- (b) For the special case of a sinusoidal modulating wave $m(t) = A_m \cos(2\pi f_m t)$, find the exact expression for the Hilbert transform of the resulting FM signal. For this case, what is the error in the approximation used in part (a)?

2.45 The *single sideband version of angle modulation* is defined by

$$s(t) = \exp[-\hat{\phi}(t)] \cos[2\pi f_c t + \phi(t)]$$

where $\hat{\phi}(t)$ is the Hilbert transform of the phase function $\phi(t)$, and f_c is the carrier frequency.

- (a) Show that the spectrum of the modulated signal $s(t)$ contains no frequency components in the interval $-f_c < f < f_c$, and is of infinite extent.
- (b) Given that the phase function

$$\phi(t) = \beta \sin(2\pi f_m t)$$

where β is the modulation index and f_m is the modulation frequency, derive the corresponding expression for the modulated wave $s(t)$.

Note: For Problems 2.44 and 2.45, you need to refer to Appendix 2 for a treatment of the Hilbert transform.

Noise in CW Modulation Systems

2.46 A DSB-SC modulated signal is transmitted over a noisy channel, with the power spectral density of the noise being as shown in Figure P2.46. The message bandwidth is 4 kHz and the carrier frequency is 200 kHz. Assuming that the average power of the modulated wave is 10 watts, determine the output signal-to-noise ratio of the receiver.

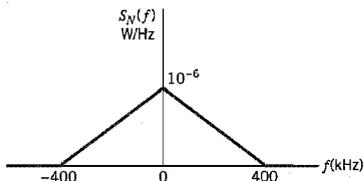


FIGURE P2.46

- 2.47 Evaluate the autocorrelation functions and cross-correlation functions of the in-phase and quadrature components of the narrowband noise at the coherent detector input for (a) the DSB-SC system, (b) an SSB system using the lower sideband, and (c) an SSB system using the upper sideband.
- 2.48 In a receiver using coherent detection, the sinusoidal wave generated by the local oscillator suffers from a phase error $\theta(t)$ with respect to the carrier wave $\cos(2\pi f_c t)$. Assuming that $\theta(t)$ is a sample function of a zero-mean Gaussian process of variance σ_θ^2 , and that most of the time the maximum value of $\theta(t)$ is small compared with unity, find the mean-square error of the receiver output for DSB-SC modulation. The mean-square error is defined as the expected value of the squared difference between the receiver output and the message signal component of the receiver output.
- 2.49 Following a procedure similar to that described in Section 2.11 for the DSB-SC receiver, extend this noise analysis to a SSB receiver. Specifically, evaluate the following:
 - (a) The output signal-to-noise ratio.
 - (b) The channel signal-to-noise ratio.

Hence, show that the figure of merit for the SSB receiver is exactly the same as that for the DSB-SC receiver. Note that unlike the DSB-SC receiver, the midband frequency of the spectral density function of the narrowband-filtered noise at the front end of the SSB

receiver is offset from the carrier frequency f_c by an amount equal to $W/2$, where W is the message bandwidth.

- 2.50 Let a message signal $m(t)$ be transmitted using single-sideband modulation. The power spectral density of $m(t)$ is

$$S_M(f) = \begin{cases} a \frac{|f|}{W}, & |f| \leq W \\ 0, & \text{otherwise} \end{cases}$$

where a and W are constants. White Gaussian noise of zero mean and power spectral density $N_0/2$ is added to the SSB modulated wave at the receiver input. Find an expression for the output signal-to-noise ratio of the receiver.

- 2.51 Consider the output of an envelope detector defined by Equation (2.92), which is reproduced here for convenience

$$y(t) = \{[A_c + A_c k_a m(t) + n_I(t)]^2 + n_Q^2(t)\}^{1/2}$$

- (a) Assume that the probability of the event

$$|n_Q(t)| > \varepsilon A_c [1 + k_a m(t)]$$

is equal to or less than δ_1 , where $\varepsilon \ll 1$. What is the probability that the effect of the quadrature component $n_Q(t)$ is negligible?

- (b) Suppose that k_a is adjusted relative to the message signal $m(t)$ such that the probability of the event

$$A_c [1 + k_a m(t)] + n_I(t) < 0$$

is equal to δ_2 . What is the probability that the approximation

$$y(t) \approx A_c [1 + k_a m(t)] + n_I(t)$$

is valid?

- (c) Comment on the significance of the result in part (b) for the case when δ_1 and δ_2 are both small compared with unity.

- 2.52 An unmodulated carrier of amplitude A_c and frequency f_c and band-limited white noise are summed and then passed through an ideal envelope detector. Assume the noise spectral density to be of height $N_0/2$ and bandwidth $2W$, centered about the carrier frequency f_c . Determine the output signal-to-noise ratio for the case when the carrier-to-noise ratio is high.

- 2.53 Let R denote the random variable obtained by observing the output of an envelope detector at some fixed time. Intuitively, the envelope detector is expected to be operating well into the threshold region if the probability that the random variable R exceeds the carrier amplitude A_c is 0.5. On the other hand, if this same probability is only 0.01, the envelope detector is expected to be relatively free of loss of message and the threshold effect.

- (a) Assuming that the narrowband noise at the detector input is white, zero-mean, Gaussian with spectral density $N_0/2$ and the message bandwidth is W , show that the probability of the event $R \geq A_c$ is

$$P(R \geq A_c) = \exp(-\rho)$$

where ρ is the carrier-to-noise ratio:

$$\rho = \frac{A_c^2}{4WN_0}$$

- (b) Using the formula for this probability, calculate the carrier-to-noise ratio when (1) the envelope detector is expected to be well into the threshold region, and (2) it is expected to be operating satisfactorily.

- 2.54 Consider a phase modulation (PM) system, with the modulated wave defined by

$$s(t) = A_c \cos[2\pi f_c t + k_p m(t)]$$

where k_p is a constant and $m(t)$ is the message signal. The additive noise $n(t)$ at the phase detector input is

$$n(t) = n_i(t) \cos(2\pi f_c t) - n_Q(t) \sin(2\pi f_c t)$$

Assuming that the carrier-to-noise ratio at the detector input is high compared with unity, determine (a) the output signal-to-noise ratio and (b) the figure of merit of the system. Compare your results with the FM system for the case of sinusoidal modulation.

- 2.55 An FDM system uses single-sideband modulation to combine 12 independent voice signals and then uses frequency modulation to transmit the composite baseband signal. Each voice signal has an average power P and occupies the frequency band 0.3 to 3.4 kHz; the system allocates it a bandwidth of 4 kHz. For each voice signal, only the lower sideband is transmitted. The subcarrier waves used for the first stage of modulation are defined by

$$c_k(t) = A_k \cos(2\pi k f_0 t), \quad 0 \leq k \leq 11$$

The received signal consists of the transmitted FM signal plus white Gaussian noise of zero mean and power spectral density $N_0/2$.

- (a) Sketch the power spectral density of the signal produced at the frequency discriminator output, showing both the signal and noise components.
 - (b) Find the relationship between the subcarrier amplitudes A_k so that the modulated voice signals have equal signal-to-noise ratios.
- 2.56 In the discussion on FM threshold effect presented in Section 2.13, we described the conditions for positive-going and negative-going clicks in terms of the envelope $r(t)$ and phase $\psi(t)$ of the narrowband noise $n(t)$. Reformulate these conditions in terms of the in-phase component $n_i(t)$ and quadrature component $n_Q(t)$ of $n(t)$.
- 2.57 By using the pre-emphasis filter shown in Figure 2.50a and with a voice signal as the modulating wave, an FM transmitter produces a signal that is essentially frequency-modulated by the lower audio frequencies and phase-modulated by the higher audio frequencies. Explain the reasons for this phenomenon.
- 2.58 Suppose that the transfer functions of the pre-emphasis and de-emphasis filters of an FM system are scaled as follows:

$$H_{pe}(f) = k \left(1 + \frac{jf}{f_0} \right)$$

and

$$H_{de}(f) = \frac{1}{k} \left(\frac{1}{1 + jf/f_0} \right)$$

The scaling factor k is to be chosen so that the average power of the emphasized message signal is the same as that of the original message signal $m(t)$.

- (a) Find the value of k that satisfies this requirement for the case when the power spectral density of the message signal $m(t)$ is

$$S_M(f) = \begin{cases} \frac{S_0}{1 + (f/f_0)^2}, & -W \leq f \leq W \\ 0, & \text{elsewhere} \end{cases}$$

- (b) What is the corresponding value of the improvement factor I produced by using this pair of pre-emphasis and de-emphasis filters? Compare this ratio with that obtained in Example 2.6. The improvement factor I is defined by Equation (2.160).

- 2.59 A phase modulation (PM) system uses a pair of pre-emphasis and de-emphasis filters defined by the transfer functions

$$H_{pe}(f) = 1 + \frac{jf}{f_0}$$

and

$$H_{de}(f) = \frac{1}{1 + (jf/f_0)}$$

Show that the improvement in output signal-to-noise ratio produced by using this pair of filters is

$$I = \frac{W/f_0}{\tan^{-1}(W/f_0)}$$

where W is the message bandwidth. Evaluate this improvement for the case when $W = 15$ kHz and $f_0 = 2.1$ kHz, and compare your result with the corresponding value for an FM system.

Computer Experiments

- 2.60 In this experiment we study the behavior of the envelope detector shown in Figure P2.5 for the following specifications:

Source resistance, $R_s = 75 \Omega$

Load resistance, $R_L = 10 \text{ k}\Omega$

Capacitance, $C = 0.01 \mu\text{F}$

The diode has a resistance of 25Ω when it is forward-biased and infinite resistance when reverse-biased.

Compute the waveform of the envelope detector output, assuming an input sinusoidal AM wave with 50 percent modulation. The modulation frequency is 1 kHz, and the carrier frequency is 20 kHz.

- 2.61 In this experiment we continue the study of the phase-locked loop considered in Section 2.14:

- (a) Compute variations in the instantaneous frequency of the voltage-controlled oscillator in the loop for the following loop parameters:

$$\text{Loop-gain parameter, } K_0 = \frac{50}{2\pi} \text{ Hz}$$

$$\text{Natural frequency, } f_n = \frac{1}{2\pi} \text{ Hz}$$

$$\text{Damping factor, } \zeta = 0.707$$

Perform the computations for the following values of frequency step: $\Delta f = 0.125,$

$$0.5, \frac{7}{12}, \frac{2}{3} \text{ Hz.}$$

- (b) For the parameters of the phase-locked loop as specified in Experiment 2 in Section 2.14, compute how variations in the relative frequency deviation $\Delta f \cdot f_n / f_n^2$ affect the peak phase error of the phase-locked loop.

PULSE MODULATION

This chapter, representing the transition from analog to digital communications, covers the following topics:

- ▶ *Sampling, which is basic to all forms of pulse modulation.*
- ▶ *Pulse-amplitude modulation, which is the simplest form of pulse modulation.*
- ▶ *Quantization, which, when combined with sampling, permits the representation of an analog signal in discrete form in both amplitude and time.*
- ▶ *Pulse-code modulation, which is the standard method for the transmission of an analog message signal by digital means.*
- ▶ *Time-division multiplexing, which provides for the time sharing of a common channel by a plurality of users by means of pulse modulation.*
- ▶ *Digital multiplexers, which combine many slow bit streams into a single faster stream.*
- ▶ *Other forms of digital pulse modulation, namely, delta modulation and differential pulse-code modulation.*
- ▶ *Linear prediction, which is basic to the encoding of analog message signals at reduced bit rates as in differential pulse-code modulation.*
- ▶ *Adaptive forms of differential pulse-code modulation and delta modulation.*
- ▶ *The MPEG-1/audio coding standard, which is a transparent, perceptually lossless compression system.*

3.1 Introduction

In *continuous-wave (CW) modulation*, which we studied in Chapter 2, some parameter of a sinusoidal carrier wave is varied continuously in accordance with the message signal. This is in direct contrast to pulse modulation, which we study in the present chapter. In *pulse modulation*, some parameter of a pulse train is varied in accordance with the message signal. We may distinguish two families of pulse modulation: *analog pulse modulation* and *digital pulse modulation*. In analog pulse modulation, a periodic pulse train is used as the carrier wave, and some characteristic feature of each pulse (e.g., amplitude, duration, or position) is varied in a continuous manner in accordance with the corresponding *sample* value of the message signal. Thus in analog pulse modulation, information is transmitted basically in analog form, but the transmission takes place at discrete times. In digital pulse modulation, on the other hand, the message signal is represented in a form that is discrete in both time and amplitude, thereby permitting its transmission in digital form as a sequence of *coded pulses*; this form of signal transmission has *no* CW counterpart.

The use of coded pulses for the transmission of analog information-bearing signals represents a basic ingredient in the application of digital communications. This chapter may therefore be viewed as a transition from analog to digital communications in our study of the principles of communication systems. We begin the discussion by describing the sampling process, which is basic to all pulse modulation systems, whether they are analog or digital.

3.2 Sampling Process

The *sampling process* is usually described in the time domain. As such, it is an operation that is basic to digital signal processing and digital communications. Through use of the sampling process, an analog signal is converted into a corresponding sequence of samples that are usually spaced uniformly in time. Clearly, for such a procedure to have practical utility, it is necessary that we choose the sampling rate properly, so that the sequence of samples uniquely defines the original analog signal. This is the essence of the sampling theorem, which is derived in what follows.

Consider an arbitrary signal $g(t)$ of finite energy, which is specified for all time. A segment of the signal $g(t)$ is shown in Figure 3.1a. Suppose that we sample the signal $g(t)$ instantaneously and at a uniform rate, once every T_s seconds. Consequently, we obtain an infinite sequence of samples spaced T_s seconds apart and denoted by $\{g(nT_s)\}$, where n takes on all possible integer values. We refer to T_s as the *sampling period*, and to its reciprocal $f_s = 1/T_s$ as the *sampling rate*. This ideal form of sampling is called *instantaneous sampling*.

Let $g_\delta(t)$ denote the signal obtained by individually weighting the elements of a periodic sequence of delta functions spaced T_s seconds apart by the sequence of numbers $\{g(nT_s)\}$, as shown by (see Figure 3.1b)

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \delta(t - nT_s) \quad (3.1)$$

We refer to $g_\delta(t)$ as the *ideal sampled signal*. The term $\delta(t - nT_s)$ represents a delta function positioned at time $t = nT_s$. From the definition of the delta function, we recall that such an idealized function has unit area; see Appendix 2. We may therefore view the multiplying factor $g(nT_s)$ in Equation (3.1) as a “mass” assigned to the delta function $\delta(t - nT_s)$. A delta function weighted in this manner is closely approximated by a rectangular pulse of

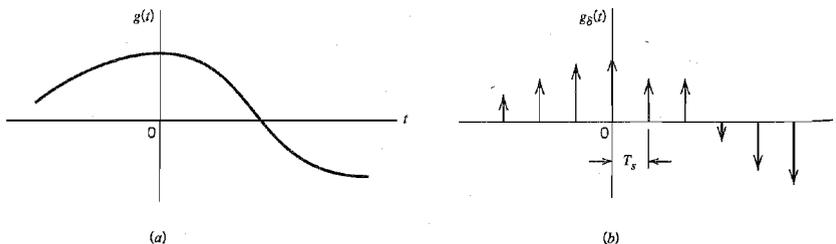


FIGURE 3.1 The sampling process. (a) Analog signal. (b) Instantaneously sampled version of the analog signal.

duration Δt and amplitude $g(nT_s)/\Delta t$; the smaller we make Δt the better will be the approximation.

Using the table of Fourier-transform pairs, we may write (see the last item of Table A6.3)

$$g_\delta(t) \Leftrightarrow f_s \sum_{m=-\infty}^{\infty} G(f - mf_s) \quad (3.2)$$

where $G(f)$ is the Fourier transform of the original signal $g(t)$, and f_s is the sampling rate. Equation (3.2) states that the process of uniformly sampling a continuous-time signal of finite energy results in a periodic spectrum with a period equal to the sampling rate.

Another useful expression for the Fourier transform of the ideal sampled signal $g_\delta(t)$ may be obtained by taking the Fourier transform of both sides of Equation (3.1) and noting that the Fourier transform of the delta function $\delta(t - nT_s)$ is equal to $\exp(-j2\pi n f T_s)$. Let $G_\delta(f)$ denote the Fourier transform of $g_\delta(t)$. We may therefore write

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s) \exp(-j2\pi n f T_s) \quad (3.3)$$

This relation is called the *discrete-time Fourier transform*. It may be viewed as a complex Fourier series representation of the periodic frequency function $G_\delta(f)$, with the sequence of samples $\{g(nT_s)\}$ defining the coefficients of the expansion.

The relations, as derived here, apply to any continuous-time signal $g(t)$ of finite energy and infinite duration. Suppose, however, that the signal $g(t)$ is *strictly band-limited*, with no frequency components higher than W Hertz. That is, the Fourier transform $G(f)$ of the signal $g(t)$ has the property that $G(f)$ is zero for $|f| \geq W$, as illustrated in Figure 3.2a; the shape of the spectrum shown in this figure is intended for the purpose of illustration only. Suppose also that we choose the sampling period $T_s = 1/2W$. Then the corresponding spectrum $G_\delta(f)$ of the sampled signal $g_\delta(t)$ is as shown in Figure 3.2b. Putting $T_s = 1/2W$ in Equation (3.3) yields

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi n f}{W}\right) \quad (3.4)$$

From Equation (3.2), we readily see that the Fourier transform of $g_\delta(t)$ may also be expressed as

$$G_\delta(f) = f_s G(f) + f_s \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} G(f - mf_s) \quad (3.5)$$

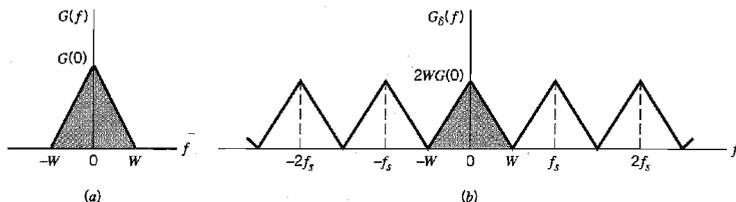


FIGURE 3.2 (a) Spectrum of a strictly band-limited signal $g(t)$. (b) Spectrum of the sampled version of $g(t)$ for a sampling period $T_s = 1/2W$.

Hence, under the following two conditions:

1. $G(f) = 0$ for $|f| \geq W$
2. $f_s = 2W$

we find from Equation (3.5) that

$$G(f) = \frac{1}{2W} G_s(f), \quad -W < f < W \quad (3.6)$$

Substituting Equation (3.4) into (3.6), we may also write

$$G(f) = \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi n f}{W}\right), \quad -W < f < W \quad (3.7)$$

Therefore, if the sample values $g(n/2W)$ of a signal $g(t)$ are specified for all n , then the Fourier transform $G(f)$ of the signal is uniquely determined by using the discrete-time Fourier transform of Equation (3.7). Because $g(t)$ is related to $G(f)$ by the inverse Fourier transform, it follows that the signal $g(t)$ is itself uniquely determined by the sample values $g(n/2W)$ for $-\infty < n < \infty$. In other words, the sequence $\{g(n/2W)\}$ has all the information contained in $g(t)$.

Consider next the problem of reconstructing the signal $g(t)$ from the sequence of sample values $\{g(n/2W)\}$. Substituting Equation (3.7) in the formula for the inverse Fourier transform defining $g(t)$ in terms of $G(f)$, we get

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} G(f) \exp(j2\pi f t) df \\ &= \int_{-W}^W \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi n f}{W}\right) \exp(j2\pi f t) df \end{aligned}$$

Interchanging the order of summation and integration:

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \frac{1}{2W} \int_{-W}^W \exp\left[j2\pi f\left(t - \frac{n}{2W}\right)\right] df \quad (3.8)$$

The integral term in Equation (3.8) is readily evaluated, yielding the final result

$$\begin{aligned} g(t) &= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \frac{\sin(2\pi W t - n\pi)}{(2\pi W t - n\pi)} \\ &= \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \operatorname{sinc}(2W t - n), \quad -\infty < t < \infty \end{aligned} \quad (3.9)$$

Equation (3.9) provides an *interpolation formula* for reconstructing the original signal $g(t)$ from the sequence of sample values $\{g(n/2W)\}$, with the sinc function $\operatorname{sinc}(2Wt)$ playing the role of an *interpolation function*. Each sample is multiplied by a delayed version of the interpolation function, and all the resulting waveforms are added to obtain $g(t)$.

We may now state the *sampling theorem* for strictly band-limited signals of finite energy in two equivalent parts, which apply to the transmitter and receiver of a pulse-modulation system, respectively:

1. A band-limited signal of finite energy, which has no frequency components higher than W Hertz, is completely described by specifying the values of the signal at instants of time separated by $1/2W$ seconds.
2. A band-limited signal of finite energy, which has no frequency components higher than W Hertz, may be completely recovered from a knowledge of its samples taken at the rate of $2W$ samples per second.

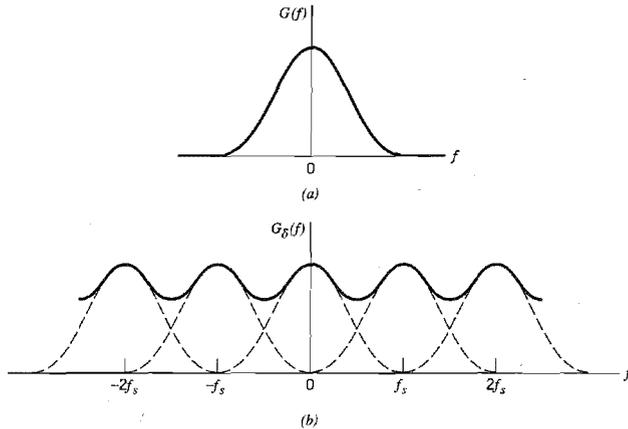


FIGURE 3.3 (a) Spectrum of a signal. (b) Spectrum of an undersampled version of the signal exhibiting the aliasing phenomenon.

The sampling rate of $2W$ samples per second, for a signal bandwidth of W Hertz, is called the *Nyquist rate*; its reciprocal $1/2W$ (measured in seconds) is called the *Nyquist interval*.

The derivation of the sampling theorem, as described herein, is based on the assumption that the signal $g(t)$ is strictly band limited. In practice, however, an information-bearing signal is *not* strictly band limited, with the result that some degree of undersampling is encountered. Consequently, some *aliasing* is produced by the sampling process. Aliasing refers to the phenomenon of a high-frequency component in the spectrum of the signal seemingly taking on the identity of a lower frequency in the spectrum of its sampled version, as illustrated in Figure 3.3. The aliased spectrum, shown by the solid curve in Figure 3.3b, pertains to an “undersampled” version of the message signal represented by the spectrum of Figure 3.3a.

To combat the effects of aliasing in practice, we may use two corrective measures, as described here:

1. Prior to sampling, a low-pass *anti-aliasing filter* is used to attenuate those high-frequency components of the signal that are not essential to the information being conveyed by the signal.
2. The filtered signal is sampled at a rate slightly higher than the Nyquist rate.

The use of a sampling rate higher than the Nyquist rate also has the beneficial effect of easing the design of the *reconstruction filter* used to recover the original signal from its sampled version. Consider the example of a message signal that has been anti-alias (low-pass) filtered, resulting in the spectrum shown in Figure 3.4a. The corresponding spectrum of the instantaneously sampled version of the signal is shown in Figure 3.4b, assuming a sampling rate higher than the Nyquist rate. According to Figure 3.4b, we readily see that the design of the reconstruction filter may be specified as follows (see Figure 3.4c):

- ▶ The reconstruction filter is low-pass with a passband extending from $-W$ to W , which is itself determined by the anti-aliasing filter.
- ▶ The filter has a transition band extending (for positive frequencies) from W to $f_s - W$, where f_s is the sampling rate.

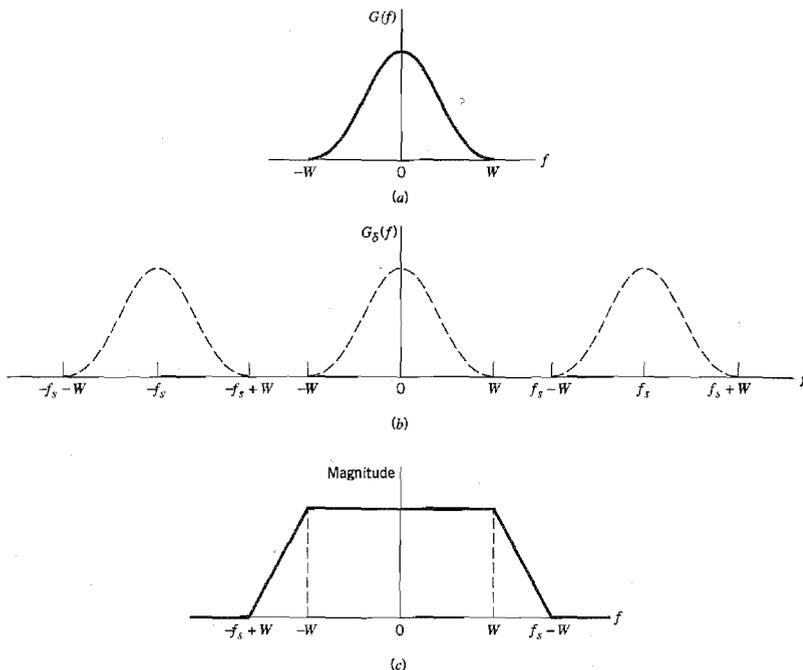


FIGURE 3.4 (a) Anti-alias filtered spectrum of an information-bearing signal. (b) Spectrum of instantaneously sampled version of the signal, assuming the use of a sampling rate greater than the Nyquist rate. (c) Magnitude response of reconstruction filter.

The fact that the reconstruction filter has a well-defined transition band means that it is physically realizable.

3.3 Pulse-Amplitude Modulation

Now that we understand the essence of the sampling process, we are ready to formally define pulse-amplitude modulation, which is the simplest and most basic form of analog pulse modulation. In *pulse-amplitude modulation (PAM)*, the amplitudes of regularly spaced pulses are varied in proportion to the corresponding sample values of a continuous message signal; the pulses can be of a rectangular form or some other appropriate shape. Pulse-amplitude modulation as defined here is somewhat similar to natural sampling, where the message signal is multiplied by a periodic train of rectangular pulses. However, in natural sampling the top of each modulated rectangular pulse varies with the message signal, whereas in PAM it is maintained flat; natural sampling is explored further in Problem 3.2.

The waveform of a PAM signal is illustrated in Figure 3.5. The dashed curve in this figure depicts the waveform of a message signal $m(t)$, and the sequence of amplitude-

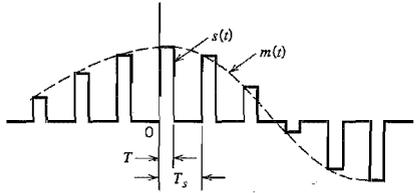


FIGURE 3.5 Flat-top samples, representing an analog signal.

modulated rectangular pulses shown as solid lines represents the corresponding PAM signal $s(t)$. There are two operations involved in the generation of the PAM signal:

1. *Instantaneous sampling* of the message signal $m(t)$ every T_s seconds, where the sampling rate $f_s = 1/T_s$ is chosen in accordance with the sampling theorem.
2. *Lengthening* the duration of each sample so obtained to some constant value T .

In digital circuit technology, these two operations are jointly referred to as “sample and hold.” One important reason for intentionally lengthening the duration of each sample is to avoid the use of an excessive channel bandwidth, since bandwidth is inversely proportional to pulse duration. However, care has to be exercised in how long we make the sample duration T , as the following analysis reveals.

Let $s(t)$ denote the sequence of flat-top pulses generated in the manner described in Figure 3.5. We may express the PAM signal as

$$s(t) = \sum_{n=-\infty}^{\infty} m(nT_s)h(t - nT_s) \quad (3.10)$$

where T_s is the *sampling period* and $m(nT_s)$ is the sample value of $m(t)$ obtained at time $t = nT_s$. The $h(t)$ is a standard rectangular pulse of unit amplitude and duration T , defined as follows (see Figure 3.6a):

$$h(t) = \begin{cases} 1, & 0 < t < T \\ \frac{1}{2}, & t = 0, t = T \\ 0, & \text{otherwise} \end{cases} \quad (3.11)$$

By definition, the instantaneously sampled version of $m(t)$ is given by

$$m_s(t) = \sum_{n=-\infty}^{\infty} m(nT_s) \delta(t - nT_s) \quad (3.12)$$

where $\delta(t - nT_s)$ is a time-shifted delta function. Therefore, convolving $m_s(t)$ with the pulse $h(t)$, we get

$$\begin{aligned} m_s(t) \star h(t) &= \int_{-\infty}^{\infty} m_s(\tau)h(t - \tau) d\tau \\ &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} m(nT_s) \delta(\tau - nT_s)h(t - \tau) d\tau \\ &= \sum_{n=-\infty}^{\infty} m(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s)h(t - \tau) d\tau \end{aligned} \quad (3.13)$$

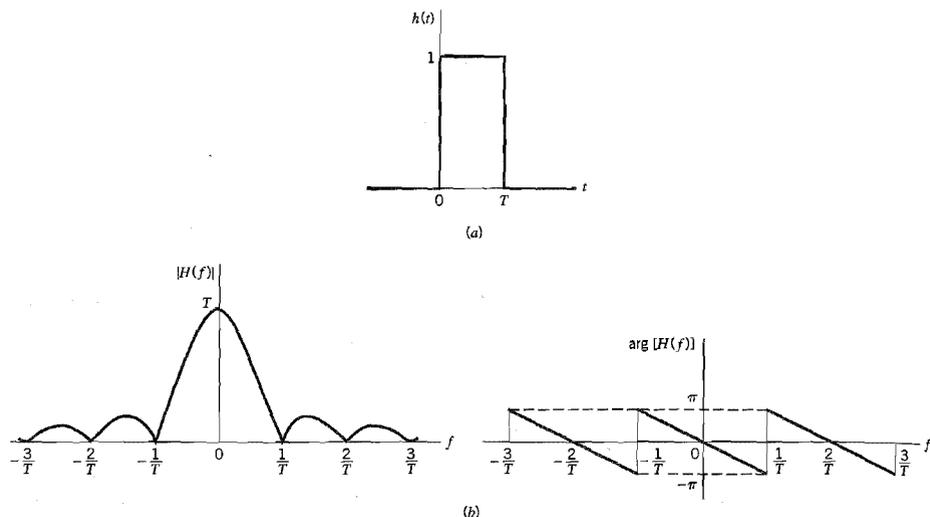


FIGURE 3.6 (a) Rectangular pulse $h(t)$. (b) Spectrum $H(f)$, made up of the magnitude $|H(f)|$, and phase $\arg[H(f)]$.

Using the sifting property of the delta function (see Appendix 2), we thus obtain

$$m_s(t) \star b(t) = \sum_{n=-\infty}^{\infty} m(nT_s)b(t - nT_s) \quad (3.14)$$

From Equations (3.10) and (3.14) it follows that the PAM signal $s(t)$ is mathematically equivalent to the convolution of $m_s(t)$, the instantaneously sampled version of $m(t)$, and the pulse $b(t)$, as shown by

$$s(t) = m_s(t) \star b(t) \quad (3.15)$$

Taking the Fourier transform of both sides of Equation (3.15) and recognizing that the convolution of two time functions is transformed into the multiplication of their respective Fourier transforms, we get

$$S(f) = M_s(f)H(f) \quad (3.16)$$

where $S(f) = F[s(t)]$, $M_s(f) = F[m_s(t)]$, and $H(f) = F[b(t)]$. Adapting Equation (3.2) to the problem at hand, we note that the Fourier transform $M_s(f)$ is related to the Fourier transform $M(f)$ of the original message signal $m(t)$ as follows:

$$M_s(f) = f_s \sum_{k=-\infty}^{\infty} M(f - kf_s) \quad (3.17)$$

where f_s is the sampling rate. Therefore, substitution of Equation (3.17) into (3.16) yields

$$S(f) = f_s \sum_{k=-\infty}^{\infty} M(f - kf_s)H(f) \quad (3.18)$$

Given a PAM signal $s(t)$ whose Fourier transform $S(f)$ is as defined in Equation (3.18), how do we recover the original message signal $m(t)$? As a first step in this recon-

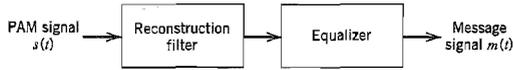


FIGURE 3.7 System for recovering message signal $m(t)$ from PAM signal $s(t)$.

struction, we may pass $s(t)$ through a low-pass filter whose frequency response is defined in Figure 3.4c; here it is assumed that the message is limited to bandwidth W and the sampling rate f_s is larger than the Nyquist rate $2W$. Then, from Equation (3.18) we find that the spectrum of the resulting filter output is equal to $M(f)H(f)$. This output is equivalent to passing the original message signal $m(t)$ through another low-pass filter of frequency response $H(f)$.

From Equation (3.11) we note that the Fourier transform of the rectangular pulse $b(t)$ is given by

$$H(f) = T \operatorname{sinc}(fT) \exp(-j\pi fT) \quad (3.19)$$

which is plotted in Figure 3.6b. We see therefore that by using flat-top samples to generate a PAM signal, we have introduced *amplitude distortion* as well as a *delay* of $T/2$. This effect is rather similar to the variation in transmission with frequency that is caused by the finite size of the scanning aperture in television. Accordingly, the distortion caused by the use of pulse-amplitude modulation to transmit an analog information-bearing signal is referred to as the *aperture effect*.

This distortion may be corrected by connecting an *equalizer* in cascade with the low-pass reconstruction filter, as shown in Figure 3.7. The equalizer has the effect of decreasing the in-band loss of the reconstruction filter as the frequency increases in such a manner as to compensate for the aperture effect. Ideally, the magnitude response of the equalizer is given by

$$\frac{1}{|H(f)|} = \frac{1}{T \operatorname{sinc}(fT)} = \frac{\pi f}{\sin(\pi fT)} \quad (3.20)$$

The amount of equalization needed in practice is usually small. Indeed, for a duty cycle $T/T_s \leq 0.1$, the amplitude distortion is less than 0.5 percent, in which case the need for equalization may be omitted altogether.

The transmission of a PAM signal imposes rather stringent requirements on the magnitude and phase responses of the channel, because of the relatively short duration of the transmitted pulses. Furthermore, the noise performance of a PAM system can never be better than baseband-signal transmission. Accordingly, we find that for transmission over long distances, PAM would be used only as a means of message processing for time-division multiplexing, from which conversion to some other form of pulse modulation is subsequently made; time-division multiplexing is discussed in Section 3.9.

3.4 Other Forms of Pulse Modulation

In a pulse modulation system we may use the increased bandwidth consumed by the pulses to improve the noise performance of the system. This can be achieved by representing the sample values of the message signals by some property of the pulse other than amplitude:

- ▶ *Pulse-duration modulation* (PDM), also referred to as *pulse-width modulation*, where samples of the message signal are used to vary the duration of the individual pulses in the carrier.

- *Pulse-position modulation* (PPM), where the position of a pulse relative to its unmodulated time of occurrence is varied in accordance with the message signal.

These two other forms of pulse modulation are illustrated in Figure 3.8 for the case of a sinusoidal modulating wave.

In PDM, long pulses expend considerable power while bearing no additional information. If this unused power is subtracted from PDM so that only time transitions are preserved, we obtain PPM. Accordingly, PPM is a more efficient form of pulse modulation than PDM.

Since in a PPM system the transmitted information is contained in the relative positions of the modulated pulses, the presence of additive noise affects the performance of such a system by falsifying the time at which the modulated pulses are judged to occur. Immunity to noise can be established by making the pulse build up so rapidly that the time interval during which noise can exert any perturbation is very short. Indeed, additive noise would have no effect on the pulse positions if the received pulses were perfectly rectangular, because the presence of noise introduces only vertical perturbations. However, the reception of perfectly rectangular pulses would require an infinite channel bandwidth, which is of course impractical. Thus with a finite channel bandwidth in practice, we find that the received pulses have a finite rise time, so the performance of the PPM receiver is affected by noise, which is to be expected.

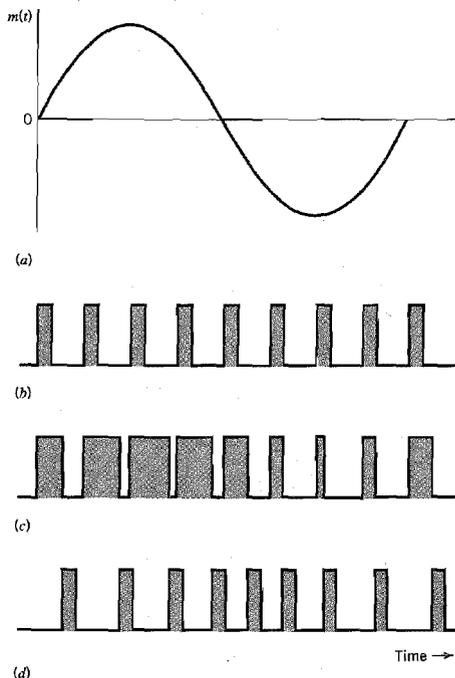


FIGURE 3.8 Illustrating two different forms of pulse-time modulation for the case of a sinusoidal modulating wave. (a) Modulating wave. (b) Pulse carrier. (c) PDM wave. (d) PPM wave.

As in a CW modulation system, the noise performance of a PPM system may be described in terms of the output signal-to-noise ratio (SNR). Also, to find the noise improvement produced by PPM over baseband transmission of a message signal, we may use the figure of merit defined as the output signal-to-noise ratio of the PPM system divided by the channel signal-to-noise ratio; see Section 2.10. Assuming that the average power of the channel noise is small compared to the peak pulse power, the figure of merit of the PPM system is proportional to the square of the transmission bandwidth B_T normalized with respect to the message bandwidth W . When, however, the input signal-to-noise ratio drops below a critical value, the system suffers a loss of the wanted message signal at the receiver output. That is, a PPM system suffers from a threshold effect of its own.

3.5 Bandwidth–Noise Trade-Off

In the context of noise performance, a PPM system is the optimum form of analog pulse modulation. The noise analysis of a PPM system reveals that pulse-position modulation (PPM) and frequency modulation (FM) systems exhibit a similar noise performance, as summarized here.¹

1. Both systems have a figure of merit proportional to the square of the transmission bandwidth normalized with respect to the message bandwidth.
2. Both systems exhibit a threshold effect as the signal-to-noise ratio is reduced.

The practical implication of point 1 is that, in terms of a trade-off of increased transmission bandwidth for improved noise performance, the best that we can do with continuous-wave (CW) modulation and analog pulse modulation systems is to follow a *square law*. A question that arises at this point in the discussion is: Can we produce a trade-off better than a square law? The answer is an emphatic yes, and *digital pulse modulation* is the way to do it. The use of such a method is a radical departure from CW modulation.

Specifically, in a basic form of digital pulse modulation known as *pulse-code modulation* (PCM),² a message signal is represented in discrete form in both time and amplitude. This form of signal representation permits the transmission of the message signal as a sequence of *coded binary pulses*. Given such a sequence, the effect of channel noise at the receiver output can be reduced to a negligible level simply by making the average power of the transmitted binary PCM wave large enough compared to the average power of the noise.

Two fundamental processes are involved in the generation of a binary PCM wave: *sampling* and *quantization*. The sampling process takes care of the discrete-time representation of the message signal; for its proper application, we have to follow the sampling theorem described in Section 3.2. The quantization process takes care of the discrete-amplitude representation of the message signal; quantization is a new process, the details of which are described in the next section. For now it suffices to say that the combined use of sampling and quantization permits the transmission of a message signal in coded form. This, in turn, makes it possible to realize an *exponential law* for the bandwidth-noise trade-off, which is also demonstrated in the next section.

3.6 Quantization Process³

A continuous signal, such as voice, has a continuous range of amplitudes and therefore its samples have a continuous amplitude range. In other words, within the finite amplitude

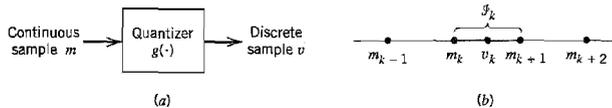


FIGURE 3.9 Description of a memoryless quantizer.

range of the signal, we find an infinite number of amplitude levels. It is not necessary in fact to transmit the exact amplitudes of the samples. Any human sense (the ear or the eye), as ultimate receiver, can detect only finite intensity differences. This means that the original continuous signal may be *approximated* by a signal constructed of discrete amplitudes selected on a minimum error basis from an available set. The existence of a finite number of discrete amplitude levels is a basic condition of pulse-code modulation. Clearly, if we assign the discrete amplitude levels with sufficiently close spacing, we may make the approximated signal practically indistinguishable from the original continuous signal.

Amplitude *quantization* is defined as the process of transforming the sample amplitude $m(nT_s)$ of a message signal $m(t)$ at time $t = nT_s$ into a discrete amplitude $v(nT_s)$ taken from a finite set of possible amplitudes. We assume that the quantization process is *memoryless* and *instantaneous*, which means that the transformation at time $t = nT_s$ is not affected by earlier or later samples of the message signal. This simple form of scalar quantization, though not optimum, is commonly used in practice.

When dealing with a memoryless quantizer, we may simplify the notation by dropping the time index. We may thus use the symbol m in place of $m(nT_s)$, as indicated in the block diagram of a quantizer shown in Figure 3.9a. Then, as shown in Figure 3.9b, the signal amplitude m is specified by the index k if it lies inside the *partition cell*

$$\mathcal{F}_k: \{m_k < m \leq m_{k+1}\}, \quad k = 1, 2, \dots, L \quad (3.21)$$

where L is the total number of amplitude levels used in the quantizer. The discrete amplitudes m_k , $k = 1, 2, \dots, L$, at the quantizer input are called *decision levels* or *decision thresholds*. At the quantizer output, the index k is transformed into an amplitude v_k that represents all amplitudes of the cell \mathcal{F}_k ; the discrete amplitudes v_k , $k = 1, 2, \dots, L$, are called *representation levels* or *reconstruction levels*, and the spacing between two adjacent representation levels is called a *quantum* or *step-size*. Thus, the quantizer output v equals v_k if the input signal sample m belongs to the interval \mathcal{F}_k . The mapping (see Figure 3.9a)

$$v = g(m) \quad (3.22)$$

is the *quantizer characteristic*, which is a staircase function by definition.

Quantizers can be of a *uniform* or *nonuniform* type. In a uniform quantizer, the representation levels are uniformly spaced; otherwise, the quantizer is nonuniform. In this section, we consider only uniform quantizers; nonuniform quantizers are considered in Section 3.7. The quantizer characteristic can also be of *midtread* or *midrise* type. Figure 3.10a shows the input–output characteristic of a uniform quantizer of the midtread type, which is so called because the origin lies in the middle of a tread of the staircaselike graph. Figure 3.10b shows the corresponding input–output characteristic of a uniform quantizer of the midrise type, in which the origin lies in the middle of a rising part of the staircaselike graph. Note that both the midtread and midrise types of uniform quantizers illustrated in Figure 3.10 are *symmetric* about the origin.

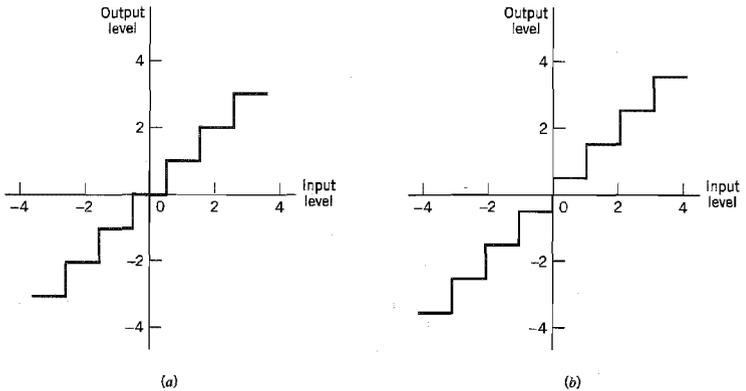


FIGURE 3.10 Two types of quantization: (a) midtread and (b) midrise.

■ QUANTIZATION NOISE

The use of quantization introduces an error defined as the difference between the input signal m and the output signal v . The error is called *quantization noise*. Figure 3.11 illustrates a typical variation of the quantization noise as a function of time, assuming the use of a uniform quantizer of the midtread type.

Let the quantizer input m be the sample value of a zero-mean random variable M . (If the input has a nonzero mean, we can always remove it by subtracting the mean from the input and then adding it back after quantization.) A quantizer $g(\cdot)$ maps the input

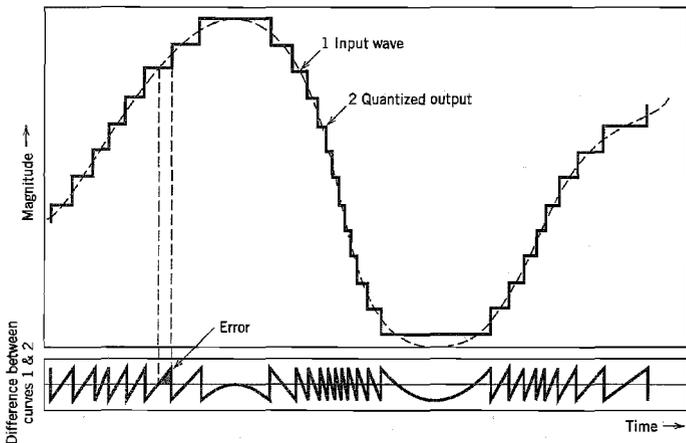


FIGURE 3.11 Illustration of the quantization process. (Adapted from Bennett, 1948, with permission of AT&T.)

random variable M of continuous amplitude into a discrete random variable V ; their respective sample values m and ν are related by Equation (3.22). Let the quantization error be denoted by the random variable Q of sample value q . We may thus write

$$q = m - \nu \quad (3.23)$$

or, correspondingly,

$$Q = M - V \quad (3.24)$$

With the input M having zero mean, and the quantizer assumed to be symmetric as in Figure 3.10, it follows that the quantizer output V and therefore the quantization error Q , will also have zero mean. Thus for a partial statistical characterization of the quantizer in terms of output signal-to-(quantization) noise ratio, we need only find the mean-square value of the quantization error Q .

Consider then an input m of continuous amplitude in the range $(-m_{\max}, m_{\max})$. Assuming a uniform quantizer of the midrise type illustrated in Figure 3.10b, we find that the step-size of the quantizer is given by

$$\Delta = \frac{2m_{\max}}{L} \quad (3.25)$$

where L is the total number of representation levels. For a uniform quantizer, the quantization error Q will have its sample values bounded by $-\Delta/2 \leq q \leq \Delta/2$. If the step-size Δ is sufficiently small (i.e., the number of representation levels L is sufficiently large), it is reasonable to assume that the quantization error Q is a *uniformly distributed* random variable, and the interfering effect of the quantization noise on the quantizer input is similar to that of thermal noise. We may thus express the probability density function of the quantization error Q as follows:

$$f_Q(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < q \leq \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases} \quad (3.26)$$

For this to be true, however, we must ensure that the incoming signal does *not* overload the quantizer. Then, with the mean of the quantization error being zero, its variance σ_Q^2 is the same as the mean-square value:

$$\begin{aligned} \sigma_Q^2 &= E[Q^2] \\ &= \int_{-\Delta/2}^{\Delta/2} q^2 f_Q(q) dq \end{aligned} \quad (3.27)$$

Substituting Equation (3.26) into (3.27), we get

$$\begin{aligned} \sigma_Q^2 &= \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} q^2 dq \\ &= \frac{\Delta^2}{12} \end{aligned} \quad (3.28)$$

Typically, the L -ary number k , denoting the k th representation level of the quantizer, is transmitted to the receiver in binary form. Let R denote the number of *bits per sample* used in the construction of the binary code. We may then write

$$L = 2^R \quad (3.29)$$

or, equivalently,

$$R = \log_2 L \quad (3.30)$$

Hence, substituting Equation (3.29) into (3.25), we get the step size

$$\Delta = \frac{2m_{\max}}{2^R} \quad (3.31)$$

Thus the use of Equation (3.31) in (3.28) yields

$$\sigma_Q^2 = \frac{1}{3}m_{\max}^2 2^{-2R} \quad (3.32)$$

Let P denote the average power of the message signal $m(t)$. We may then express the *output signal-to-noise ratio* of a uniform quantizer as

$$\begin{aligned} (\text{SNR})_O &= \frac{P}{\sigma_Q^2} \\ &= \left(\frac{3P}{m_{\max}^2} \right) 2^{2R} \end{aligned} \quad (3.33)$$

Equation (3.33) shows that the output signal-to-noise ratio of the quantizer increases *exponentially* with increasing number of bits per sample, R . Recognizing that an increase in R requires a proportionate increase in the channel (transmission) bandwidth B_T , we thus see that the use of a binary code for the representation of a message signal (as in pulse-code modulation) provides a more efficient method than either frequency modulation (FM) or pulse-position modulation (PPM) for the trade-off of increased channel bandwidth for improved noise performance. In making this statement, we presume that the FM and PPM systems are limited by receiver noise, whereas the binary-coded modulation system is limited by quantization noise. We have more to say on the latter issue in Section 3.8.

► EXAMPLE 3.1 Sinusoidal Modulating Signal

Consider the special case of a full-load sinusoidal modulating signal of amplitude A_m , which utilizes all the representation levels provided. The average signal power is (assuming a load of 1 ohm)

$$P = \frac{A_m^2}{2}$$

The total range of the quantizer input is $2A_m$, because the modulating signal swings between $-A_m$ and A_m . We may therefore set $m_{\max} = A_m$, in which case the use of Equation (3.32) yields the average power (variance) of the quantization noise as

$$\sigma_Q^2 = \frac{1}{3}A_m^2 2^{-2R}$$

Thus the output signal-to-noise ratio of a uniform quantizer, for a full-load test tone, is

$$(\text{SNR})_O = \frac{A_m^2/2}{A_m^2 2^{-2R}/3} = \frac{3}{2} (2^{2R}) \quad (3.34)$$

Expressing the signal-to-noise ratio in decibels, we get

$$10 \log_{10}(\text{SNR})_O = 1.8 + 6R \quad (3.35)$$

TABLE 3.1 Signal-to-(quantization) noise ratio for varying number of representation levels for sinusoidal modulation

Number of Representation Levels, L	Number of Bits per Sample, R	Signal-to-Noise Ratio (dB)
32	5	31.8
64	6	37.8
128	7	43.8
256	8	49.8

For various values of L and R , the corresponding values of signal-to-noise ratio are as given in Table 3.1. From Table 3.1 we can make a quick estimate of the number of bits per sample required for a desired output signal-to-noise ratio, assuming sinusoidal modulation. ◀

Thus far in this section we have focused on how to characterize memoryless scalar quantizers and assess their performance. In so doing, however, we avoided the optimum design of quantizers, that is, the issue of selecting the representation levels and partition cells so as to minimize the average quantization power for a prescribed number of representation levels. Unfortunately, this optimization problem does not lend itself to a closed-form solution because of the highly *nonlinear* nature of the quantization process. Rather, we have effective algorithms for finding the optimum design in an iterative manner. A well-known algorithm that deserves to be mentioned in this context is the Lloyd-Max quantizer, which is discussed next.

■ CONDITIONS FOR OPTIMALITY OF SCALAR QUANTIZERS

In designing a scalar quantizer the challenge is how to select the representation levels and surrounding partition cells so as to minimize the average quantization power for a fixed number of representation levels.

To state the problem in mathematical terms, consider a message signal $m(t)$ drawn from a stationary process $M(t)$. Let $-A \leq m \leq A$ denote the dynamic range of $m(t)$, which is partitioned into a set of L cells, as depicted in Figure 3.12. The boundaries of the partition cells are defined by a set of real numbers m_1, m_2, \dots, m_{L+1} that satisfy the following three conditions:

$$\begin{aligned} m_1 &= -A \\ m_{L+1} &= A \\ m_k &\leq m_{k+1} \text{ for } k = 1, 2, \dots, L \end{aligned}$$

The k th partition cell is defined by

$$\mathcal{F}_k: m_k < m \leq m_{k+1} \text{ for } k = 1, 2, \dots, L \quad (3.36)$$

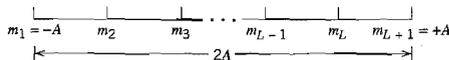


FIGURE 3.12 Illustrating the partitioning of the dynamic range $-A \leq m \leq A$ of a message signal $m(t)$ into a set of L cells.

Let the representation levels (i.e., quantization values) be denoted by v_k , $k = 1, 2, \dots, L$. Then, assuming that $d(m, v_k)$ denotes a *distortion measure* for using v_k to represent all those values of the input m that lie inside the partition cell \mathcal{F}_k , the goal is to find the two sets, $\{v_k\}_{k=1}^L$ and $\{\mathcal{F}_k\}_{k=1}^L$, that minimize the *average distortion*

$$D = \sum_{k=1}^L \int_{m \in \mathcal{F}_k} d(m, v_k) f_M(m) dm \quad (3.37)$$

where $f_M(m)$ is the probability density function of the random variable M with sample value m .

A commonly used distortion measure is

$$d(m, v_k) = (m - v_k)^2 \quad (3.38)$$

in which case we speak of the *mean-square distortion*. In any event, the optimization problem stated herein is nonlinear, defying an explicit, closed-form solution. To get around this difficulty, we resort to an algorithmic approach for solving the problem in an iterative manner.

Structurally speaking, the quantizer consists of two components with interrelated design parameters:

- ▶ An encoder characterized by the set of partition cells $\{\mathcal{F}_k\}_{k=1}^L$; it is located in the transmitter.
- ▶ A decoder characterized by the set of representation levels $\{v_k\}_{k=1}^L$; it is located in the receiver.

Accordingly, we may identify two critically important conditions that provide the mathematical basis for all algorithmic solutions to the optimum quantization problem. One condition assumes that we are given a decoder and the problem is to find the optimum encoder in the transmitter. The other condition assumes that we are given an encoder and the problem is to find the optimum decoder in the receiver. Henceforth, these two conditions are referred to as condition I and condition II, respectively.

Condition I. Optimality of the Encoder for a Given Decoder

The availability of a decoder means that we have a certain *codebook* in mind. Let the codebook be defined by

$$\mathcal{C}: \{v_k\}_{k=1}^L \quad (3.39)$$

Given the codebook \mathcal{C} , the problem is to find the set of partition cells $\{\mathcal{F}_k\}_{k=1}^L$ that minimizes the average distortion D . That is, we wish to find the encoder defined by the nonlinear mapping

$$g(m) = v_k, \quad k = 1, 2, \dots, L \quad (3.40)$$

such that we have

$$D = \int_{-A}^A d(m, g(m)) f_M(m) dm \geq \sum_{k=1}^L \int_{m \in \mathcal{F}_k} [\min_{v_k \in \mathcal{C}} d(m, v_k)] f_M(m) dm \quad (3.41)$$

For the lower bound specified in Equation (3.41) to be attained, we require that the nonlinear mapping of Equation (3.40) be satisfied only if the condition

$$d(m, v_k) \leq d(m, v_j) \quad \text{holds for all } j \neq k \quad (3.42)$$

The necessary condition described in Equation (3.42) for optimality of the encoder for a specified codebook \mathcal{C} is recognized as the *nearest neighbor condition*. In words, the nearest neighbor condition requires that the partition cell \mathcal{F}_k should embody all those values of the input m that are closer to v_k than any other element of the codebook \mathcal{C} . This optimality condition is indeed intuitively satisfying.

Condition II. Optimality of the Decoder for a Given Encoder

Consider next the reverse situation to that described under condition I, which may be stated as follows: Optimize the codebook $\mathcal{C} = \{v_k\}_{k=1}^L$ for the decoder, given that the set of partition cells $\{\mathcal{F}_k\}_{k=1}^L$ characterizing the encoder is fixed. The criterion for optimization is the average (mean-square) distortion:

$$D = \sum_{k=1}^L \int_{m \in \mathcal{F}_k} (m - v_k)^2 f_M(m) dm \quad (3.43)$$

The probability density function $f_M(m)$ is clearly independent of the codebook \mathcal{C} . Hence, differentiating D with respect to the representation level v_k , we readily obtain

$$\frac{\partial D}{\partial v_k} = -2 \sum_{k=1}^L \int_{m \in \mathcal{F}_k} (m - v_k) f_M(m) dm \quad (3.44)$$

Setting $\partial D / \partial v_k$ equal to zero and then solving for v_k , we obtain the optimum value

$$v_{k, \text{opt}} = \frac{\int_{m \in \mathcal{F}_k} m f_M(m) dm}{\int_{m \in \mathcal{F}_k} f_M(m) dm} \quad (3.45)$$

The denominator in Equation (3.45) is just the probability, p_k , that the random variable M with sample value m lies in the partition cell \mathcal{F}_k , as shown by

$$\begin{aligned} p_k &= P(m_k < M \leq m_{k+1}) \\ &= \int_{m \in \mathcal{F}_k} f_M(m) dm \end{aligned} \quad (3.46)$$

Accordingly, we may interpret the optimality condition of Equation (3.45) as choosing the representation level v_k to equal the *conditional mean* of the random variable M , given that M lies in the partition cell \mathcal{F}_k . We can thus formally state the condition for optimality of the decoder for a given encoder as follows:

$$v_{k, \text{opt}} = E[M | m_k < M \leq m_{k+1}] \quad (3.47)$$

where E is the expectation operator. Equation (3.47) is also intuitively satisfying.

Note that the nearest neighbor condition (condition I) for optimality of the encoder for a given decoder was proved for a generic average distortion. However, the conditional mean requirement (condition II) for optimality of the decoder for a given encoder was proved for the special case of a mean-square distortion. In any event, these two conditions are necessary for optimality of a scalar quantizer. Basically, the algorithm for designing the quantizer consists of alternately optimizing the encoder in accordance with condition I, then optimizing the decoder in accordance with condition II, and continuing in this

manner until the average distortion D reaches a minimum. An optimum quantizer designed in this manner is called a *Lloyd-Max quantizer*.⁴

3.7 Pulse-Code Modulation

With the sampling and quantization processes at our disposal, we are now ready to describe pulse-code modulation, which, as mentioned previously, is the most basic form of digital pulse modulation. In *pulse-code modulation (PCM)*, a message signal is represented by a sequence of coded pulses, which is accomplished by representing the signal in discrete form in both time and amplitude. The basic operations performed in the transmitter of a PCM system are *sampling*, *quantizing*, and *encoding*, as shown in Figure 3.13a; the low-pass filter prior to sampling is included to prevent aliasing of the message signal. The quantizing and encoding operations are usually performed in the same circuit, which is called an *analog-to-digital converter*. The basic operations in the receiver are *regeneration* of impaired signals, *decoding*, and *reconstruction* of the train of quantized samples, as shown in Figure 3.13c. Regeneration also occurs at intermediate points along the transmission path as necessary, as indicated in Figure 3.13b. When time-division multiplexing is used, it becomes necessary to synchronize the receiver to the transmitter for the overall system to operate satisfactorily, as discussed in Section 3.9. In what follows, we describe the various operations that constitute a basic PCM system.

■ SAMPLING

The incoming message signal is sampled with a train of narrow rectangular pulses so as to closely approximate the instantaneous sampling process. To ensure perfect reconstruction of the message signal at the receiver, the sampling rate must be greater than twice the highest frequency component W of the message signal in accordance with the sampling theorem. In practice, a low-pass anti-aliasing filter is used at the front end of the sampler to exclude frequencies greater than W before sampling. Thus the application of sampling

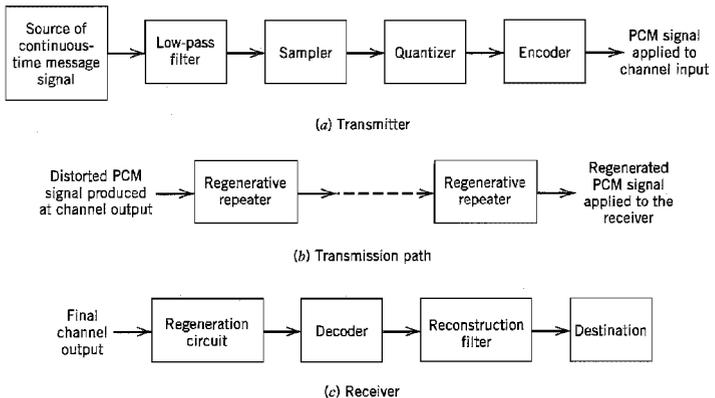


FIGURE 3.13 The basic elements of a PCM system.

permits the reduction of the continuously varying message signal (of some finite duration) to a limited number of discrete values per second.

■ QUANTIZATION

The sampled version of the message signal is then quantized, thereby providing a new representation of the signal that is discrete in both time and amplitude. The quantization process may follow a uniform law as described in Section 3.6. In telephonic communication, however, it is preferable to use a variable separation between the representation levels. For example, the range of voltages covered by voice signals, from the peaks of loud talk to the weak passages of weak talk, is on the order of 1000 to 1. By using a *nonuniform quantizer* with the feature that the step-size increases as the separation from the origin of the input-output amplitude characteristic is increased, the large end steps of the quantizer can take care of possible excursions of the voice signal into the large amplitude ranges that occur relatively infrequently. In other words, the weak passages, which need more protection, are favored at the expense of the loud passages. In this way, a nearly uniform percentage precision is achieved throughout the greater part of the amplitude range of the input signal, with the result that fewer steps are needed than would be the case if a uniform quantizer were used.

The use of a nonuniform quantizer is equivalent to passing the baseband signal through a *compressor* and then applying the compressed signal to a uniform quantizer. A particular form of compression law that is used in practice is the so-called μ -law,⁵ which is defined by

$$|v| = \frac{\log(1 + \mu|m|)}{\log(1 + \mu)} \quad (3.48)$$

where m and v are the normalized input and output voltages, and μ is a positive constant. In Figure 3.14a, we have plotted the μ -law for three different values of μ . The case of uniform quantization corresponds to $\mu = 0$. For a given value of μ , the reciprocal slope

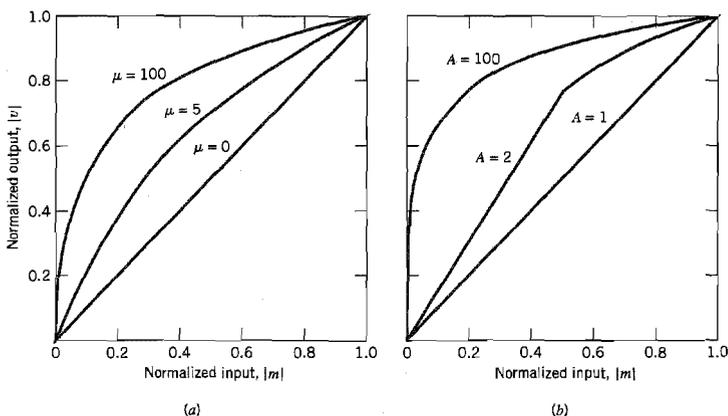


FIGURE 3.14 Compression laws. (a) μ -law. (b) A-law.

of the compression curve, which defines the quantum steps, is given by the derivative of $|m|$ with respect to $|v|$; that is,

$$\frac{d|m|}{d|v|} = \frac{\log(1 + \mu)}{\mu} (1 + \mu|m|) \quad (3.49)$$

We see therefore that the μ -law is neither strictly linear nor strictly logarithmic, but it is approximately linear at low input levels corresponding to $\mu|m| \ll 1$, and approximately logarithmic at high input levels corresponding to $\mu|m| \gg 1$.

Another compression law that is used in practice is the so-called *A-law* defined by

$$|v| = \begin{cases} \frac{A|m|}{1 + \log A}, & 0 \leq |m| \leq \frac{1}{A} \\ \frac{1 + \log(A|m|)}{1 + \log A}, & \frac{1}{A} \leq |m| \leq 1 \end{cases} \quad (3.50)$$

which is plotted in Figure 3.14*b* for varying A . The case of uniform quantization corresponds to $A = 1$. The reciprocal slope of this second compression curve is given by the derivative of $|m|$ with respect to $|v|$, as shown by (depending on the value assigned to the normalized input $|m|$)

$$\frac{d|m|}{d|v|} = \begin{cases} \frac{1 + \log A}{A}, & 0 \leq |m| \leq \frac{1}{A} \\ (1 + A)|m|, & \frac{1}{A} \leq |m| \leq 1 \end{cases} \quad (3.51)$$

To restore the signal samples to their correct relative level, we must, of course, use a device in the receiver with a characteristic complementary to the compressor. Such a device is called an *expander*. Ideally, the compression and expansion laws are exactly inverse so that, except for the effect of quantization, the expander output is equal to the compressor input. The combination of a *compressor* and an *expander* is called a *componder*.

For both the μ -law and A -law, the dynamic range capability of the compander improves with increasing μ and A , respectively. The SNR for low-level signals increases at the expense of the SNR for high-level signals. To accommodate these two conflicting requirements (i.e., a reasonable SNR for both low- and high-level signals), a compromise is usually made in choosing the value of parameter μ for the μ -law and parameter A for the A -law. The typical values used in practice are: $\mu = 255$ and $A = 87.6$.

It is also of interest to note that in actual PCM systems, the companding circuitry does not produce an exact replica of the nonlinear compression curves shown in Figure 3.14. Rather, it provides a *piecewise linear* approximation to the desired curve. By using a large enough number of linear segments, the approximation can approach the true compression curve very closely. This form of approximation is illustrated in Example 3.2.

■ ENCODING

In combining the processes of sampling and quantization, the specification of a continuous message (baseband) signal becomes limited to a discrete set of values, but not in the form best suited to transmission over a telephone line or radio path. To exploit the advantages of sampling and quantizing for the purpose of making the transmitted signal more robust to noise, interference and other channel impairments, we require the use of an *encoding*

TABLE 3.2 Binary number system
for $R = 4$ bits/sample

Ordinal Number of Representation Level	Level Number Expressed as Sum of Powers of 2	Binary Number
0		0000
1	2^0	0001
2	2^1	0010
3	$2^1 + 2^0$	0011
4	2^2	0100
5	$2^2 + 2^0$	0101
6	$2^2 + 2^1$	0110
7	$2^2 + 2^1 + 2^0$	0111
8	2^3	1000
9	$2^3 + 2^0$	1001
10	$2^3 + 2^1$	1010
11	$2^3 + 2^1 + 2^0$	1011
12	$2^3 + 2^2$	1100
13	$2^3 + 2^2 + 2^0$	1101
14	$2^3 + 2^2 + 2^1$	1110
15	$2^3 + 2^2 + 2^1 + 2^0$	1111

process to translate the discrete set of sample values to a more appropriate form of signal. Any plan for representing each of this discrete set of values as a particular arrangement of discrete events is called a *code*. One of the discrete events in a code is called a *code element* or *symbol*. For example, the presence or absence of a pulse is a symbol. A particular arrangement of symbols used in a code to represent a single value of the discrete set is called a *code word* or *character*.

In a *binary code*, each symbol may be either of two distinct values or kinds, such as the presence or absence of a pulse. The two symbols of a binary code are customarily denoted as 0 and 1. In a *ternary code*, each symbol may be one of three distinct values or kinds, and so on for other codes. However, *the maximum advantage over the effects of noise in a transmission medium is obtained by using a binary code, because a binary symbol withstands a relatively high level of noise and is easy to regenerate*. Suppose that, in a binary code, each code word consists of R bits: *bit* is an acronym for *binary digit*; thus R denotes the number of *bits per sample*. Then, using such a code, we may represent a total of 2^R distinct numbers. For example, a sample quantized into one of 256 levels may be represented by an 8-bit code word.

There are several ways of establishing a one-to-one correspondence between representation levels and code words. A convenient method is to express the ordinal number of the representation level as a binary number. In the binary number system, each digit has a place-value that is a power of 2, as illustrated in Table 3.2 for the case of four bits per sample (i.e., $R = 4$).

Line Codes

Any of several line codes can be used for the electrical representation of a binary data stream. Figure 3.15 displays the waveforms of five important line codes for the example data stream 01101001. Figure 3.16 displays their individual power spectra (for

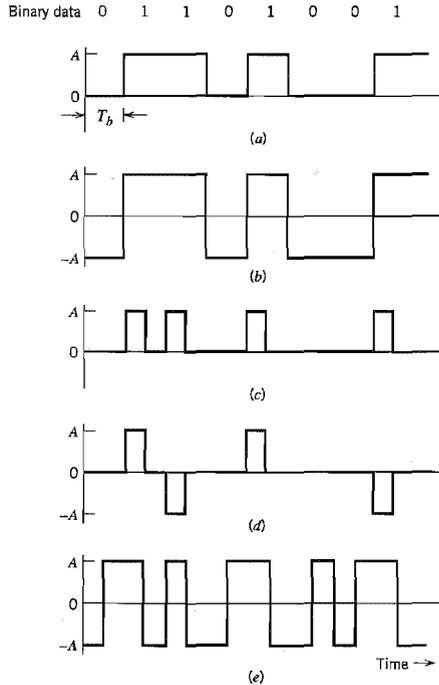


FIGURE 3.15 Line codes for the electrical representations of binary data. (a) Unipolar NRZ signaling. (b) Polar NRZ signaling. (c) Unipolar RZ signaling. (d) Bipolar RZ signaling. (e) Split-phase or Manchester code.

positive frequencies) for randomly generated binary data, assuming that (1) symbols 0 and 1 are equiprobable, (2) the average power is normalized to unity, and (3) the frequency f is normalized with respect to the bit rate $1/T_b$. (For the formulas used to plot the power spectra of Figure 3.16, the reader is referred to Problem 3.11.) The five line codes illustrated in Figure 3.15 are described here:

1. Unipolar nonreturn-to-zero (NRZ) signaling

In this line code, symbol 1 is represented by transmitting a pulse of amplitude A for the duration of the symbol, and symbol 0 is represented by switching off the pulse, as in Figure 3.15a. This line code is also referred to as *on-off signaling*. Disadvantages of on-off signaling are the waste of power due to the transmitted DC level and the fact that the power spectrum of the transmitted signal does not approach zero at zero frequency.

2. Polar nonreturn-to-zero (NRZ) signaling

In this second line code, symbols 1 and 0 are represented by transmitting pulses of amplitudes $+A$ and $-A$, respectively, as illustrated in Figure 3.15b. This line code is relatively easy to generate but its disadvantage is that the power spectrum of the signal is large near zero frequency.

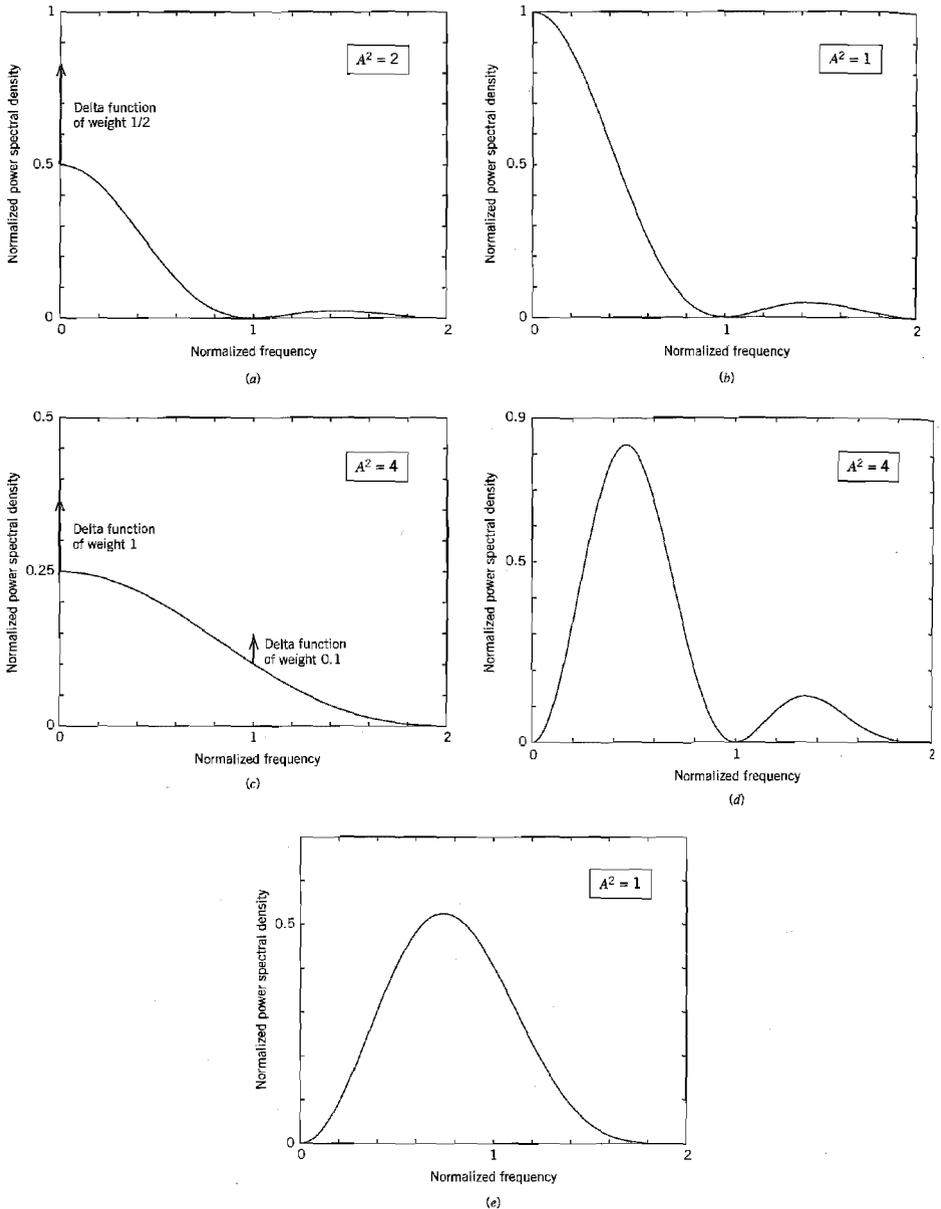


FIGURE 3.16 Power spectra of line codes: (a) Unipolar NRZ signal. (b) Polar NRZ signal. (c) Unipolar RZ signal. (d) Bipolar RZ signal. (e) Manchester-encoded signal. The frequency is normalized with respect to the bit rate $1/T_b$, and the average power is normalized to unity.

3. Unipolar return-to-zero (RZ) signaling

In this other line code, symbol 1 is represented by a rectangular pulse of amplitude A and half-symbol width, and symbol 0 is represented by transmitting no pulse, as illustrated in Figure 3.15c. An attractive feature of this line code is the presence of delta functions at $f = 0, \pm 1/T_b$ in the power spectrum of the transmitted signal, which can be used for bit-timing recovery at the receiver. However, its disadvantage is that it requires 3 dB more power than polar return-to-zero signaling for the same probability of symbol error; this issue is addressed in Chapter 4 under Problem 4.10.

4. Bipolar return-to-zero (BRZ) signaling

This line code uses three amplitude levels as indicated in Figure 3.15d. Specifically, positive and negative pulses of equal amplitude (i.e., $+A$ and $-A$) are used alternately for symbol 1, with each pulse having a half-symbol width; no pulse is always used for symbol 0. A useful property of the BRZ signaling is that the power spectrum of the transmitted signal has no DC component and relatively insignificant low-frequency components for the case when symbols 1 and 0 occur with equal probability. This line code is also called *alternate mark inversion* (AMI) signaling.

5. Split-phase (Manchester code)

In this method of signaling, illustrated in Figure 3.15e, symbol 1 is represented by a positive pulse of amplitude A followed by a negative pulse of amplitude $-A$, with both pulses being half-symbol wide. For symbol 0, the polarities of these two pulses are reversed. The Manchester code suppresses the DC component and has relatively insignificant low-frequency components, regardless of the signal statistics. This property is essential in some applications.

Differential Encoding

This method is used to encode information in terms of *signal transitions*. In particular, a transition is used to designate symbol 0 in the incoming binary data stream, while no transition is used to designate symbol 1, as illustrated in Figure 3.17. In Figure 3.17b we show the differentially encoded data stream for the example data specified in Figure 3.17a. The original binary data stream used here is the same as that used in Figure 3.15. The waveform of the differentially encoded data is shown in Figure 3.17c, assuming the use of unipolar nonreturn-to-zero signaling. From Figure 3.17 it is apparent that a differentially encoded signal may be inverted without affecting its interpretation. The original binary information is recovered simply by comparing the polarity of adjacent binary symbols to establish whether or not a transition has occurred. Note that differential encoding requires the use of a *reference bit* before initiating the encoding process. In Figure 3.17, symbol 1 is used as the reference bit.

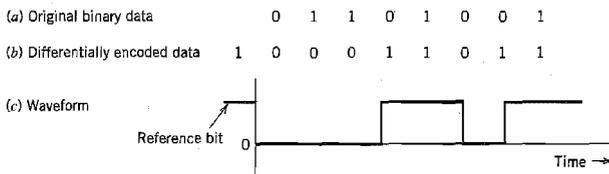


FIGURE 3.17 (a) Original binary data. (b) Differentially encoded data, assuming reference bit 1. (c) Waveform of differentially encoded data using unipolar NRZ signaling.

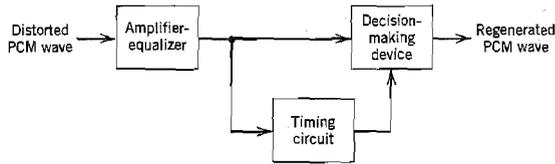


FIGURE 3.18 Block diagram of regenerative repeater.

■ REGENERATION

The most important feature of PCM systems lies in the ability to control the effects of distortion and noise produced by transmitting a PCM signal through a channel. This capability is accomplished by reconstructing the PCM signal by means of a chain of *regenerative repeaters* located at sufficiently close spacing along the transmission route. As illustrated in Figure 3.18, three basic functions are performed by a regenerative repeater: *equalization*, *timing*, and *decision making*. The equalizer shapes the received pulses so as to compensate for the effects of amplitude and phase distortions produced by the nonideal transmission characteristics of the channel. The timing circuitry provides a periodic pulse train, derived from the received pulses, for sampling the equalized pulses at the instants of time where the signal-to-noise ratio is a maximum. Each sample so extracted is compared to a predetermined *threshold* in the decision-making device. In each bit interval, a decision is then made whether the received symbol is a 1 or a 0 on the basis of whether the threshold is exceeded or not. If the threshold is exceeded, a clean new pulse representing symbol 1 is transmitted to the next repeater. Otherwise, another clean new pulse representing symbol 0 is transmitted. In this way, the accumulation of distortion and noise in a repeater span is completely removed, provided that the disturbance is not too large to cause an error in the decision-making process. Ideally, except for delay, the regenerated signal is exactly the same as the signal originally transmitted. In practice, however, the regenerated signal departs from the original signal for two main reasons:

1. The unavoidable presence of channel noise and interference causes the repeater to make wrong decisions occasionally, thereby introducing *bit errors* into the regenerated signal.
2. If the spacing between received pulses deviates from its assigned value, a *jitter* is introduced into the regenerated pulse position, thereby causing distortion.

■ DECODING

The first operation in the receiver is to regenerate (i.e., reshape and clean up) the received pulses one last time. These clean pulses are then regrouped into code words and decoded (i.e., mapped back) into a quantized PAM signal. The *decoding* process involves generating a pulse the amplitude of which is the linear sum of all the pulses in the code word, with each pulse being weighted by its place value ($2^0, 2^1, 2^2, \dots, 2^{R-1}$) in the code, where R is the number of bits per sample.

■ FILTERING

The final operation in the receiver is to recover the message signal by passing the decoder output through a low-pass reconstruction filter whose cutoff frequency is equal to the message bandwidth W . Assuming that the transmission path is error free, the recovered

signal includes no noise with the exception of the initial distortion introduced by the quantization process.

3.8 Noise Considerations in PCM Systems

The performance of a PCM system is influenced by two major sources of noise:

1. *Channel noise*, which is introduced anywhere between the transmitter output and the receiver input. Channel noise is always present, once the equipment is switched on.
2. *Quantization noise*, which is introduced in the transmitter and is carried all the way along to the receiver output. Unlike channel noise, quantization noise is *signal-dependent* in the sense that it disappears when the message signal is switched off.

Naturally, these two sources of noise appear simultaneously once the PCM system is in operation. However, the traditional practice is to consider them separately, so that we may develop insight into their individual effects on the system performance.

The main effect of channel noise is to introduce *bit errors* into the received signal. In the case of a binary PCM system, the presence of a bit error causes symbol 1 to be mistaken for symbol 0, or vice versa. Clearly, the more frequently bit errors occur, the more dissimilar the receiver output becomes compared to the original message signal. The fidelity of information transmission by PCM in the presence of channel noise may be measured in terms of the *average probability of symbol error*, which is defined as the probability that the reconstructed symbol at the receiver output differs from the transmitted binary symbol, on the average. The average probability of symbol error, also referred to as the *bit error rate* (BER), assumes that all the bits in the original binary wave are of equal importance. When, however, there is more interest in reconstructing the analog waveform of the original message signal, different symbol errors may need to be *weighted* differently; for example, an error in the most significant bit in a code word (representing a quantized sample of the message signal) is more harmful than an error in the least significant bit.

To optimize system performance in the presence of channel noise, we need to minimize the average probability of symbol error. For this evaluation, it is customary to model the channel noise as additive, white, and Gaussian. The effect of channel noise can be made practically negligible by ensuring the use of an adequate signal energy-to-noise density ratio through the provision of short-enough spacing between the regenerative repeaters in the PCM system. In such a situation, the performance of the PCM system is essentially limited by quantization noise acting alone.

From the discussion of quantization noise presented in Section 3.6, we recognize that quantization noise is essentially under the designer's control. It can be made negligibly small through the use of an adequate number of representation levels in the quantizer and the selection of a companding strategy matched to the characteristics of the type of message signal being transmitted. We thus find that the use of PCM offers the possibility of building a communication system that is *rugged* with respect to channel noise on a scale that is beyond the capability of any CW modulation or analog pulse modulation system.

■ ERROR THRESHOLD

The underlying theory of bit error rate calculation in a PCM system is deferred until Chapter 4. For the present, it suffices to say that the average probability of symbol error in a binary encoded PCM receiver due to additive white Gaussian noise depends solely on

E_b/N_0 , which is defined as *the ratio of the transmitted signal energy per bit, E_b , to the noise spectral density, N_0* . Note that the ratio E_b/N_0 is dimensionless even though the quantities E_b and N_0 have different physical meaning. In Table 3.3 we present a summary of this dependence for the case of a binary PCM system using polar nonreturn-to-zero signaling. The results presented in the last column of the table assume a bit rate of 10^5 b/s.

From Table 3.3 it is clear that there is an *error threshold* (at about 11 dB). For E_b/N_0 below the error threshold the receiver performance involves significant numbers of errors, and above it the effect of channel noise is practically negligible. In other words, provided that the ratio E_b/N_0 exceeds the error threshold, channel noise has virtually no effect on the receiver performance, which is precisely the goal of PCM. When, however, E_b/N_0 drops below the error threshold, there is a sharp increase in the rate at which errors occur in the receiver. Because decision errors result in the construction of incorrect code words, we find that when the errors are frequent, the reconstructed message at the receiver output bears little resemblance to the original message.

Comparing the figure of 11 dB for the error threshold in a PCM system using polar NRZ signaling with the 60–70 dB required for high-quality transmission of speech using amplitude modulation, we see that PCM requires much less power, even though the average noise power in the PCM system is increased by the R -fold increase in bandwidth, where R is the number of bits in a code word (i.e., bits per sample).

In most transmission systems, the effects of noise and distortion from the individual links accumulate. For a given quality of overall transmission, the longer the physical separation between the transmitter and the receiver, the more severe are the requirements on each link in the system. In a PCM system, however, because the signal can be regenerated as often as necessary, the effects of amplitude, phase, and nonlinear distortions in one link (if not too severe) have practically no effect on the regenerated input signal to the next link. We have also seen that the effect of channel noise can be made practically negligible by using a ratio E_b/N_0 above threshold. For all practical purposes, then, the transmission requirements for a PCM link are almost independent of the physical length of the communication channel.

Another important characteristic of a PCM system is its *ruggedness to interference*, caused by stray impulses or cross-talk. The combined presence of channel noise and interference causes the error threshold necessary for satisfactory operation of the PCM system to increase. If an adequate margin over the error threshold is provided in the first place, however, the system can withstand the presence of relatively large amounts of interference. In other words, a PCM system is *robust* to channel noise and interference.

TABLE 3.3 Influence of E_b/N_0 on the probability of error

E_b/N_0	Probability of Error P_e	For a Bit Rate of 10^5 b/s, This Is About One Error Every
4.3 dB	10^{-2}	10^{-3} second
8.4	10^{-4}	10^{-1} second
10.6	10^{-6}	10 seconds
12.0	10^{-8}	20 minutes
13.0	10^{-10}	1 day
14.0	10^{-12}	3 months

3.9 Time-Division Multiplexing

The sampling theorem provides the basis for transmitting the information contained in a band-limited message signal $m(t)$ as a sequence of samples of $m(t)$ taken uniformly at a rate that is usually slightly higher than the Nyquist rate. An important feature of the sampling process is a *conservation of time*. That is, the transmission of the message samples engages the communication channel for only a fraction of the sampling interval on a periodic basis, and in this way some of the time interval between adjacent samples is cleared for use by other independent message sources on a time-shared basis. We thereby obtain a *time-division multiplex* (TDM) system, which enables the joint utilization of a common communication channel by a plurality of independent message sources without mutual interference among them.

The concept of TDM is illustrated by the block diagram shown in Figure 3.19. Each input message signal is first restricted in bandwidth by a low-pass anti-aliasing filter to remove the frequencies that are nonessential to an adequate signal representation. The low-pass filter outputs are then applied to a *commutator*, which is usually implemented using electronic switching circuitry. The function of the commutator is twofold: (1) to take a narrow sample of each of the N input messages at a rate f_s that is slightly higher than $2W$, where W is the cutoff frequency of the anti-aliasing filter, and (2) to sequentially interleave these N samples inside the sampling interval T_s . Indeed, this latter function is the essence of the time-division multiplexing operation. Following the commutation process, the multiplexed signal is applied to a *pulse modulator*, the purpose of which is to transform the multiplexed signal into a form suitable for transmission over the common channel. It is clear that the use of time-division multiplexing introduces a bandwidth expansion factor N , because the scheme must squeeze N samples derived from N independent message sources into a time slot equal to one sampling interval. At the receiving end of the system, the received signal is applied to a *pulse demodulator*, which performs the reverse operation of the pulse modulator. The narrow samples produced at the pulse demodulator output are distributed to the appropriate low-pass reconstruction filters by means of a *decommutator*, which operates in *synchronism* with the commutator in the transmitter. This synchronization is essential for a satisfactory operation of the system. The way this synchronization is implemented depends naturally on the method of pulse modulation used to transmit the multiplexed sequence of samples.

The TDM system is highly sensitive to dispersion in the common channel, that is, to variations of amplitude with frequency or lack of proportionality of phase with frequency. Accordingly, accurate equalization of both magnitude and phase responses of the channel is necessary to ensure a satisfactory operation of the system; this issue is discussed in

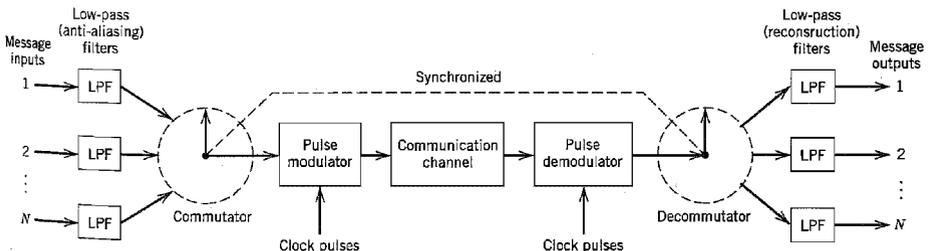


FIGURE 3.19 Block diagram of TDM system.

Chapter 4. However, unlike FDM, to a first-order approximation TDM is immune to nonlinearities in the channel as a source of cross-talk. The reason for this behavior is that different message signals are not simultaneously applied to the channel.

■ SYNCHRONIZATION

In applications using PCM, it is natural to multiplex different message sources by time division, whereby each source keeps its individuality throughout the journey from the transmitter to the receiver. This individuality accounts for the comparative ease with which message sources may be dropped or reinserted in a time-division multiplex system. As the number of independent message sources is increased, the time interval that may be allotted to each source has to be reduced, since all of them must be accommodated into a time interval equal to the reciprocal of the sampling rate. This, in turn, means that the allowable duration of a code word representing a single sample is reduced. However, pulses tend to become more difficult to generate and to transmit as their duration is reduced. Furthermore, if the pulses become too short, impairments in the transmission medium begin to interfere with the proper operation of the system. Accordingly, in practice, it is necessary to restrict the number of independent message sources that can be included within a time-division group.

In any event, for a PCM system with time-division multiplexing to operate satisfactorily, it is necessary that the timing operations at the receiver, except for the time lost in transmission and regenerative repeating, follow closely the corresponding operations at the transmitter. In a general way, this amounts to requiring a local clock at the receiver to keep the same time as a distant standard clock at the transmitter, except that the local clock is somewhat slower by an amount corresponding to the time required to transport the message signals from the transmitter to the receiver. One possible procedure to synchronize the transmitter and receiver clocks is to set aside a code element or pulse at the end of a *frame* (consisting of a code word derived from each of the independent message sources in succession) and to transmit this pulse every other frame only. In such a case, the receiver includes a circuit that would search for the pattern of 1s and 0s alternating at half the frame rate, and thereby establish synchronization between the transmitter and receiver.

When the transmission path is interrupted, it is highly unlikely that transmitter and receiver clocks will continue to indicate the same time for long. Accordingly, in carrying out a synchronization process, we must set up an orderly procedure for detecting the synchronizing pulse. The procedure consists of observing the code elements one by one until the synchronizing pulse is detected. That is, after observing a particular code element long enough to establish the absence of the synchronizing pulse, the receiver clock is set back by one code element and the next code element is observed. This *searching process* is repeated until the synchronizing pulse is detected. Clearly, the time required for synchronization depends on the epoch at which proper transmission is re-established.

▼ EXAMPLE 3.2 The T1 System

In this example, we describe the important characteristics of a PCM system known as the *T1 system*,⁶ which carries 24 voice channels over separate pairs of wires with regenerative repeaters spaced at approximately 2-km intervals. The T1 carrier system is basic to the North American Digital Switching Hierarchy described in Section 3.10.

A voice signal (male or female) is essentially limited to a band from 300 to 3100 Hz¹⁰ that frequencies outside this band do not contribute much to articulation efficiency. Indeed,

telephone circuits that respond to this range of frequencies give quite satisfactory service. Accordingly, it is customary to pass the voice signal through a low-pass filter with a cutoff frequency of about 3.1 kHz prior to sampling. Hence, with $W = 3.1$ kHz, the nominal value of the Nyquist rate is 6.2 kHz. The filtered voice signal is usually sampled at a slightly higher rate, namely, 8 kHz, which is the *standard* sampling rate in telephone systems.

For companding, the T1 system uses a *piecewise-linear* characteristic (consisting of 15 linear segments) to approximate the logarithmic μ -law of Equation (3.48) with the constant $\mu = 255$. This approximation is constructed in such a way that the segment end points lie on the compression curve computed from Equation (3.48), and their projections onto the vertical axis are spaced uniformly. Table 3.4 gives the projections of the segment end points onto the horizontal axis and the step-sizes of the individual segments. The table is normalized to 8159, so that all values are represented as integer numbers. Segment 0 of the approximation is a colinear segment, passing through the origin; it contains a total of 30 uniform decision levels. Linear segments $1a, 2a, \dots, 7a$ lie above the horizontal axis, whereas linear segments $1b, 2b, \dots, 7b$ lie below the horizontal axis; each of these 14 segments contains 16 uniform decision levels. For colinear segment 0 the decision levels at the quantizer input are $\pm 1, \pm 3, \dots, \pm 31$, and the corresponding representation levels at the quantizer output are $0, \pm 1, \dots, \pm 15$. For linear segments $1a$ and $1b$, the decision levels at the quantizer input are $\pm 31, \pm 35, \dots, \pm 95$, and the corresponding representation levels at the quantizer output are $\pm 16, \pm 17, \dots, \pm 31$, and so on for the other linear segments.

There are a total of $31 + (14 \times 16) = 255$ representation levels associated with the 15-segment companding characteristic described above. To accommodate this number of representation levels, each of the 24 voice channels uses a binary code with an 8-bit word. The first bit indicates whether the input voice sample is positive or negative; this bit is a 1 if positive and a 0 if negative. The next three bits of the code word identify the particular segment inside which the amplitude of the input voice sample lies, and the last four bits identify the actual representation level inside that segment.

With a sampling rate of 8 kHz, each frame of the multiplexed signal occupies a period of 125 μ s. In particular, it consists of twenty-four 8-bit words, plus a single bit that is added at the end of the frame for the purpose of synchronization. Hence, each frame consists of a total of $(24 \times 8) + 1 = 193$ bits. Correspondingly, the duration of each bit equals 0.647 μ s, and the resulting transmission rate is 1.544 megabits per second (Mb/s).

In addition to the voice signal, a telephone system must also pass special supervisory signals to the far end. This *signaling information* is needed to transmit dial pulses, as well as

TABLE 3.4 The 15-segment companding characteristic ($\mu = 255$)

Linear Segment Number	Step-Size	Projections of Segment End Points onto the Horizontal Axis
0	2	± 31
$1a, 1b$	4	± 95
$2a, 2b$	8	± 223
$3a, 3b$	16	± 479
$4a, 4b$	32	± 991
$5a, 5b$	64	± 2015
$6a, 6b$	128	± 4063
$7a, 7b$	256	± 8159

telephone off-hook/on-hook signals. In the T1 system, this requirement is accomplished as follows. Every sixth frame, the least significant (that is, the eighth) bit of each voice channel is deleted and a *signaling bit* is inserted in its place, thereby yielding an average $7\frac{1}{2}$ -bit operation for each voice input. The sequence of signaling bits is thus transmitted at a rate equal to sampling rate of 8 kHz divided by six, that is, 1.333 kb/s. This signaling rate applies to each of the 24 input channels. ◀

3.10 Digital Multiplexers

In Section 3.9 we introduced the idea of time-division multiplexing whereby a group of analog signals (e.g., voice signals) are sampled sequentially in time at a *common* sampling rate and then multiplexed for transmission over a common line. In this section we consider the multiplexing of digital signals at different bit rates. This enables us to combine several digital signals, such as computer outputs, digitized voice signals, digitized facsimile, and television signals, into a single data stream (at a considerably higher bit rate than any of the inputs). Figure 3.20 shows a conceptual diagram of the *digital multiplexing-demultiplexing* operation.

The multiplexing of digital signals is accomplished by using a *bit-by-bit interleaving procedure* with a selector switch that sequentially takes a bit from each incoming line and then applies it to the high-speed common line. At the receiving end of the system the output of this common line is separated out into its low-speed individual components and then delivered to their respective destinations.

Digital multiplexers are categorized into two major groups. One group of multiplexers is used to take relatively low bit-rate data streams originating from digital computers and multiplex them for TDM transmission over the public switched telephone network. The implementation of this first group of multiplexers requires the use of modems (modulators-demodulators), which are discussed in Chapter 6.

The second group of digital multiplexers forms part of the data transmission service provided by telecommunication carriers such as AT&T. In particular, these multiplexers constitute a *digital hierarchy* that time-division multiplexes low-rate bit streams into much higher-rate bit streams. The details of the bit rates that are accommodated in the hierarchy vary from one country to another. However, a worldwide feature of the hierarchy is that it starts at 64 kb/s, which corresponds to the standard PCM representation of a voice signal. An incoming bit stream at this rate, irrespective of its origin, is called a *digital signal zero (DS0)*. In the *United States, Canada, and Japan*⁷ the hierarchy follows the North American digital TDM hierarchy as described here:

- ▶ The *first-level hierarchy* combines twenty-four DS0 bit streams to obtain a *digital signal one (DS1)* at 1.544 Mb/s, which is carried on the T1 system described in

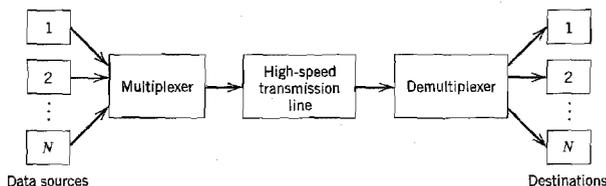


FIGURE 3.20 Conceptual diagram of multiplexing-demultiplexing.

Example 3.2. These bit streams are called the *primary rate* in the digital hierarchy, because it is the lowest bit rate that exists outside a digital switch. The *digital switch* is a device consisting of memory and logic, the function of which is merely the switching of digital signals, hence the name.

- ▶ The *second-level multiplexer* combines four DS1 bit streams to obtain a *digital signal two (DS2)* at 6.312 Mb/s.
- ▶ The *third-level multiplexer* combines seven DS2 bit streams to obtain a *digital signal three (DS3)* at 44.736 Mb/s.
- ▶ The *fourth-level multiplexer* combines six DS3 bit streams to obtain a *digital signal four (DS4)* at 274.176 Mb/s.
- ▶ The *fifth-level multiplexer*, the final one in the hierarchy, combines two DS4 bit streams to obtain a *digital signal five (DS5)* at 560.160 Mb/s.

Note that the bit rate of a digital signal produced by any one of these multiplexers is slightly higher than the prescribed multiple of the incoming bit rate because of bit stuffing built into the design of each multiplexer; bit stuffing is discussed in the sequel.

Moreover, it is important to recognize that the functions of a digital transmission facility is merely to carry a bit stream without interpreting what the bits themselves mean. However, the digital switches at the two ends of the facility do have a common understanding of how to interpret the bits within the stream, such as whether the bits represent voice or data, framing format, signaling format, and so on.

There are some basic problems involved in the design of a digital multiplexer, irrespective of its grouping:

1. Digital signals cannot be directly interleaved into a format that allows for their eventual separation unless their bit rates are locked to a common clock. Rather, provision has to be made for *synchronization* of the incoming digital signals, so that they can be properly interleaved.
2. The multiplexed signal must include some form of *framing* so that its individual components can be identified at the receiver.
3. The multiplexer has to handle small variations in the bit rates of the incoming digital signals. For example, a 1000-km coaxial cable carrying 3×10^8 pulses per second will have about one million pulses in transit, with each pulse occupying about one meter of the cable. A 0.01 percent variation in the propagation delay, produced by a 1°F decrease in temperature, will result in 100 fewer pulses in the cable. Clearly, these pulses must be absorbed by the multiplexer.

To tailor the requirements of synchronization and rate adjustment to accommodate small variations in the input data rates, we may use a technique known as *bit stuffing*. The idea here is to have the outgoing bit rate of the multiplexer slightly higher than the sum of the maximum expected bit rates of the input channels by stuffing in additional non-information carrying pulses. All incoming digital signals are stuffed with a number of bits sufficient to raise each of their bit rates to equal that of a locally generated clock. To accomplish bit stuffing, each incoming digital signal or bit stream is fed into an *elastic store* at the multiplexer. The elastic store is a device that stores a bit stream in such a manner that the stream may be read out at a rate different from the rate at which it is read in. At the demultiplexer, the stuffed bits must obviously be removed from the multiplexed signal. This requires a method that can be used to identify the stuffed bits. To illustrate one such method, and also show one method of providing frame synchronization, we describe the signal format of the AT&T *M12 multiplexer*, which is designed to combine four DS1 bit

streams into one DS2 bit stream. This is the second level of the digital hierarchy discussed earlier.

▶ **EXAMPLE 3.3 Signal Format of the AT&T M12 Multiplexer**

Figure 3.21 illustrates the signal format of the M12 multiplexer. Each frame is subdivided into four subframes. The first subframe (first line in Figure 3.21) is transmitted, then the second, the third, and the fourth, in that order.

Bit-by-bit interleaving of the incoming four DS1 bit streams is used to accumulate a total of 48 bits, 12 from each input. A *control bit* is then inserted by the multiplexer. Each frame contains a total of 24 control bits, separated by sequences of 48 data bits. Three types of control bits are used in the M12 multiplexer to provide synchronization and frame indication, and to identify which of the four input signals has been stuffed. These control bits are labeled *F*, *M*, and *C* in Figure 3.21. Their functions are as follows:

1. The *F*-control bits, two per subframe, constitute the *main framing pulses*. The subscripts on the *F*-control bits denote the actual bit (0 or 1) transmitted. Thus the main framing sequence is $F_0F_1F_0F_1F_0F_1F_0F_1$ or 01010101.
2. The *M*-control bits, one per subframe, form *secondary framing pulses* to identify the four subframes. Here again the subscripts on the *M*-control bits denote the actual bit (0 or 1) transmitted. Thus the secondary framing sequence is $M_0M_1M_1M_1$ or 0111.
3. The *C*-control bits, three per subframe, are *stuffing indicators*. In particular, C_I refers to input channel I, C_{II} refers to input channel II, and so forth. For example, the three *C*-control bits in the first subframe following M_0 in the first subframe are stuffing indicators for the first DS1 bit stream. The insertion of a stuffed bit in this DS1 bit stream is indicated by setting all three *C*-control bits to 1. To indicate no stuffing, all three are set to 0. If the three *C*-control bits indicate stuffing, the stuffed bit is located in the position of the first information bit associated with the first DS1 bit stream that follows the F_1 -control bit in the same subframe. In a similar way, the second, third, and fourth DS1 bit streams may be stuffed, as required. By using *majority logic decoding* in the receiver, a single error in any of the three *C*-control bits can be detected. This form of decoding means simply that the majority of the *C*-control bits determine whether an all-one or all-zero sequence was transmitted. Thus three 1s or combinations of two 1s and a 0 indicate that a stuffed bit is present in the information sequence, following the control bit F_1 in the pertinent subframe. On the other hand, three 0s or combinations of two 0s and a 1 indicate that no stuffing is used.

The demultiplexer at the receiving M12 unit first searches for the main framing sequence $F_0F_1F_0F_1F_0F_1F_0F_1$. This establishes identity for the four input DS1 bit streams and also for the *M*- and *C*-control bits. From the $M_0M_1M_1M_1$ sequence, the correct framing of the *C*-control bits is verified. Finally, the four DS1 bit streams are properly demultiplexed and destuffed.

The signal format described above has two safeguards:

1. It is possible, although unlikely, that with just the $F_0F_1F_0F_1F_0F_1F_0F_1$ sequence, one of the incoming DS1 bit streams may contain a similar sequence. This could then cause

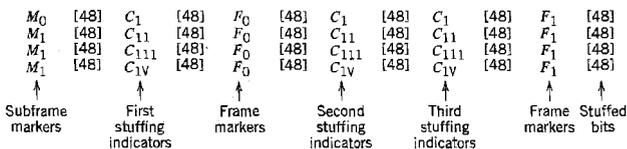


FIGURE 3.21 Signal format of AT&T M12 multiplexer.

the receiver to lock onto the wrong sequence. The presence of the $M_0M_1M_1M_1$ sequence provides verification of the genuine $F_0F_1F_0F_1F_0F_1$ sequence, thereby ensuring that the four DS1 bit streams are properly demultiplexed.

2. The single-error correction capability built into the C-control bits ensures that the four DS1 bit streams are properly destuffed.

The capacity of the M12 multiplexer to accommodate small variations in the input data rates can be calculated from the format of Figure 3.21. In each M frame, defined as the interval containing one cycle of $M_0M_1M_1M_1$ bits, one bit can be stuffed into each of four input DS1 bit streams. Each such signal has $12 \times 6 \times 4 = 288$ positions in each M frame. Also, the T1 bit stream has a bit rate equal to 1.544 Mb/s. Hence, each input can be incremented by

$$1.544 \times 10^3 \times \frac{1}{288} = 5.4 \text{ kb/s}$$

This result is much larger than the expected change in the bit rate of the incoming DS1 bit stream. It follows therefore that the use of only one stuffed bit per input channel in each frame is sufficient to accommodate expected variations in the input signal rate.

The local clock that determines the outgoing bit rate also determines the nominal *stuffing rate* S , defined as the average number of bits stuffed per channel in any frame. The M12 multiplexer is designed for $S = 1/3$. Accordingly, the nominal bit rate of the DS2 bit stream is

$$1.544 \times 4 \times \frac{49}{48} \times \frac{288}{288-S} = 6.312 \text{ Mb/s}$$

This also ensures that the nominal DS2 clock frequency is a multiple of 8 kHz (the nominal sampling rate of a voice signal), which is a desirable feature. ◀

3.11 Virtues, Limitations, and Modifications of PCM

In a generic sense, pulse-code modulation (PCM) has emerged as the most favored modulation scheme for the transmission of analog information-bearing signals such as voice and video signals. The advantages of PCM may all be traced to the use of *coded pulses for the digital representation of analog signals*, a feature that distinguishes it from all other analog methods of modulation. We may summarize the important advantages of PCM as follows:

1. *Robustness* to channel noise and interference.
2. Efficient *regeneration* of the coded signal along the transmission path.
3. Efficient *exchange* of increased channel bandwidth for improved signal-to-noise ratio, obeying an exponential law.
4. A *uniform format* for the transmission of different kinds of baseband signals, hence their integration with other forms of digital data in a common network.
5. Comparative *ease* with which message sources may be dropped or reinserted in a time-division multiplex system.
6. *Secure* communication through the use of special modulation schemes or encryption; the encryption and decryption of data are discussed in Appendix 5.

These advantages, however, are attained at the cost of increased system complexity and increased channel bandwidth. These two issues are considered in the sequel in turn.

Although the use of PCM involves many complex operations, today they can all be implemented in a cost-effective fashion using commercially available and/or custom-made *very-large-scale integrated* (VLSI) chips. In other words, the requisite device technology for the implementation of a PCM system is already in place. Moreover, with continuing improvements in VLSI technology, we are likely to see an ever-expanding use of PCM for the digital transmission of analog signals.

If, however, the simplicity of implementation is a necessary requirement, then we may use delta modulation as an alternative to pulse-code modulation. In delta modulation, the baseband signal is intentionally “oversampled” to permit the use of a simple quantizing strategy for constructing the encoded signal; delta modulation is discussed in Section 3.12.

Turning next to the issue of bandwidth, we do recognize that the increased bandwidth requirement of PCM may have been a reason for justifiable concern in the past. Today, however, it is of no real concern for two different reasons. First, the increasing availability of *wideband communication channels* means that bandwidth is no longer a system constraint in the traditional way it used to be. Liberation from the bandwidth constraint has been made possible by the deployment of communication satellites for broadcasting and the ever-increasing use of fiber optics for networking; a discussion of these communication channel concepts was presented in the Background and Preview chapter.

The second reason is that through the use of sophisticated *data compression* techniques, it is indeed possible to remove the redundancy inherently present in a PCM signal and thereby reduce the bit rate of the transmitted data without serious degradation in system performance. In effect, increased processing complexity (and therefore increased cost of implementation) is traded off for a reduced bit rate and therefore reduced bandwidth requirement. A major motivation for bit-rate reduction is for secure communication over radio channels that are inherently of low capacity.

3.12 Delta Modulation

In *delta modulation*⁸ (DM), an incoming message signal is oversampled (i.e., at a rate much higher than the Nyquist rate) to purposely increase the correlation between adjacent samples of the signal. This is done to permit the use of a simple quantizing strategy for constructing the encoded signal.

In its basic form, DM provides a *staircase approximation* to the oversampled version of the message signal, as illustrated in Figure 3.22a. The difference between the input and the approximation is quantized into only two levels, namely, $\pm\Delta$, corresponding to positive and negative differences. Thus if the approximation falls below the signal at any sampling epoch, it is increased by Δ . If on the other hand, the approximation lies above the signal, it is diminished by Δ . Provided that the signal does not change too rapidly from sample to sample, we find that the staircase approximation remains within $\pm\Delta$ of the input signal.

Let $m(t)$ denote the input (message) signal, and $m_s(t)$ denote its staircase approximation. For convenience of presentation, we adopt the following notation that is commonly used in the digital signal processing literature:

$$m[n] = m(nT_s), \quad n = 0, \pm 1, \pm 2, \dots$$

where T_s is the sampling period and $m(nT_s)$ is a sample of the signal $m(t)$ taken at time $t = nT_s$, and likewise for the samples of other continuous-time signals. We may then

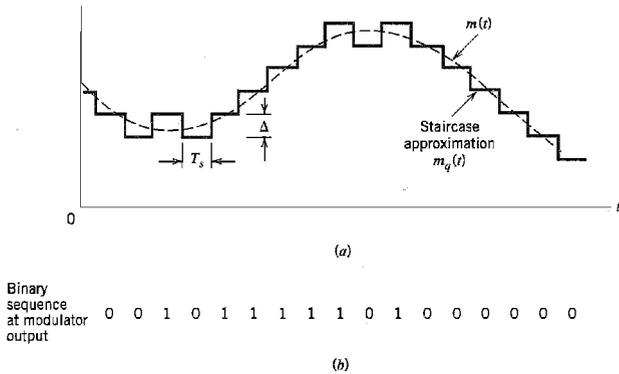


FIGURE 3.22 Illustration of delta modulation.

formalize the basic principles of delta modulation in the following set of discrete-time relations:

$$e[n] = m[n] - m_q[n - 1] \quad (3.52)$$

$$e_q = \Delta \operatorname{sgn}(e[n]) \quad (3.53)$$

$$m_q[n] = m_q[n - 1] + e_q[n] \quad (3.54)$$

where $e[n]$ is an *error signal* representing the difference between the present sample $m[n]$ of the input signal and the latest approximation $m_q[n - 1]$ to it, $e_q[n]$ is the quantized version of $e[n]$, and $\operatorname{sgn}(\cdot)$ is the signum function. Finally, the quantizer output $m_q[n]$ is coded to produce the DM signal.

Figure 3.22a illustrates the way in which the staircase approximation $m_q(t)$ follows variations in the input signal $m(t)$ in accordance with Equations (3.52)–(3.54), and Figure 3.22b displays the corresponding binary sequence at the delta modulator output. It is apparent that in a delta modulation system the rate of information transmission is simply equal to the sampling rate $f_s = 1/T_s$.

The principal virtue of delta modulation is its simplicity. It may be generated by applying the sampled version of the incoming message signal to a modulator that involves a *comparator*, *quantizer*, and *accumulator* interconnected as shown in Figure 3.23a. The block labeled z^{-1} inside the accumulator represents a *unit delay*, that is, a delay equal to one sampling period. (The variable z is commonly used in the z -transform, which is basic to the analysis of discrete-time signals and systems.) Details of the modulator follow directly from Equations (3.52)–(3.54). The comparator computes the difference between its two inputs. The quantizer consists of a *hard limiter* with an input-output relation that is a scaled version of the signum function. The quantizer output is then applied to an accumulator, producing the result

$$\begin{aligned} m_q[n] &= \Delta \sum_{i=1}^n \operatorname{sgn}(e[i]) \\ &= \sum_{i=1}^n e_q[i] \end{aligned} \quad (3.55)$$

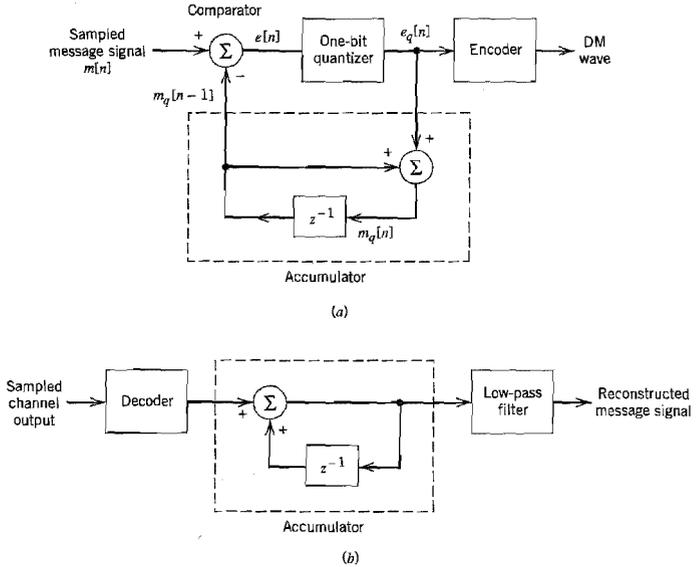


FIGURE 3.23 DM system. (a) Transmitter. (b) Receiver.

which is obtained by solving Equations (3.53) and (3.54) for $m_q[n]$. Thus, at the sampling instant nT_s , the accumulator increments the approximation by a step Δ in a positive or negative direction, depending on the algebraic sign of the error sample $e[n]$. If the input sample $m[n]$ is greater than the most recent approximation $m_q[n]$, a positive increment $+\Delta$ is applied to the approximation. If, on the other hand, the input sample is smaller, a negative increment $-\Delta$ is applied to the approximation. In this way, the accumulator does the best it can to track the input samples by one step (of amplitude $+\Delta$ or $-\Delta$) at a time. In the receiver shown in Figure 3.23b, the staircase approximation $m_q(t)$ is reconstructed by passing the sequence of positive and negative pulses, produced at the decoder output, through an accumulator in a manner similar to that used in the transmitter. The out-of-band quantization noise in the high-frequency staircase waveform $m_q(t)$ is rejected by passing it through a low-pass filter, as in Figure 3.23b, with a bandwidth equal to the original message bandwidth.

Delta modulation is subject to two types of quantization error: slope overload distortion and granular noise. We will discuss the case of slope overload distortion first.

We observe that Equation (3.54) is the digital equivalent of integration in the sense that it represents the accumulation of positive and negative increments of magnitude Δ . Also, denoting the quantization error by $q[n]$, as shown by

$$m_q[n] = m[n] + q[n] \tag{3.56}$$

we observe from Equation (3.52) that the input to the quantizer is

$$e[n] = m[n] - m[n - 1] - q[n - 1] \tag{3.57}$$

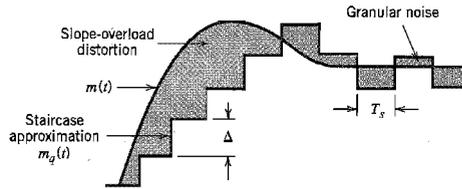


FIGURE 3.24 Illustration of the two different forms of quantization error in delta modulation.

Thus except for the quantization error $q[n - 1]$, the quantizer input is a *first backward difference* of the input signal, which may be viewed as a digital approximation to the derivative of the input signal or, equivalently, as the inverse of the digital integration process. If we consider the maximum slope of the original input waveform $m(t)$, it is clear that in order for the sequence of samples $\{m_q[n]\}$ to increase as fast as the input sequence of samples $\{m[n]\}$ in a region of maximum slope of $m(t)$, we require that the condition

$$\frac{\Delta}{T_s} \geq \max \left| \frac{dm(t)}{dt} \right| \quad (3.58)$$

be satisfied. Otherwise, we find that the step-size Δ is too small for the staircase approximation $m_q(t)$ to follow a steep segment of the input waveform $m(t)$, with the result that $m_q(t)$ falls behind $m(t)$, as illustrated in Figure 3.24. This condition is called *slope overload*, and the resulting quantization error is called *slope-overload distortion (noise)*. Note that since the maximum slope of the staircase approximation $m_q(t)$ is fixed by the step size Δ , increases and decreases in $m_q(t)$ tend to occur along straight lines. For this reason, a delta modulator using a fixed step size is often referred to as a *linear delta modulator*.

In contrast to slope-overload distortion, *granular noise* occurs when the step size Δ is too large relative to the local slope characteristics of the input waveform $m(t)$, thereby causing the staircase approximation $m_q(t)$ to hunt around a relatively flat segment of the input waveform; this phenomenon is also illustrated in Figure 3.24. Granular noise is analogous to quantization noise in a PCM system.

We thus see that there is a need to have a large step-size to accommodate a wide dynamic range, whereas a small step size is required for the accurate representation of relatively low-level signals. It is therefore clear that the choice of the optimum step size that minimizes the mean-square value of the quantization error in a linear delta modulator will be the result of a compromise between slope-overload distortion and granular noise. To satisfy such a requirement, we need to make the delta modulator “adaptive,” in the sense that the step size is made to vary in accordance with the input signal; this issue is discussed further in a computer experiment presented in Section 3.16.

■ DELTA-SIGMA MODULATION

As mentioned earlier, the quantizer input in the conventional form of delta modulation may be viewed as an approximation to the *derivative* of the incoming message signal. This behavior leads to a drawback of delta modulation in that transmission disturbances such as noise result in an accumulative error in the demodulated signal. This drawback can be

overcome by *integrating* the message signal prior to delta modulation. The use of integration in the manner described here has also the following beneficial effects:

- ▼ The low-frequency content of the input signal is pre-emphasized.
- ▼ Correlation between adjacent samples of the delta modulator input is increased, which tends to improve overall system performance by reducing the variance of the error signal at the quantizer input.
- ▼ Design of the receiver is simplified.

A delta modulation scheme that incorporates integration at its input is called *delta-sigma modulation* (D- Σ M).⁹ To be more precise, however, it should be called *sigma-delta modulation*, because the integration is in fact performed before the delta modulation. Nevertheless, the former terminology is the one commonly used in the literature.

Figure 3.25a shows the block diagram of a delta-sigma modulation system. In this diagram, the message signal $m(t)$ is defined in its continuous-time form, which means that the pulse modulator now consists of a hard-limiter followed by a multiplier; the latter component is also fed from an external pulse generator (clock) to produce a 1-bit encoded signal. The use of integration at the transmitter input clearly requires an inverse signal emphasis, namely, differentiation, at the receiver. The need for this differentiation is, however, eliminated because of its cancellation by integration in the conventional DM receiver.

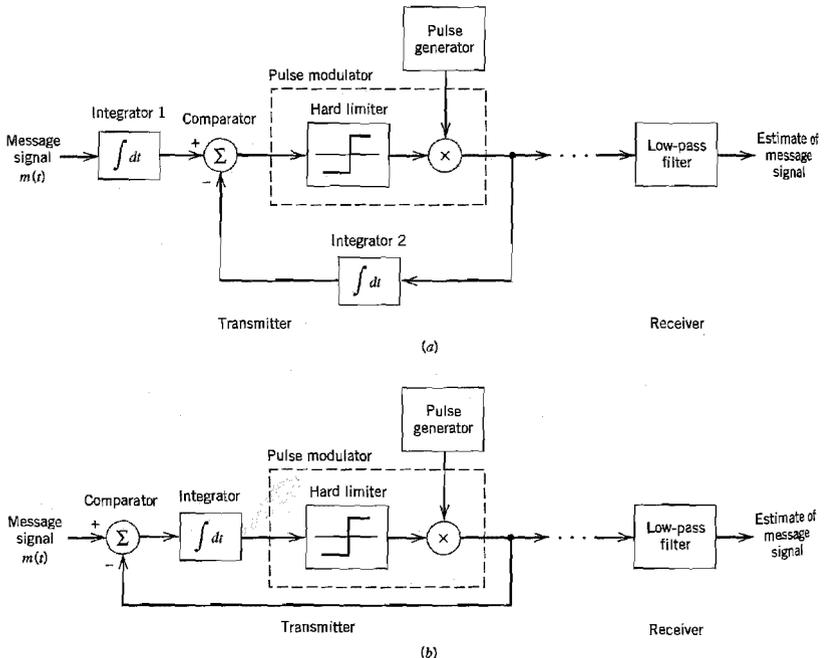


FIGURE 3.25 Two equivalent versions of delta-sigma modulation system.

Thus the receiver of a delta-sigma modulation system consists simply of a low-pass filter, as indicated in Figure 3.25a.

Moreover, we note that integration is basically a linear operation. Accordingly, we may simplify the design of the transmitter by combining the two integrators 1 and 2 of Figure 3.25a into a single integrator placed after the comparator, as shown in Figure 3.25b. This latter form of the delta-sigma modulation system is not only simpler than that of Figure 3.25a, but it also provides an interesting interpretation of delta-sigma modulation as a “smoothed” version of 1-bit pulse-code modulation: The term *smoothness* refers to the fact that the comparator output is integrated prior to quantization, and the term *1-bit* merely restates that the quantizer consists of a hard-limiter with only two representation levels.

In delta modulation, simplicity of implementations of both the transmitter and receiver is attained by using a sampling rate far in excess of that needed for pulse-code modulation. The price paid for this benefit is a corresponding increase in the transmission and therefore channel bandwidth. There are, however, applications where channel bandwidth is at a premium, in which case we have the opposite requirement to that in delta modulation. Specifically, we may wish to trade increased system complexity for a reduced channel bandwidth. A signal-processing operation basic to the attainment of this latter design objective is prediction, the linear form of which is discussed next.

3.13 Linear Prediction

Consider a *finite-duration impulse response (FIR) discrete-time filter* configured as in Figure 3.26, which involves the use of three functional blocks:

1. Set of p unit-delay elements, each of which is represented by z^{-1} .
2. Set of multipliers involving the filter coefficients w_1, w_2, \dots, w_p .
3. Set of “adders” used to sum the scaled versions of the delayed inputs $x[n-1], x[n-2], \dots, x[n-p]$ to produce the output $\hat{x}[n]$. The filter output $\hat{x}[n]$ or more precisely, the *linear prediction* of the input, is thus defined by the *convolution sum*

$$\hat{x}[n] = \sum_{k=1}^p w_k x[n-k] \quad (3.59)$$

where p , the number of unit-delay elements, is called the *prediction order*.

The actual sample at time nT_s is $x[n]$. The *prediction error*, denoted by $e[n]$, is defined as the difference between $x[n]$ and the prediction $\hat{x}[n]$, as shown by

$$e[n] = x[n] - \hat{x}[n] \quad (3.60)$$

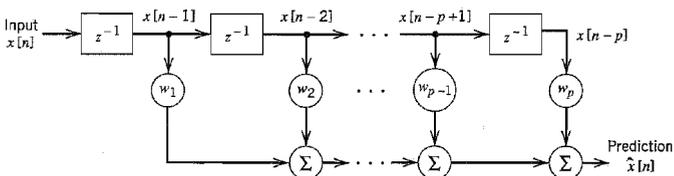


FIGURE 3.26 Block diagram of a linear prediction filter of order p .

The design objective is to choose the filter coefficients w_1, w_2, \dots, w_p so as to minimize an index of performance, J , defined as the mean-square error:

$$J = E[e^2[n]] \tag{3.61}$$

Substituting Equations (3.59) and (3.60) into (3.61) and then expanding terms, we may reformulate the index of performance as

$$J = E[x^2[n]] - 2 \sum_{k=1}^p w_k E[x[n]x[n-k]] + \sum_{j=1}^p \sum_{k=1}^p w_j w_k E[x[n-j]x[n-k]] \tag{3.62}$$

We assume that the input signal $x(t)$ is the sample function of a stationary process $X(t)$ of zero mean; that is, $E[x[n]]$ is zero for all n . Define

$$\begin{aligned} \sigma_X^2 &= \text{variance of a sample of the process } X(t) \text{ at time } nT_s \\ &= E[x^2[n]] - (E[x[n]])^2 \\ &= E[x^2[n]] \\ R_X(kT_s) &= \text{autocorrelation of the process } X(t) \text{ for a lag of } kT_s \\ &= R_X[k] \\ &= E[x[n]x[n-k]] \end{aligned}$$

Accordingly, we may rewrite Equation (3.62) in the simplified form

$$J = \sigma_X^2 - 2 \sum_{k=1}^p w_k R_X[k] + \sum_{j=1}^p \sum_{k=1}^p w_j w_k R_X[k-j] \tag{3.63}$$

Hence differentiating the index of performance J with respect to the filter coefficient w_k , setting the result equal to zero, and then rearranging terms, we obtain

$$\sum_{j=1}^p w_j R_X[k-j] = R_X[k] = R_X[-k], \quad k = 1, 2, \dots, p \tag{3.64}$$

The optimality equations (3.64) are called the *Wiener-Hopf equations* for linear prediction.

We find it convenient to reformulate the Wiener-Hopf equations (3.64) in matrix form. Let

$$\begin{aligned} \mathbf{w}_o &= p\text{-by-1 optimum coefficient vector} \\ &= [w_1, w_2, \dots, w_p]^T \\ \mathbf{r}_X &= p\text{-by-1 autocorrelation vector} \\ &= [R_X[1], R_X[2], \dots, R_X[p]]^T \\ \mathbf{R}_X &= p\text{-by-}p \text{ autocorrelation matrix} \\ &= \begin{bmatrix} R_X[0] & R_X[1] & \cdots & R_X[p-1] \\ R_X[1] & R_X[0] & \cdots & R_X[p-2] \\ \vdots & \vdots & \ddots & \vdots \\ R_X[p-1] & R_X[p-2] & \cdots & R_X[0] \end{bmatrix} \end{aligned}$$

We may thus simplify the set of equations (3.64) as

$$\mathbf{R}_X \mathbf{w}_o = \mathbf{r}_X \tag{3.65}$$

We assume that the autocorrelation matrix \mathbf{R}_X is *nonsingular*, so that its inverse exists. We may then solve Equation (3.65) for the coefficient vector \mathbf{w}_o by multiplying both sides of this equation by the *inverse matrix* \mathbf{R}_X^{-1} , obtaining the optimum solution

$$\mathbf{w}_o = \mathbf{R}_X^{-1} \mathbf{r}_X \quad (3.66)$$

Note that all the elements on the main diagonal of the autocorrelation matrix \mathbf{R}_X are equal to $R_X[0] = \sigma_X^2$, and the elements on any other diagonal parallel to the main diagonal are also equal. A square matrix having this property is said to be *Toeplitz*, which is a direct consequence of the assumption that the input signal $x(t)$ is drawn from a stationary process. The practical significance of the Toeplitz property is that the correlation matrix \mathbf{R}_X is uniquely defined by the set of autocorrelation values $R_X[0], R_X[1], \dots, R_X[p-1]$. The autocorrelation vector \mathbf{r}_X is defined by the set of autocorrelation values $R_X[1], R_X[2], \dots, R_X[p]$. It follows therefore that the p filter coefficients of the linear optimum predictor are uniquely defined by the variance $\sigma_X^2 = R_X[0]$ and p values of the autocorrelation function of the process $X(t)$ for lags of $T_s, 2T_s, \dots, pT_s$.

The minimum mean-square value of the prediction error is obtained by substituting Equation (3.64) into (3.63), which yields (after simplification)

$$J_{\min} = \sigma_X^2 - \mathbf{r}_X^T \mathbf{R}_X^{-1} \mathbf{r}_X \quad (3.67)$$

The quadratic term $\mathbf{r}_X^T \mathbf{R}_X^{-1} \mathbf{r}_X$ is always positive. Accordingly, the mean-square error J_{\min} of the optimum linear predictor defined by Equation (3.67) is always less than the variance σ_X^2 of the input sample that is being predicted.

▣ LINEAR ADAPTIVE PREDICTION

The use of Equation (3.66) for calculating the weight vector of a linear predictor requires knowledge of the autocorrelation function $R_X[k]$ of the input sequence $\{x[n]\}$ for lags $k = 0, 1, \dots, p$, where p is the prediction order. What if knowledge of $R_X[k]$ for varying k is not available? In these situations, which occur frequently in practice, we may resort to the use of an *adaptive predictor*.

The predictor is adaptive in the following sense:

- ▶ Computation of the tap weights $w_k, k = 1, 2, \dots, p$, proceeds in a “recursive” manner, starting from some arbitrary initial values of the tap weights.
- ▶ The algorithm used to adjust the tap weights (from one iteration to the next) is “self-designed,” operating solely on the basis of available data.

The aim of the algorithm is to find the minimum point of the bowl-shaped *error surface* that describes the dependence of the cost function J on the tap weights. It is therefore intuitively reasonable that successive adjustments to the tap-weights of the predictor be made in the direction of the steepest descent of the error surface, that is, in a direction opposite to the *gradient vector* whose elements are defined by

$$g_k = \frac{\partial J}{\partial w_k}, \quad k = 1, 2, \dots, p \quad (3.68)$$

This is indeed the idea behind the *method of steepest descent*. Let $w_k[n]$ denote the value of the k th tap-weight at iteration n . Then the updated value of this weight at iteration $n + 1$ is defined by

$$w_k[n + 1] = w_k[n] - \frac{1}{2} \mu g_k, \quad k = 1, 2, \dots, p \quad (3.69)$$

where μ is a *step-size parameter* that controls the speed of adaptation, and the factor $1/2$ is included for convenience of presentation. Differentiating the cost function J of Equation (3.63) with respect to w_k , we readily find that

$$\begin{aligned}
 g_k &= -2R_x[k] + 2 \sum_{j=1}^p w_j R_x[k-j] \\
 &= -2E[x[n]x[n-k]] + 2 \sum_{j=1}^p w_j E[x[n-j]x[n-k]], \quad k = 1, 2, \dots, p
 \end{aligned} \tag{3.70}$$

This formula for g_k could do with further simplification, which is achieved by using *instantaneous values as estimates* of the autocorrelation functions $R_x[k]$ and $R_x[k-j]$. That is, we ignore the expectation operators in Equation (3.70) to facilitate the adaptive process on a step-by-step basis. We may thus express the corresponding *estimate* of g_k at iteration n as

$$\hat{g}_k[n] = -2x[n]x[n-k] + 2 \sum_{j=1}^p w_j[n]x[n-j]x[n-k], \quad k = 1, 2, \dots, p \tag{3.71}$$

Note that for an input $x[n]$ drawn from a stationary process the gradient g_k is a deterministic quantity, whereas the estimate $\hat{g}_k[n]$ is the sample value of a random variable.

In any event, substituting Equation (3.71) into (3.69) and factoring the common term $x[n-k]$, we may write

$$\begin{aligned}
 \hat{w}_k[n+1] &= \hat{w}_k[n] + \mu x[n-k] \left(x[n] - \sum_{j=1}^p \hat{w}_j[n]x[n-j] \right) \\
 &= \hat{w}_k[n] + \mu x[n-k]e[n], \quad k = 1, 2, \dots, p
 \end{aligned} \tag{3.72}$$

where $e[n]$ is the *prediction error* defined as

$$e[n] = x[n] - \sum_{j=1}^p \hat{w}_j[n]x[n-j] \tag{3.73}$$

In Equations (3.72) and (3.73), we have used \hat{w}_k as an *estimate* of the k th tap-weight to distinguish it from the actual value w_k . Note also that $x[n]$ plays the role of a “desired response” for computing the recursive adjustments applied to the tap-weights of the predictor.

Equations (3.72) and (3.73) constitute the popular *least-mean-square (LMS) algorithm* for linear adaptive prediction, the operation of which is depicted in Figure 3.27. The reason for popularity of this adaptive filtering algorithm is the simplicity of its implementation. In particular, the computational complexity of the algorithm, measured in terms of the number of additions and multiplications, is *linear* in the prediction order p .

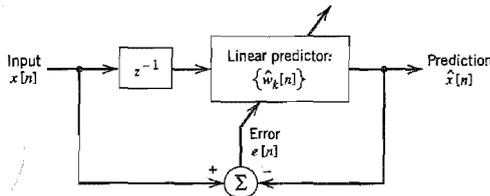


FIGURE 3.27 Block diagram illustrating the linear adaptive prediction process.

The LMS algorithm is a *stochastic* adaptive filtering algorithm, stochastic in the sense that, starting from the *initial condition* defined by $\{w_k[0]\}_{k=1}^L$, it seeks to find the minimum point of the error surface by following a zig-zag path. Moreover, it never finds this minimum point exactly. Rather, it executes a random motion around the minimum point of the error surface, once steady-state conditions are established.

With this material on linear prediction at hand, we are ready to discuss practical improvements on the performance of pulse-code modulation.

3.14 Differential Pulse-Code Modulation

When a voice or video signal is sampled at a rate slightly higher than the Nyquist rate as usually done in pulse-code modulation, the resulting sampled signal is found to exhibit a high degree of correlation between adjacent samples. The meaning of this high correlation is that, in an average sense, the signal does not change rapidly from one sample to the next, and as a result, the difference between adjacent samples has a variance that is smaller than the variance of the signal itself. When these highly correlated samples are encoded, as in the standard PCM system, the resulting encoded signal contains *redundant information*. This means that symbols that are not absolutely essential to the transmission of information are generated as a result of the encoding process. By removing this redundancy before encoding, we obtain a more efficient coded signal, which is the basic idea behind differential pulse-code modulation.

Now if we know the past behavior of a signal up to a certain point in time, we may use prediction to make an estimate of a future value of the signal as described in Section 3.13. Suppose then a baseband signal $m(t)$ is sampled at the rate $f_s = 1/T_s$ to produce the sequence $\{m[n]\}$ whose samples are T_s seconds apart. The fact that it is possible to predict future values of the signal $m(t)$ provides motivation for the *differential quantization* scheme shown in Figure 3.28a. In this scheme, the input signal to the quantizer is defined by

$$e[n] = m[n] - \hat{m}[n] \quad (3.74)$$

which is the difference between the unquantized input sample $m[n]$ and a prediction of it, denoted by $\hat{m}[n]$. This predicted value is produced by using a linear prediction filter whose input, as we will see, consists of a quantized version of the input sample $m[n]$. The difference signal $e[n]$ is the prediction error, since it is the amount by which the prediction filter fails to predict the input exactly. By encoding the quantizer output, as in Figure 3.28a, we obtain a variant of PCM known as *differential pulse-code modulation*¹⁰ (DPCM).

The quantizer output may be expressed as

$$e_q[n] = e[n] + q[n] \quad (3.75)$$

where $q[n]$ is the quantization error. According to Figure 3.28a, the quantizer output $e_q[n]$ is added to the predicted value $\hat{m}[n]$ to produce the prediction-filter input

$$m_q[n] = \hat{m}[n] + e_q[n] \quad (3.76)$$

Substituting Equation (3.75) into (3.76), we get

$$m_q[n] = \hat{m}[n] + e[n] + q[n] \quad (3.77)$$

However, from Equation (3.74) we observe that the sum term $\hat{m}[n] + e[n]$ is equal to the input sample $m[n]$. Therefore, we may simplify Equation (3.77) as

$$m_q[n] = m[n] + q[n] \quad (3.78)$$

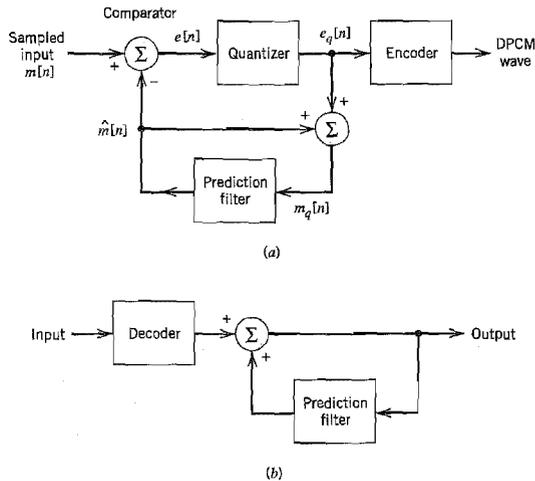


FIGURE 3.28 DPCM system. (a) Transmitter. (b) Receiver.

which represents a quantized version of the input sample $m[n]$. That is, irrespective of the properties of the prediction filter, the quantized sample $m_q[n]$ at the prediction filter input differs from the original input sample $m[n]$ by the quantization error $q[n]$. Accordingly, if the prediction is good, the variance of the prediction error $e[n]$ will be smaller than the variance of $m[n]$, so that a quantizer with a given number of levels can be adjusted to produce a quantization error with a smaller variance than would be possible if the input sample $m[n]$ were quantized directly as in a standard PCM system.

The receiver for reconstructing the quantized version of the input is shown in Figure 3.28b. It consists of a decoder to reconstruct the quantized error signal. The quantized version of the original input is reconstructed from the decoder output using the same prediction filter used in the transmitter of Figure 3.28a. In the absence of channel noise, we find that the encoded signal at the receiver input is identical to the encoded signal at the transmitter output. Accordingly, the corresponding receiver output is equal to $m_q[n]$, which differs from the original input $m[n]$ only by the quantization error $q[n]$ incurred as a result of quantizing the prediction error $e[n]$.

From the foregoing analysis we observe that, in a noise-free environment, the prediction filters in the transmitter and receiver operate on the same sequence of samples, $m_q[n]$. It is with this purpose in mind that a feedback path is added to the quantizer in the transmitter, as shown in Figure 3.28a.

Differential pulse-code modulation includes delta modulation as a special case. In particular, comparing the DPCM system of Figure 3.28 with the DM system of Figure 3.23, we see that they are basically similar, except for two important differences: the use of a one-bit (two-level) quantizer in the delta modulator and the replacement of the prediction filter by a single delay element (i.e., zero prediction order). Simply put, DM is the 1-bit version of DPCM. Note that unlike a standard PCM system, the transmitters of both the DPCM and DM involve the use of *feedback*.

DPCM, like DM, is subject to slope-overload distortion whenever the input signal changes too rapidly for the prediction filter to track it. Also, like PCM, DPCM suffers from quantization noise.

■ PROCESSING GAIN

The output signal-to-noise ratio of the DPCM system shown in Figure 3.28 is, by definition,

$$(\text{SNR})_O = \frac{\sigma_M^2}{\sigma_Q^2} \quad (3.79)$$

where σ_M^2 is the variance of the original input sample $m[n]$, assumed to be of zero mean, and σ_Q^2 is the variance of the quantization error $q[n]$. We may rewrite Equation (3.79) as the product of two factors as follows:

$$\begin{aligned} (\text{SNR})_O &= \left(\frac{\sigma_M^2}{\sigma_E^2} \right) \left(\frac{\sigma_E^2}{\sigma_Q^2} \right) \\ &= G_p (\text{SNR})_Q \end{aligned} \quad (3.80)$$

where σ_E^2 is the variance of the prediction error. The factor $(\text{SNR})_Q$ is the *signal-to-quantization noise ratio*, which is defined by

$$(\text{SNR})_Q = \frac{\sigma_E^2}{\sigma_Q^2} \quad (3.81)$$

The other factor G_p is the *processing gain* produced by the differential quantization scheme; it is defined by

$$G_p = \frac{\sigma_M^2}{\sigma_E^2} \quad (3.82)$$

The quantity G_p , when greater than unity, represents a gain in signal-to-noise ratio that is due to the differential quantization scheme of Figure 3.28. Now, for a given baseband (message) signal, the variance σ_M^2 is fixed, so that G_p is maximized by minimizing the variance σ_E^2 of the prediction error $e[n]$. Accordingly, our objective should be to design the prediction filter so as to minimize σ_E^2 .

In the case of voice signals, it is found that the optimum signal-to-quantization noise advantage of DPCM over standard PCM is in the neighborhood of 4 to 11 dB. The greatest improvement occurs in going from no prediction to first-order prediction, with some additional gain resulting from increasing the order of the prediction filter up to 4 or 5, after which little additional gain is obtained. Since 6 dB of quantization noise is equivalent to 1 bit per sample by virtue of Equation (3.35), the advantage of DPCM may also be expressed in terms of bit rate. For a constant signal-to-quantization noise ratio, and assuming a sampling rate of 8 kHz, the use of DPCM may provide a saving of about 8 to 16 kb/s (i.e., 1 to 2 bits per sample) compared to the standard PCM.

3.15 Adaptive Differential Pulse-Code Modulation

The use of PCM for speech coding at the standard rate of 64 kb/s demands a high channel bandwidth for its transmission. In certain applications, however, such as secure transmission over radio channels that are inherently of low capacity, channel bandwidth is at a premium. In applications of this kind, there is a definite need for *speech coding at low bit rates, while maintaining acceptable fidelity or quality of reproduction*.

For coding speech at low bit rates, a waveform coder of prescribed configuration is optimized by exploiting both *statistical characterization of speech waveforms and properties of hearing*. In particular, the design philosophy has two aims in mind:

1. To remove redundancies from the speech signal as far as possible.
2. To assign the available bits to code the nonredundant parts of the speech signal in a perceptually efficient manner.

As we strive to reduce the bit rate from 64 kb/s (used in standard PCM) to 32, 16, 8, and 4 kb/s, the schemes used for redundancy removal and bit assignment become increasingly more sophisticated. As a rule of thumb, in the 64 to 8 kb/s range, the computational complexity (measured in terms of multiply-add operations) required to code speech increases by an order of magnitude when the bit rate is halved, for approximately equal speech quality.

In this section, we describe *adaptive differential pulse-code modulation (ADPCM)*,¹¹ which permits the coding of speech at 32 kb/s through the combined use of *adaptive quantization* and *adaptive prediction*; the number of eight bits per sample required in the standard PCM is thereby reduced to four. The term *adaptive* used herein means being responsive to changing level and spectrum of the input speech signal. The variation of performance with speakers and speech material, together with variations in signal level inherent in the speech communication process, make the combined use of adaptive quantization and adaptive prediction necessary to achieve best performance over a wide range of speakers and speaking situations.

Adaptive quantization refers to a quantizer that operates with a *time-varying* step-size $\Delta[n]$. At any given sampling instant identified by the index n , the adaptive quantizer is assumed to have a uniform transfer characteristic. The step-size $\Delta[n]$ is varied so as to match the variance σ_M^2 of the input sample $m[n]$. In particular, we write

$$\Delta[n] = \phi \hat{\sigma}_M[n] \quad (3.83)$$

where ϕ is a constant, and $\hat{\sigma}_M[n]$ is an *estimate* of the standard deviation $\sigma_M[n]$ (i.e., square root of the variance σ_M^2). For a nonstationary input, $\sigma_M[n]$ is time varying. The problem of adaptive quantization according to Equation (3.83) is, therefore, one of computing the estimate $\hat{\sigma}_M[n]$ continuously.

The implementation of Equation (3.83) may proceed in one of two ways:

1. *Adaptive quantization with forward estimation (AQF)*, in which unquantized samples of the input signal are used to derive forward estimates of $\sigma_M[n]$.
2. *Adaptive quantization with backward estimation (AQB)*, in which samples of the quantizer output are used to derive backward estimates of $\sigma_M[n]$.

The AQF scheme requires the use of a *buffer* to store unquantized samples of the input speech signal needed for the learning period. It also requires the explicit transmission of level information (typically, about 5 to 6 bits per step-size sample) to a remote decoder, thereby burdening the system with additional *side information* that has to be transmitted to the receiver. Moreover, a processing *delay* (on the order of 16 ms for speech) in the encoding operation results from the use of AQF, which is unacceptable in some applications. The problems of level transmission, buffering, and delay intrinsic to AQF are all avoided in AQB. In the latter scheme, the recent history of the quantizer output is used to extract information for the computation of the step size $\Delta[n]$. In practice, AQB is therefore usually preferred over AQF.

Figure 3.29 shows the block diagram of an adaptive quantizer with backward estimation. It represents a nonlinear feedback system; hence, it is not obvious that the system

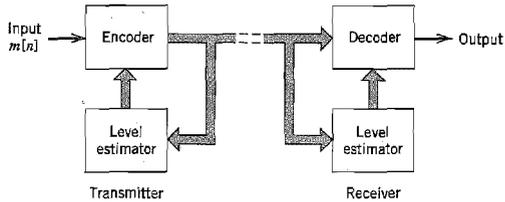


FIGURE 3.29 Adaptive quantization with backward estimation (AQE).

will be stable. However, if the quantizer input $m[n]$ is *bounded*, then the backward estimate $\hat{\sigma}_M[n]$ and the corresponding step size $\Delta[n]$ are as well; under such a condition, the system is indeed *stable*.

The use of adaptive prediction in ADPCM is justified because speech signals are inherently *nonstationary*, a phenomenon that manifests itself in the fact that the autocorrelation function and power spectral density of speech signals are time-varying functions of their respective arguments. This implies that the design of predictors for such inputs should likewise be time varying, that is, adaptive. As with adaptive quantization, there are two schemes for performing adaptive prediction:

1. *Adaptive prediction with forward estimation (APF)*, in which unquantized samples of the input signal are used to derive estimates of the predictor coefficients.
2. *Adaptive prediction with backward estimation (APB)*, in which samples of the quantizer output and the prediction error are used to derive estimates of the predictor coefficients.

However, APF suffers from the same intrinsic disadvantages (side information, buffering, and delay) as AQF. These disadvantages are eliminated by using the APB scheme shown in Figure 3.30, where the box labeled “logic for adaptive prediction” represents the algorithm for updating the predictor coefficients. In the latter scheme, the optimum predictor coefficients are estimated on the basis of quantized and transmitted data; they can therefore be updated as frequently as desired, say, from sample to sample. Accordingly, APB is the preferred method of prediction for ADPCM.

The LMS algorithm for the predictor, described in Section 3.13, and an adaptive scheme for the quantizer, based on Equation (3.83), have been combined in a synchronous

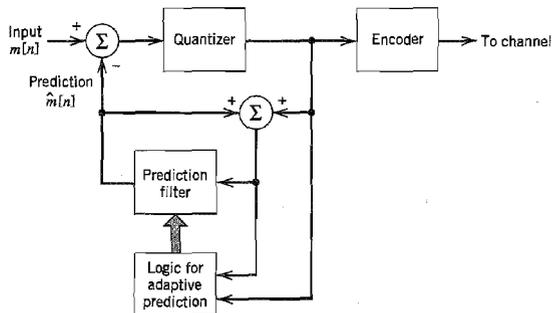


FIGURE 3.30 Adaptive prediction with backward estimation (APB).

fashion for the design of both the encoder and decoder. The performance of this combination is so impressive at 32 kb/s that ADPCM is now accepted internationally as a standard coding technique for voice signals, along with 64 kb/s using standard PCM.

3.16 Computer Experiment: Adaptive Delta Modulation

A simple form of AQB is to be found in the modification of linear delta modulation (LDM) to form *adaptive delta modulation* (ADM). The principle underlying all ADM algorithms is two-fold:

1. If successive errors are of opposite polarity, then the delta modulator is operating in its granular mode; in this case, it may be advantageous to reduce the step size.
2. If, however, successive errors are of the same polarity, then the delta modulator is operating in its slope-overload mode; in this second case, the step size should be increased.

Thus by varying the step-size in accordance with this principle, the delta modulator is enabled to cope with changes in the input signal.

Figure 3.31 shows the block diagram of an ADM based on increasing or decreasing the step size by a factor of 50 percent at each iteration of the adaptive process. The algorithm for adaptation of the step size is defined by¹²

$$\Delta[n] = \begin{cases} \frac{|\Delta[n-1]|}{m_q[n]} (m_q[n] + 0.5m_q[n-1]) & \text{if } \Delta[n-1] \geq \Delta_{\min} \\ \Delta_{\min} & \text{if } \Delta[n-1] < \Delta_{\min} \end{cases} \quad (3.84)$$

where $\Delta[n]$ is the step size at iteration (time step) n of the algorithm, and $m_q[n]$ is the 1-bit quantizer output that equals ± 1 .

In this experiment we use a sinusoidal input signal to demonstrate the reconstruction performance of the ADM algorithm based on Equation (3.84), and compare it to the performance of a corresponding linear delta modulator (LDM). Details of the experiment are as follows:

Input signal:

$$m(t) = A \sin(2\pi f_m t)$$

where amplitude $A = 10$, frequency $f_m = f_s/100$, and $f_s =$ sampling frequency.

Linear delta modulation (LDM):

$$\text{Step size } \Delta[n] = 1 \text{ for all } n$$

Adaptive delta modulation (ADM):

$$\Delta_{\min} = \frac{1}{8}$$

The results of the experiment are plotted in Figure 3.32. Part *a* of Figure 3.32 is for LDM, and part *b* of the figure is for ADM. From the waveforms presented here, we may make the following observations:

- ▶ ADM *tracks* changes in the sinusoidal input signal much better than LDM. This improvement in the performance of ADM is due to adaptation of the step size in

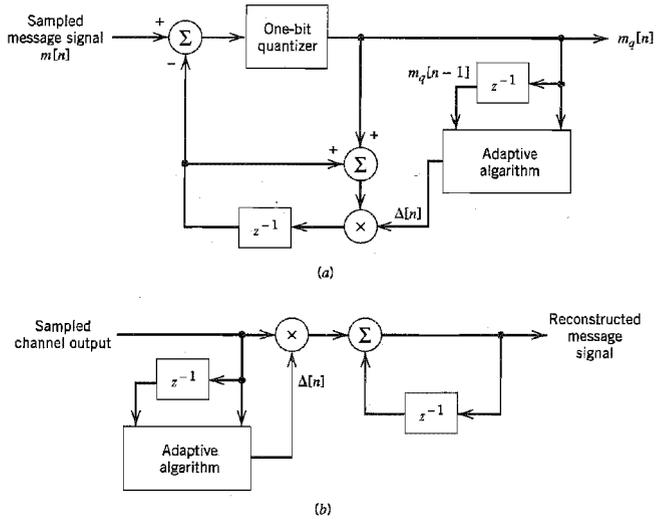


FIGURE 3.31 Adaptive delta modulation system: (a) Transmitter. (b) Receiver.

successive iterations of the algorithm. In particular, the reduced step size of the ADM results in smaller quantization errors near the extremities of the input signal than the LDM. However, both modulation schemes produce comparable quantization errors in regions of the input signal where the slope is moderately high.

- ▶ The improved tracking performance of the ADM results in an output signal with a much lower bit rate, on the average, than the LDM.

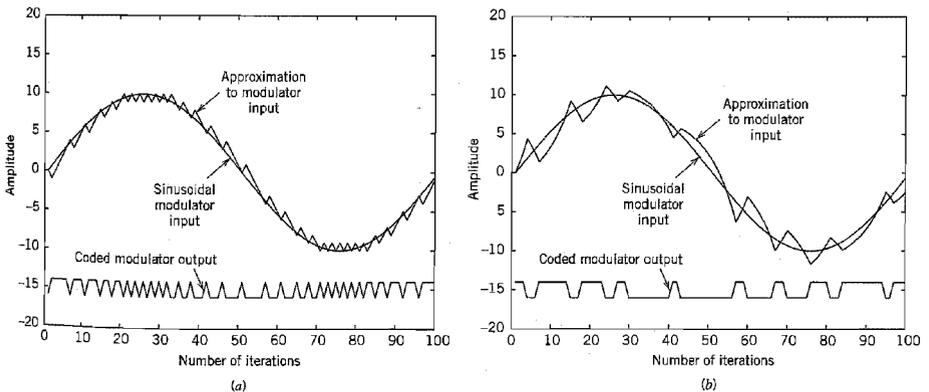


FIGURE 3.32 Waveforms resulting from the computer experiment on delta modulation: (a) Linear delta modulation. (b) Adaptive delta modulation.

3.17 MPEG Audio Coding Standard

Speech (voice) and audio signals are similar in that, in both cases, the quality of a coding scheme is based on the properties of human auditory perception. In the case of speech signals, we have efficient coding schemes (e.g., ADPCM) because a speech production model is available. Unfortunately, nothing similar exists for audio signals.

In this section, we revisit the *MPEG-1/audio coding standard* briefly described in the Background and Preview chapter; MPEG stands for Motion Picture Experts Group, and the suffix 1 is intended to mean it is the first in a series of several standards.¹³ Like ADPCM, the MPEG-1/audio coding standard is a lossy compression system, but it differs from ADPCM in an important practical respect: The MPEG-1 standard is capable of achieving *transparent, perceptually lossless compression* of stereophonic audio signals at high sampling rates. In particular, subjective listening tests performed by the MPEG/audio committee, under very difficult listening conditions, have shown that even with a 6-to-1 compression ratio, the coded and original audio signals are perceptually indistinguishable.

The MPEG-1/audio coding standard achieves this remarkable performance by exploiting two psychoacoustic characteristics of the human auditory system:

1. Critical bands.

The inner ear¹⁴ of the auditory system represents the power spectra of incoming signals on a nonlinear scale in the form of limited frequency bands called the *critical bands*. The audible frequency band, extending up to 20 kHz, is covered by 25 critical bands, whose individual bandwidths increase with frequency. Loosely speaking, the auditory system may be modeled as a *band-pass filter bank*, consisting of 25 overlapping band-pass filters with bandwidths less than 100 Hz for the lowest audible frequencies and up to 5 kHz for the highest audible frequencies.

2. Auditory masking.

Auditory masking or noise masking is a frequency-domain phenomenon that arises when a low-level signal (the maskee) and a high-level signal (the masker) occur simultaneously and are close enough to each other in frequency. If the low-level signal lies below a *masking threshold*, it is made inaudible (i.e., masked) by the stronger signal. The auditory-masking phenomenon is most pronounced when both signals lie in the same critical band, and less effective when they lie in neighboring bands.

Figure 3.33 illustrates the definition of masking threshold and related parameters for a pair of adjacent frequency bands; it is assumed that the masker (i.e., the high-level signal) lies inside the dark-shaded critical band. The low-level signals lying inside this dark area and below the masking threshold are masked by the stronger signal. From Figure 3.33 we see that the masking threshold varies with frequency across the critical band. Accordingly, we may define a *minimum masking threshold* for a critical band, below which all low-level signals that lie inside that band are made inaudible by the stronger signal. The power difference, expressed in decibels, between the masker and the minimum masking threshold, is termed the *signal-to-mask ratio* (SMR). Figure 3.33 also includes the *signal-to-noise ratio* (SNR) for an *R*-bit quantizer. The difference between SMR and SNR is the *noise-to-mask ratio* (NMR) for an *R*-bit quantizer as shown by

$$\text{NMR} = \text{SMR} - \text{SNR} \quad (3.85)$$

where all three terms are expressed in dBs. Within a critical band, the quantization noise is inaudible as long as the NMR for the pertinent quantizer is negative.

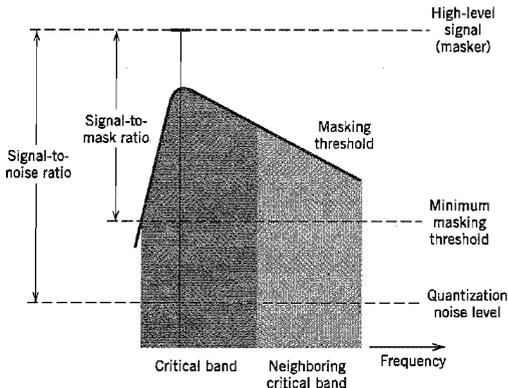


FIGURE 3.33 Illustrating the definitions of masking threshold and related parameters. The high-level signal (masker) lies inside the darker-shaded critical band, hence the masking is more effective in this band than in the neighboring band shown in lighter shading. (Adapted from Noll (1998) with permission of the CRC Press.)

With this background on the psychoacoustics of the auditory system, we are now ready to describe the operation of the MPEG-1/audio coding standard. Figure 3.34 shows the basic block diagrams of the encoder and decoder. The encoder consists of four functional units: time-to-frequency mapping network, psychoacoustic model, quantizer and coder, and frame-packing unit. The decoder consists of three functional units: frame-unpacking unit, frequency-sample reconstruction network, and frequency-to-time mapping network. The psychoacoustic model is thus only necessary in the encoder.

Starting with a description of the encoder first, the function of the *time-to-frequency mapping network* is to decompose the input audio signal into multiple subbands for coding. The mapping is performed in three layers, labeled I, II, and III, which are of increasing complexity, delay, and subjective perceptual performance. The algorithm in layer I uses a

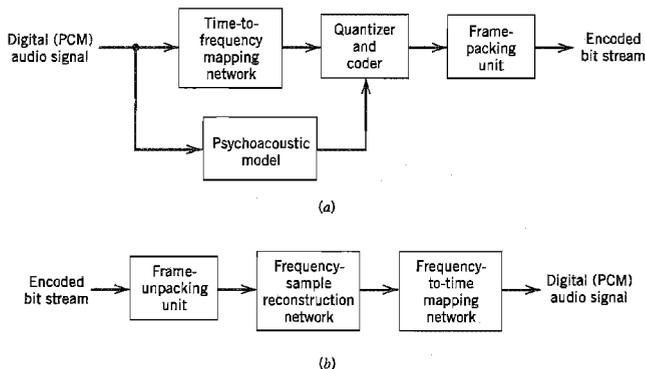


FIGURE 3.34 MPEG/Audio coding system. (a) Transmitter. (b) Receiver.

band-pass filter bank that divides the audio signal into 32 *constant-width subbands*; this filter bank is also found in layers II and III. In light of our previous remarks on the non-uniformly spaced critical bands, the design of this filter bank is a compromise between computational efficiency and perceptual performance. The algorithm in layer II is a simple enhancement of layer I; it improves the compression performance by coding the data in larger groups. Finally, the layer III algorithm is much more refined in that it is designed to achieve frequency resolutions closer to the partitions between the critical bands.

The *psychoacoustic model* is the key component in the encoder. Its function is to analyze the spectral content of the input audio signal and thereby compute the signal-to-mask ratio for each subband in each of the three layers. This information is, in turn, used by the quantizer-coder to decide how to apportion the available number of bits for the quantization of the subband signals. This dynamic allocation of bits is performed so as to minimize the audibility of the quantization noise. Finally, the *frame-packing unit* assembles the quantized audio samples into a decodable bit stream.

The decoder simply reverses the signal-processing operations performed in the encoder, converting the received stream of encoded bits into a time-domain audio signal.

To sum up, the MPEG-1/audio coding standard represents the state of the art in the coding of audio signals. Layer I achieves a compression ratio of 4 at an approximate stereo bit rate of 384 kb/s for transparent quality of performance. The corresponding compression ratios for layers II and III are 8 and 12 at approximate stereo bit rates of 192 kb/s and 128 kb/s, respectively. The subjective quality of the MPEG-1/audio coding standard is equivalent to compact disc quality (16-bit PCM) for many types of music; the compact disc (CD) is today's de facto standard of digital audio representation.

3.18 Summary and Discussion

In this chapter we introduced two fundamental and complementary processes:

- ▶ *Sampling*, which operates in the time domain; the sampling process is the link between an analog waveform and its discrete-time representation.
- ▶ *Quantization*, which operates in the amplitude domain; the quantization process is the link between an analog waveform and its discrete-amplitude representation.

The sampling process builds on the *sampling theorem*, which states that a strictly band-limited signal with no frequency components higher than W Hz is represented uniquely by a sequence of samples taken at a uniform rate equal to or greater than the Nyquist rate of $2W$ samples per second. The quantization process exploits the fact that any human sense, as ultimate receiver, can only detect finite intensity differences.

The sampling process is basic to the operation of all pulse modulation systems, which may be classified into analog pulse modulation and digital pulse modulation. The distinguishing feature between them is that analog pulse modulation systems maintain a continuous amplitude representation of the message signal, whereas digital pulse modulation systems also employ quantization to provide a representation of the message signal that is discrete in both time and amplitude.

Analog pulse modulation results from varying some parameter of the transmitted pulses, such as amplitude, duration, or position, in which case we speak of pulse-amplitude modulation (PAM), pulse-duration modulation (PDM), or pulse-position modulation (PPM), respectively. In time-division multiplexing (TDM) of several channels, signal processing usually begins with PAM. To use PDM or PPM in such an application, we have

to ensure that full-scale modulation will not cause a pulse from one message signal to enter a time slot belonging to another message signal. This restriction results in a wasteful use of time space in telephone systems that are characterized by high peak factors, which is one reason for not using PDM or PPM in telephony. Also, despite the fact that PPM is more efficient than PDM, they both fall short of the ideal system for exchanging transmission bandwidth for improved noise performance.

Digital pulse modulation systems transmit analog message signals as a sequence of coded pulses, which is made possible through the combined use of sampling and quantization. Pulse-code modulation is an important form of digital pulse modulation that is endowed with some unique system advantages, which, in turn, have made it the standard method of modulation for the transmission of such analog signals as voice and video signals. The advantages of pulse-code modulation include robustness to noise and interference, efficient regeneration of the coded pulses along the transmission path, and a uniform format for different kinds of baseband signals.

Indeed, it is because of this list of advantages unique to pulse-code modulation that it has become the method of choice for the construction of public switched telephone networks (PSTNs). In this context, the reader should carefully note that the telephone channel viewed from the PSTN to an Internet service provider, for example, is *nonlinear* due to the use of companding and, most importantly, it is *entirely digital*. This observation has a significant impact on the design of high-speed modems for communication between a computer user and server, as discussed in Chapter 6.

Delta modulation and differential pulse-code modulation are two other useful forms of digital pulse modulation. The principal advantage of delta modulation is the simplicity of its circuitry. In contrast, differential pulse-code modulation employs increased circuit complexity to reduce channel bandwidth. The improvement is achieved by using the idea of prediction to remove redundant symbols from an incoming data stream. A further improvement in the operation of differential pulse-code modulation can be made through the use of adaptivity to account for statistical variations in the input data. By so doing, bandwidth requirement is reduced significantly without serious degradation in system performance.

Unlike adaptive differential pulse-code modulation, the MPEG audio coding standard achieves the compression of stereophonic audio signals in a transparent, perceptually lossless manner. This impressive performance is realized by exploiting certain psychoacoustic properties of the auditory system.

At this point in the discussion, it is informative to take a critical look at the different forms of pulse modulation that we have described in this chapter. In a strict sense, the term *pulse modulation* is a misnomer in that all of its different forms, be they analog or digital, are in fact *source coding* techniques. We say this for the simple reason that a message signal remains a baseband signal after undergoing all the changes involved in a pulse modulation process. The baseband nature of a pulse-modulated signal is exemplified by the fact that, irrespective of its exact description, it can be transmitted over a baseband channel of adequate bandwidth. Indeed, the material presented in the next chapter is devoted to the baseband transmission of data represented by a sequence of pulses.

It is also important to recognize that pulse modulation techniques are *lossy* in the sense that some information is lost as a result of the signal representation that they perform. For example, in pulse-amplitude modulation, the customary practice is to use low-pass anti-alias filtering prior to sampling; in so doing, information is lost by virtue of the fact that high-frequency components considered to be unessential are removed by the filter. The lossy nature of pulse modulation is most vividly seen in pulse-code modulation that is characterized by the generation of quantization noise (i.e., distortion); the transmitted

sequence of encoded pulses does not have the infinite precision needed to represent continuous samples exactly. Nevertheless, the loss of information incurred by the use of a pulse-modulation process is *under the designer's control* in that it can be made small enough for it to be nondiscernible by the end user.

The material presented in this chapter on pulse modulation has been from a signal processing perspective. We will revisit pulse-code modulation in Chapter 9, which is devoted to information-theoretic considerations of communication systems. In so doing we will develop deeper insight into its operation as a source coding technique.

NOTES AND REFERENCES

1. The classic book on pulse modulation is Black (1953). A more detailed treatment of the subject is presented in the book by Rowe (1965). For the noise analysis of a PPM system, see the third edition of the book by Haykin (1994).
2. Pulse-code modulation was invented by Reeves in 1937. For a historical account of this invention, see the paper by Reeves (1975). The book by Jayant and Noll (1984) presents detailed treatment of pulse-code modulation, differential pulse-code modulation, delta modulation, and their variants. The book edited by Jayant (1976) provides a collection of early papers written on waveform quantization and coding.
3. For a detailed discussion of quantization noise in PCM systems, see the paper by Bennett (1948) and also the book by Rowe (1965, pp. 311–321).
4. The two necessary conditions of Equations (3.42) and (3.47) for optimality of a scalar quantizer were reported independently by Lloyd (1957) and Max (1960), hence the name “Lloyd-Max quantizer.” The derivation of these two optimality conditions presented in this chapter follows the book by Gersho and Gray (1992).
5. The μ -law used for signal compression is described in Smith (1957). The μ -law is used in the United States, Canada, and Japan. In Europe, the A-law is used for signal compression; this compression law is described in Cattermode (1969, pp. 133–140). For a discussion of the μ -law and A-law, see also the paper by Kaneko (1970).
6. For a description of the original version of the T1-carrier system, see the paper by Fultz and Penick (1965). The description given in Example 3.2 is based on an updated version of this system; see Henning and Pan (1972).
7. The North America/Japan standards for digital multiplexers were originally adopted by AT&T. Another set of standards has been adopted by CCITT for the rest of the world. The CCITT digital hierarchy is similar to that described in Section 3.10, except for certain changes in the specifications of the number of channel inputs to the five digital multiplexers and their individual bit rates. For details of the CCITT digital hierarchy, see Couch (1997).
8. For the original papers on delta modulation, see Schouten, DeJager, and Greefkes (1952) and DeJager (1952). For a review paper on delta modulation, see the paper by Schindler (1970).
9. Delta-sigma modulation is described in the book by Jayant and Noll (1984, pp. 399–400); see also the paper by Inose, Yasuda, and Murakami (1962).
10. Differential pulse-code modulation was invented by Cutler; the invention is described in a patent issued in 1952. For a comparison of the noise performances of PCM and DPCM, see the paper by Jayant (1974); see also Rabiner and Schafer (1978, Chapter 5).
11. For a discussion of adaptive differential pulse-code modulation, see Jayant and Noll (1984).

12. The adaptive delta modulation algorithm (ADM) of Equation (3.84) is the corrected version of an algorithm presented in Sklar (1988, p. 641). Sklar's algorithm was adapted from an earlier paper by Song et al. (1971), where an optimum ADM system is derived; the highly nonlinear equations characterizing the optimum system are approximated in the latter paper by piecewise-linear equations for the purpose of implementation.
13. The MPEG-1/audio coding standard is described in the papers by Brandenburg and Stoll (1994), Pan (1993), and the article by Peter Noll in the handbook on Digital Signal Processing edited by Madisetti and Williams (1998); the latter article also discusses the follow-up standards to MPEG-1. In particular, the MPEG-2 offers stereophonic audio coding at sampling rates lower than MPEG-1.
14. The ear, the organ of hearing, responds to incoming acoustical waves. It has three main parts, with their functions as summarized here:
 - ▶ The *outer ear* aids in the collection of sounds.
 - ▶ The *middle ear* provides an acoustic impedance match between the air and the cochlea fluids, thereby conveying the vibrations of the tympanic membrane (eardrum) due to the incoming sounds to the inner ear in an efficient manner.
 - ▶ The *inner ear* converts the mechanical vibrations from the middle ear to an electrochemical or neural signal for transmission to the brain for processing.

PROBLEMS

Sampling Process

- 3.1 A narrowband signal has a bandwidth of 10 kHz centered on a carrier frequency of 100 kHz. It is proposed to represent this signal in discrete-time form by sampling its in-phase and quadrature components individually. What is the minimum sampling rate that can be used for this representation? Justify your answer. How would you reconstruct the original narrowband signal from the sampled versions of its in-phase and quadrature components?
- 3.2 In *natural sampling*, an analog signal $g(t)$ is multiplied by a periodic train of rectangular pulses $c(t)$. Given that the pulse repetition frequency of this periodic train is f_s and the duration of each rectangular pulse is T (with $f_s T \ll 1$), do the following:
 - (a) Find the spectrum of the signal $s(t)$ that results from the use of natural sampling; you may assume that time $t = 0$ corresponds to the midpoint of a rectangular pulse in $c(t)$.
 - (b) Show that the original signal $m(t)$ may be recovered exactly from its naturally sampled version, provided that the conditions embodied in the sampling theorem are satisfied.
- 3.3 Specify the Nyquist rate and the Nyquist interval for each of the following signals:
 - (a) $g(t) = \text{sinc}(200t)$
 - (b) $g(t) = \text{sinc}^2(200t)$
 - (c) $g(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$
- 3.4 (a) Plot the spectrum of a PAM wave produced by the modulating signal

$$m(t) = A_m \cos(2\pi f_m t)$$

assuming a modulation frequency $f_m = 0.25$ Hz, sampling period $T_s = 1$ s, and pulse duration $T = 0.45$ s.

- (b) Using an ideal reconstruction filter, plot the spectrum of the filter output. Compare this result with the output that would be obtained if there were no aperture effect.

Pulse-Amplitude Modulation

- 3.5 Figure P3.5 shows the idealized spectrum of a message signal $m(t)$. The signal is sampled at a rate equal to 1 kHz using flat-top pulses, with each pulse being of unit amplitude and duration 0.1 ms. Determine and sketch the spectrum of the resulting PAM signal.

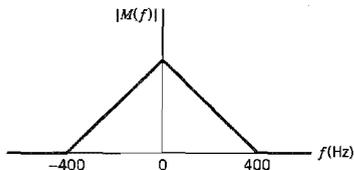


FIGURE P3.5

- 3.6 In this problem, we evaluate the equalization needed for the aperture effect in a PAM system. The operating frequency $f = f_s/2$, which corresponds to the highest frequency component of the message signal for a sampling rate equal to the Nyquist rate. Plot $1/\text{sinc}(0.5T/T_s)$ versus T/T_s , and hence find the equalization needed when $T/T_s = 0.1$.
- 3.7 Consider a PAM wave transmitted through a channel with white Gaussian noise and minimum bandwidth $B_T = 1/2T_s$, where T_s is the sampling period. The noise is of zero mean and power spectral density $N_0/2$. The PAM signal uses a standard pulse $g(t)$ with its Fourier transform defined by

$$G(f) = \begin{cases} \frac{1}{2B_T}, & |f| < B_T \\ 0, & |f| > B_T \end{cases}$$

By considering a full-load sinusoidal modulating wave, show that PAM and baseband-signal transmission have equal signal-to-noise ratios for the same average transmitted power.

- 3.8 Twenty-four voice signals are sampled uniformly and then time-division multiplexed. The sampling operation uses flat-top samples with $1 \mu\text{s}$ duration. The multiplexing operation includes provision for synchronization by adding an extra pulse of sufficient amplitude and also $1 \mu\text{s}$ duration. The highest frequency component of each voice signal is 3.4 kHz.
- (a) Assuming a sampling rate of 8 kHz, calculate the spacing between successive pulses of the multiplexed signal.
- (b) Repeat your calculation assuming the use of Nyquist rate sampling.
- 3.9 Twelve different message signals, each with a bandwidth of 10 kHz, are to be multiplexed and transmitted. Determine the minimum bandwidth required for each method if the multiplexing/modulation method used is
- (a) FDM, SSB.
- (b) TDM, PAM.
- 3.10 A PAM telemetry system involves the multiplexing of four input signals: $s_i(t)$, $i = 1, 2, 3, 4$. Two of the signals $s_1(t)$ and $s_2(t)$ have bandwidths of 80 Hz each, whereas the remaining two signals $s_3(t)$ and $s_4(t)$ have bandwidths of 1 kHz each. The signals $s_3(t)$ and $s_4(t)$ are each sampled at the rate of 2400 samples per second. This sampling rate is divided by 2^R (i.e., an integer power of 2) to derive the sampling rate for $s_1(t)$ and $s_2(t)$.
- (a) Find the maximum value of R .
- (b) Using the value of R found in part (a), design a multiplexing system that first multiplexes $s_1(t)$ and $s_2(t)$ into a new sequence, $s_5(t)$, and then multiplexes $s_3(t)$, $s_4(t)$, and $s_5(t)$.

Line Codes

3.11 In this problem we derive the formulas used to compute the power spectra of Figure 3.16 for the five line codes described in Section 3.7. In the case of each line code, the bit duration is T_b and the pulse amplitude A is conditioned to normalize the average power of the line code to unity as indicated in Figure 3.16. Assume that the data stream is randomly generated, and symbols 0 and 1 are equally likely.

Derive the power spectral densities of these line codes as summarized here:

(a) Unipolar nonreturn-to-zero signals:

$$S(f) = \frac{A^2 T_b}{4} \operatorname{sinc}^2(f T_b) \left(1 + \frac{1}{T_b} \delta(f) \right)$$

(b) Polar nonreturn-to-zero signals:

$$S(f) = A^2 T_b \operatorname{sinc}^2(f T_b)$$

(c) Unipolar return-to-zero signals:

$$S(f) = \frac{A^2 T_b}{16} \operatorname{sinc}^2\left(\frac{f T_b}{2}\right) \left[1 + \frac{1}{T_b} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_b}\right) \right]$$

(d) Bipolar return-to-zero signals:

$$S(f) = \frac{A^2 T_b}{4} \operatorname{sinc}^2\left(\frac{f T_b}{2}\right) \sin^2(\pi f T_b)$$

(e) Manchester-encoded signals:

$$S(f) = A^2 T_b \operatorname{sinc}^2\left(\frac{f T_b}{2}\right) \sin^2\left(\frac{\pi f T_b}{2}\right)$$

Hence, confirm the spectral plots displayed in Figure 3.16.

3.12 Suppose a random binary data stream (with equiprobable symbols) is differentially encoded and then transmitted using one of the five line codes described in Problem 3.11. How is the power spectral density of the transmitted data affected by the use of differential encoding? Justify your answer.

3.13 A randomly generated data stream consists of equiprobable binary symbols 0 and 1. It is encoded into a polar nonreturn-to-zero waveform with each binary symbol being defined as follows:

$$s(t) = \begin{cases} \cos\left(\frac{\pi t}{T_b}\right), & -\frac{T_b}{2} < t \leq \frac{T_b}{2} \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch the waveform so generated, assuming that the data stream is 00101110.

(b) Derive an expression for the power spectral density of this signal, and sketch it. *Hint:* use Equation (1.52).

(c) Compare the power spectral density of this random waveform with that defined in part (b) of Problem 3.11.

3.14 Given the data stream 1110010100, sketch the transmitted sequence of pulses for each of the following line codes:

(a) Unipolar nonreturn-to-zero

(b) Polar nonreturn-to-zero

(c) Unipolar return-to-zero

(d) Bipolar return-to-zero

(e) Manchester code

- 3.15 Suppose the binary data stream considered in Problem 3.14 is differentially encoded and then transmitted using one of the five line codes considered therein. Sketch each of the transmitted data streams, assuming the use of symbol 1 for the reference bit. How is the result affected if symbol 0 is used for the reference bit?

Pulse-Code Modulation

- 3.16 A speech signal has a total duration of 10 s. It is sampled at the rate of 8 kHz and then encoded. The signal-to-(quantization) noise ratio is required to be 40 dB. Calculate the minimum storage capacity needed to accommodate this digitized speech signal.
- 3.17 Consider a uniform quantizer characterized by the input-output relation illustrated in Figure 3.10a. Assume that a Gaussian-distributed random variable with zero mean and unit variance is applied to this quantizer input.
- What is the probability that the amplitude of the input lies outside the range -4 to $+4$?
 - Using the result of part (a), show that the output signal-to-noise ratio of the quantizer is given by

$$(\text{SNR})_O = 6R - 7.2 \text{ dB}$$

where R is the number of bits per sample. Specifically, you may assume that the quantizer input extends from -4 to $+4$. Compare the result of part (b) with that obtained in Example 3.1.

- 3.18 A PCM system uses a uniform quantizer followed by a 7-bit binary encoder. The bit rate of the system is equal to 50×10^6 b/s.
- What is the maximum message bandwidth for which the system operates satisfactorily?
 - Determine the output signal-to-(quantization) noise ratio when a full-load sinusoidal modulating wave of frequency 1 MHz is applied to the input.
- 3.19 Show that, with a nonuniform quantizer, the mean-square value of the quantization error is approximately equal to $(1/12)\sum_i \Delta_i^2 p_i$, where Δ_i is the i th step size and p_i is the probability that the input signal amplitude lies within the i th interval. Assume that the step size Δ_i is small compared with the excursion of the input signal.
- 3.20 (a) A sinusoidal signal, with an amplitude of 3.25 volts, is applied to a uniform quantizer of the midtread type whose output takes on the values 0, ± 1 , ± 2 , ± 3 volts. Sketch the waveform of the resulting quantizer output for one complete cycle of the input.
- (b) Repeat this evaluation for the case when the quantizer is of the midrise type whose output takes on the values $=0.5, \pm 1.5, \pm 2.5, \pm 3.5$ volts.
- 3.21 The signal

$$m(t) = 6 \sin(2\pi t) \text{ volts}$$

is transmitted using a 4-bit binary PCM system. The quantizer is of the midrise type, with a step size of 1 volt. Sketch the resulting PCM wave for one complete cycle of the input. Assume a sampling rate of four samples per second, with samples taken at $t = \pm 1/8, \pm 3/8, \pm 5/8, \dots$, seconds.

- 3.22 Figure P3.22 shows a PCM signal in which the amplitude levels of $+1$ volt and -1 volt are used to represent binary symbols 1 and 0, respectively. The code word used consists of three bits. Find the sampled version of an analog signal from which this PCM signal is derived.

with $\Delta/2$ for PCM being replaced by Δ for DM.) The DM system is designed to handle analog message signals limited to bandwidth W .

(a) Show that the average quantization noise power produced by the system is

$$N = \frac{4\pi^2 A^2 f_m^2 W}{3f_s^3}$$

where it is assumed that the step size Δ has been chosen in accordance with the formula used in Problem 3.27 so as to avoid slope overload.

(b) Hence determine the signal-to-(quantization) noise ratio of the DM system for a sinusoidal input.

3.29 Consider a DM system designed to accommodate analog message signals limited to bandwidth $W = 5$ kHz. A sinusoidal test signal of amplitude $A = 1$ volt and frequency $f_m = 1$ kHz is applied to the system. The sampling rate of the system is 50 kHz.

(a) Calculate the step size Δ required to minimize slope overload.

(b) Calculate the signal-to-(quantization) noise ratio of the system for the specified sinusoidal test signal.

For these calculations, use the formulas derived in Problems 3.27 and 3.28.

3.30 Consider a low-pass signal with a bandwidth of 3 kHz. A linear delta modulation system, with step size $\Delta = 0.1V$, is used to process this signal at a sampling rate ten times the Nyquist rate.

(a) Evaluate the maximum amplitude of a test sinusoidal signal of frequency 1 kHz, which can be processed by the system without slope-overload distortion.

(b) For the specifications given in part (a), evaluate the output signal-to-noise ratio under (i) prefiltered, and (ii) postfiltered conditions.

Linear Prediction

3.31 A one-step linear predictor operates on the sampled version of a sinusoidal signal. The sampling rate is equal to $10f_0$ where f_0 is the frequency of the sinusoid. The predictor has a single coefficient denoted by w_1 .

(a) Determine the optimum value of w_1 required to minimize the prediction error variance.

(b) Determine the minimum value of the prediction error variance.

3.32 A stationary process $X(t)$ has the following values for its autocorrelation function:

$$R_X(0) = 1$$

$$R_X(1) = 0.8$$

$$R_X(2) = 0.6$$

$$R_X(3) = 0.4$$

(a) Calculate the coefficients of an optimum linear predictor involving the use of three unit-delays.

(b) Calculate the variance of the resulting prediction error.

3.33 Repeat the calculations of Problem 3.32, but this time use a linear predictor with two unit-delays. Compare the performance of this second optimum linear predictor with that considered in Problem 3.32.

Differential Pulse-Code Modulation

3.34 A DPCM system uses a linear predictor with a single tap. The normalized autocorrelation function of the input signal for a lag of one sampling interval is 0.75. The predictor is

designed to minimize the prediction error variance. Determine the processing gain attained by the use of this predictor.

3.35 Calculate the improvement in processing gain of a DPCM system using the optimized three-tap linear predictor of Problem 3.32 over that of the optimized two-tap linear predictor of Problem 3.33. For this calculation, use the autocorrelation function values of the input signal specified in Problem 3.32.

3.36 In this problem, we compare the performance of a DPCM system with that of an ordinary PCM system using companding.

For a sufficiently large number of representation levels, the signal-to-(quantization) noise ratio of PCM systems, in general, is defined by

$$10 \log_{10}(\text{SNR})_O = \alpha + 6n \text{ dB}$$

where 2^n is the number of representation levels. For a companded PCM system using the μ -law, the constant α is itself defined by

$$\alpha \approx 4.77 - 20 \log_{10}(\log(1 + \mu)) \text{ dB}$$

For a DPCM system, on the other hand, the constant α lies in the range $-3 < \alpha < 15$ dBs. The formulas quoted herein apply to telephone-quality speech signals.

Compare the performance of the DPCM system against that of the μ -companded PCM system with $\mu = 255$ for each of the following scenarios:

- (a) The improvement in $(\text{SNR})_O$ realized by DPCM over companded PCM for the same number of bits per sample.
- (b) The reduction in the number of bits per sample required by DPCM, compared to the companded PCM for the same $(\text{SNR})_O$.

3.37 In the DPCM system depicted in Figure P3.37, show that in the absence of channel noise, the transmitting and receiving prediction filters operate on slightly different input signals.

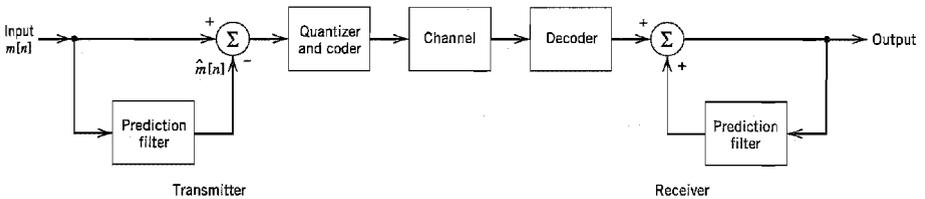


FIGURE P3.37

Computer Experiments

3.38 A sinusoidal signal of frequency $f_0 = 10^4/2\pi$ Hz is sampled at the rate of 8 kHz and then applied to a sample-and-hold circuit to produce a flat-topped PAM signal $s(t)$ with pulse duration $T = 500 \mu\text{s}$.

- (a) Compute the waveform of the PAM signal $s(t)$.
- (b) Compute $|S(f)|$, denoting the magnitude spectrum of the PAM signal $s(t)$.
- (c) Compute the envelope of $|S(f)|$. Hence confirm that the frequency at which this envelope goes through zero for the first time is equal to $(1/T) = 20$ kHz.

3.39 In this problem, we use computer simulation to compare the performance of a companded PCM system using the μ -law against that of the corresponding system using a uniform

quantizer. The simulation is to be performed for a sinusoidal input signal of varying amplitude.

- (a) Using the μ -law described in Table 3.4, plot the output signal-to-noise ratio as a function of the input signal-to-noise ratio, both ratios being expressed in decibels.
- (b) Compare the results of your computation in part (a) with a uniform quantizer having 256 representation levels.

3.40 In this experiment we study the linear adaptive prediction of a signal $x[n]$ governed by the following recursion:

$$x[n] = 0.8x[n-1] - 0.1x[n-2] + 0.1v[n]$$

where $v[n]$ is drawn from a discrete-time white noise process of zero mean and unit variance. (A process generated in this manner is referred to as an *autoregressive process of order two*.) Specifically, the adaptive prediction is performed using the *normalized LMS algorithm* defined by

$$\begin{aligned}\hat{x}[n] &= \sum_{k=1}^p w_k[n]x[n-k] \\ e[n] &= x[n] - \hat{x}[n] \\ w_k[n+1] &= w_k[n] + \left(\frac{\mu}{\sum_{k=1}^p x^2[n-k]} \right) x[n-k]e[n], \quad k = 1, 2, \dots, p\end{aligned}$$

where p is the prediction order and μ is the normalized step-size parameter. The important point to note here is that μ is dimensionless and stability of the algorithm is assured by choosing it in accordance with the formula

$$0 < \mu < 2$$

The algorithm is initiated by setting

$$w_k[0] = 0 \quad \text{for all } k$$

The *learning curve* of the algorithm is defined as a plot of the mean-square error versus the number of iterations n for specified parameter values, which is obtained by averaging the plot of $e^2[n]$ versus n over a large number of different realizations of the algorithm.

- (a) Plot the learning curves for the adaptive prediction of $x[n]$ for a fixed prediction order $p = 5$ and three different values of step-size parameter: $\mu = 0.0075$, 0.05 , and 0.5 .
- (b) What observations can you make from the learning curves of part (a)?

BASEBAND PULSE TRANSMISSION

This chapter discusses the transmission of digital data over a baseband channel, with emphasis on the following topics:

- ▶ *The matched filter, which is the optimum system for detecting a known signal in additive white Gaussian noise.*
- ▶ *Calculation of the bit error rate due to the presence of channel noise.*
- ▶ *Intersymbol interference, which arises when the channel is dispersive as is commonly the case in practice.*
- ▶ *Nyquist's criterion for distortionless baseband data transmission.*
- ▶ *Correlative-level coding or partial-response signaling for combatting the effects of intersymbol interference.*
- ▶ *Digital subscriber lines.*
- ▶ *Equalization of a dispersive baseband channel.*
- ▶ *The eye pattern for displaying the combined effects of intersymbol interference and channel noise in data transmission.*

4.1 Introduction

In Chapter 3 we described techniques for converting an analog information-bearing signal into digital form. There is another way in which digital data can arise in practice: The data may represent the output of a source of information that is inherently discrete in nature (e.g., a digital computer). In this chapter we study the transmission of digital data (of whatever origin) over a *baseband channel*.¹ Data transmission over a band-pass channel using modulation is covered in Chapter 6.

Digital data have a broad spectrum with a significant low-frequency content. Baseband transmission of digital data therefore requires the use of a low-pass channel with a bandwidth large enough to accommodate the essential frequency content of the data stream. Typically, however, the channel is *dispersive* in that its frequency response deviates from that of an ideal low-pass filter. The result of data transmission over such a channel is that each received pulse is affected somewhat by adjacent pulses, thereby giving rise to a common form of interference called *intersymbol interference* (ISI). Intersymbol interference is a major source of bit errors in the reconstructed data stream at the receiver output. To correct for it, control has to be exercised over the pulse shape in the overall system. Thus much of the material covered in this chapter is devoted to *pulse shaping* in one form or another.

Another source of bit errors in a baseband data transmission system is the ubiquitous *channel noise*. Naturally, noise and ISI arise in the system simultaneously. However, to understand how they affect the performance of the system, we first consider them separately; later on in the chapter, we study their combined effects.

We thus begin the chapter by describing a fundamental result in communication theory, which deals with the *detection* of a pulse signal of known waveform that is immersed in additive white noise. The device for the optimum detection of such a pulse involves the use of a linear-time-invariant filter known as a *matched filter*,² which is so called because its impulse response is matched to the pulse signal.

4.2 Matched Filter

A basic problem that often arises in the study of communication systems is that of *detecting* a pulse transmitted over a channel that is corrupted by channel noise (i.e., additive noise at the front end of the receiver). For the purpose of the discussion presented in this section, we assume that the major source of system limitation is the channel noise.

Consider then the receiver model shown in Figure 4.1, involving a linear time-invariant filter of impulse response $h(t)$. The filter input $x(t)$ consists of a pulse signal $g(t)$ corrupted by additive channel noise $w(t)$, as shown by

$$x(t) = g(t) + w(t), \quad 0 \leq t \leq T \quad (4.1)$$

where T is an arbitrary observation interval. The pulse signal $g(t)$ may represent a binary symbol 1 or 0 in a digital communication system. The $w(t)$ is the sample function of a white noise process of zero mean and power spectral density $N_0/2$. It is assumed that the receiver has knowledge of the waveform of the pulse signal $g(t)$. The source of uncertainty lies in the noise $w(t)$. The function of the receiver is to detect the pulse signal $g(t)$ in an optimum manner, given the received signal $x(t)$. To satisfy this requirement, we have to optimize the design of the filter so as to minimize the effects of noise at the filter output in some statistical sense, and thereby enhance the detection of the pulse signal $g(t)$.

Since the filter is linear, the resulting output $y(t)$ may be expressed as

$$y(t) = g_o(t) + n(t) \quad (4.2)$$

where $g_o(t)$ and $n(t)$ are produced by the signal and noise components of the input $x(t)$, respectively. A simple way of describing the requirement that the output signal component $g_o(t)$ be considerably greater than the output noise component $n(t)$ is to have the filter make the instantaneous power in the output signal $g_o(t)$, measured at time $t = T$, as large as possible compared with the average power of the output noise $n(t)$. This is equivalent to maximizing the *peak pulse signal-to-noise ratio*, defined as

$$\eta = \frac{|g_o(T)|^2}{E[n^2(t)]} \quad (4.3)$$

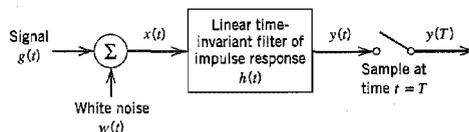


FIGURE 4.1 Linear receiver.

where $|g_o(T)|^2$ is the instantaneous power in the output signal, E is the statistical expectation operator, and $E[\eta^2(t)]$ is a measure of the average output noise power. The requirement is to specify the impulse response $h(t)$ of the filter such that the output signal-to-noise ratio in Equation (4.3) is maximized.

Let $G(f)$ denote the Fourier transform of the known signal $g(t)$, and $H(f)$ denote the frequency response of the filter. Then the Fourier transform of the output signal $g_o(t)$ is equal to $H(f)G(f)$, and $g_o(t)$ is itself given by the inverse Fourier transform

$$g_o(t) = \int_{-\infty}^{\infty} H(f)G(f) \exp(j2\pi ft) df \quad (4.4)$$

Hence, when the filter output is sampled at time $t = T$, we have (in the absence of channel noise)

$$|g_o(T)|^2 = \left| \int_{-\infty}^{\infty} H(f)G(f) \exp(j2\pi fT) df \right|^2 \quad (4.5)$$

Consider next the effect on the filter output due to the noise $w(t)$ acting alone. The power spectral density $S_N(f)$ of the output noise $n(t)$ is equal to the power spectral density of the input noise $w(t)$ times the squared magnitude response $|H(f)|^2$ (see Section 1.7). Since $w(t)$ is white with constant power spectral density $N_0/2$, it follows that

$$S_N(f) = \frac{N_0}{2} |H(f)|^2 \quad (4.6)$$

The average power of the output noise $n(t)$ is therefore

$$\begin{aligned} E[\eta^2(t)] &= \int_{-\infty}^{\infty} S_N(f) df \\ &= \frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df \end{aligned} \quad (4.7)$$

Thus substituting Equations (4.5) and (4.7) into (4.3), we may rewrite the expression for the peak pulse signal-to-noise ratio as

$$\eta = \frac{\left| \int_{-\infty}^{\infty} H(f)G(f) \exp(j2\pi fT) df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H(f)|^2 df} \quad (4.8)$$

Our problem is to find, for a given $G(f)$, the particular form of the frequency response $H(f)$ of the filter that makes η a maximum. To find the solution to this optimization problem, we apply a mathematical result known as Schwarz's inequality to the numerator of Equation (4.8).

A derivation of *Schwarz's inequality* is given in Chapter 5. For now it suffices to say that if we have two complex functions $\phi_1(x)$ and $\phi_2(x)$ in the real variable x , satisfying the conditions

$$\int_{-\infty}^{\infty} |\phi_1(x)|^2 dx < \infty$$

and

$$\int_{-\infty}^{\infty} |\phi_2(x)|^2 dx < \infty$$

then we may write

$$\left| \int_{-\infty}^{\infty} \phi_1(x) \phi_2(x) dx \right|^2 \leq \int_{-\infty}^{\infty} |\phi_1(x)|^2 dx \int_{-\infty}^{\infty} |\phi_2(x)|^2 dx \quad (4.9)$$

The equality in (4.9) holds if, and only if, we have

$$\phi_1(x) = k \phi_2^*(x) \quad (4.10)$$

where k is an arbitrary constant, and the asterisk denotes complex conjugation.

Returning to the problem at hand, we readily see that by invoking Schwarz's inequality (4.9), and setting $\phi_1(x) = H(f)$ and $\phi_2(x) = G(f) \exp(j\pi fT)$, the numerator in Equation (4.8) may be rewritten as

$$\left| \int_{-\infty}^{\infty} H(f) G(f) \exp(j2\pi fT) df \right|^2 \leq \int_{-\infty}^{\infty} |H(f)|^2 df \int_{-\infty}^{\infty} |G(f)|^2 df \quad (4.11)$$

Using this relation in Equation (4.8), we may redefine the peak pulse signal-to-noise ratio as

$$\eta \leq \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df \quad (4.12)$$

The right-hand side of this relation does not depend on the frequency response $H(f)$ of the filter but only on the signal energy and the noise power spectral density. Consequently, the peak pulse signal-to-noise ratio η will be a maximum when $H(f)$ is chosen so that the equality holds; that is,

$$\eta_{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 df \quad (4.13)$$

Correspondingly, $H(f)$ assumes its optimum value denoted by $H_{\text{opt}}(f)$. To find this optimum value we use Equation (4.10), which, for the situation at hand, yields

$$H_{\text{opt}}(f) = k G^*(f) \exp(-j2\pi fT) \quad (4.14)$$

where $G^*(f)$ is the complex conjugate of the Fourier transform of the input signal $g(t)$, and k is a scaling factor of appropriate dimensions. This relation states that, except for the factor $k \exp(-j2\pi fT)$, the frequency response of the optimum filter is the same as the complex conjugate of the Fourier transform of the input signal.

Equation (4.14) specifies the optimum filter in the frequency domain. To characterize it in the time domain, we take the inverse Fourier transform of $H_{\text{opt}}(f)$ in Equation (4.14) to obtain the impulse response of the optimum filter as

$$h_{\text{opt}}(t) = k \int_{-\infty}^{\infty} G^*(f) \exp[-j2\pi f(T-t)] df \quad (4.15)$$

Since for a real signal $g(t)$ we have $G^*(f) = G(-f)$, we may rewrite Equation (4.15) as

$$\begin{aligned} h_{\text{opt}}(t) &= k \int_{-\infty}^{\infty} G(-f) \exp[-j2\pi f(T-t)] df \\ &= k \int_{-\infty}^{\infty} G(f) \exp[j2\pi f(T-t)] df \\ &= k g(T-t) \end{aligned} \quad (4.16)$$

Equation (4.16) shows that the impulse response of the optimum filter, except for the scaling factor k , is a time-reversed and delayed version of the input signal $g(t)$; that is, it

is “matched” to the input signal. A linear time-invariant filter defined in this way is called a matched filter. Note that in deriving the matched filter the only assumption we have made about the input noise $w(t)$ is that it is stationary and white with zero mean and power spectral density $N_0/2$. In other words, no assumption was made on the statistics of the channel noise $w(t)$.

▣ PROPERTIES OF MATCHED FILTERS

We note that a filter, which is matched to a pulse signal $g(t)$ of duration T , is characterized by an impulse response that is a time-reversed and delayed version of the input $g(t)$, as shown by

$$h_{\text{opt}}(t) = kg(T - t)$$

In other words, the impulse response $h_{\text{opt}}(t)$ is uniquely defined, except for the delay T and the scaling factor k , by the waveform of the pulse signal $g(t)$ to which the filter is matched. In the frequency domain, the matched filter is characterized by a frequency response that is, except for a delay factor, the complex conjugate of the Fourier transform of the input $g(t)$, as shown by

$$H_{\text{opt}}(f) = kG^*(f) \exp(-j2\pi fT)$$

The most important result in the calculation of the performance of signal processing systems using matched filters is perhaps the following:

The peak pulse signal-to-noise ratio of a matched filter depends only on the ratio of the signal energy to the power spectral density of the white noise at the filter input.

To demonstrate this property, consider a filter matched to a known signal $g(t)$. The Fourier transform of the resulting matched filter output $g_o(t)$ is

$$\begin{aligned} G_o(f) &= H_{\text{opt}}(f)G(f) \\ &= kG^*(f)G(f) \exp(-j2\pi fT) \\ &= k|G(f)|^2 \exp(-j2\pi fT) \end{aligned} \quad (4.17)$$

Using Equation (4.17) in the formula for the inverse Fourier transform, we find that the matched filter output at time $t = T$ is

$$\begin{aligned} g_o(T) &= \int_{-\infty}^{\infty} G_o(f) \exp(j2\pi fT) df \\ &= k \int_{-\infty}^{\infty} |G(f)|^2 df \end{aligned}$$

According to *Rayleigh's energy theorem*, the integral of the squared magnitude spectrum of a pulse signal with respect to frequency is equal to the signal energy E :

$$E = \int_{-\infty}^{\infty} g^2(t) dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

Hence

$$g_o(T) = kE \quad (4.18)$$

Substituting Equation (4.14) into (4.7), we find that the average output noise power is

$$\begin{aligned} E[n^2(t)] &= \frac{k^2 N_0}{2} \int_{-\infty}^{\infty} |G(f)|^2 df \\ &= k^2 N_0 E/2 \end{aligned} \quad (4.19)$$

where again we have made use of Rayleigh’s energy theorem. Therefore, the peak pulse signal-to-noise ratio has the maximum value

$$\eta_{\max} = \frac{(kE)^2}{(k^2N_0E/2)} = \frac{2E}{N_0} \tag{4.20}$$

From Equation (4.20) we see that dependence on the waveform of the input $g(t)$ has been completely removed by the matched filter. Accordingly, in evaluating the ability of a matched-filter receiver to combat additive white noise, we find that all signals that have the same energy are equally effective. Note that the signal energy E is in joules and the noise spectral density $N_0/2$ is in watts per Hertz, so that the ratio $2E/N_0$ is dimensionless; however, the two quantities have different physical meaning. We refer to E/N_0 as the *signal energy-to-noise spectral density ratio*.

► **EXAMPLE 4.1 Matched Filter for Rectangular Pulse**

Consider a signal $g(t)$ in the form of a rectangular pulse of amplitude A and duration T , as shown in Figure 4.2a. In this example, the impulse response $h(t)$ of the matched filter has exactly the same waveform as the signal itself. The output signal $g_o(t)$ of the matched filter produced in response to the input signal $g(t)$ has a triangular waveform, as shown in Figure 4.2b.

The maximum value of the output signal $g_o(t)$ is equal to kA^2T , which is the energy of the input signal $g(t)$ scaled by the factor k ; this maximum value occurs at $t = T$, as indicated in Figure 4.2b.

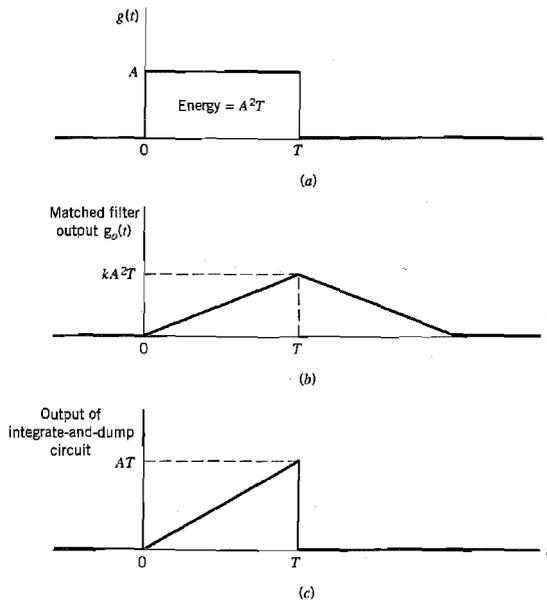


FIGURE 4.2 (a) Rectangular pulse. (b) Matched filter output. (c) Integrator output.



FIGURE 4.3 Integrate-and-dump circuit.

For the special case of a rectangular pulse, the matched filter may be implemented using a circuit known as the *integrate-and-dump circuit*, a block diagram of which is shown in Figure 4.3. The integrator computes the area under the rectangular pulse, and the resulting output is then sampled at time $t = T$, where T is the duration of the pulse. Immediately after $t = T$, the integrator is restored to its initial condition; hence the name of the circuit. Figure 4.2c shows the output waveform of the integrate-and-dump circuit for the rectangular pulse of Figure 4.2a. We see that for $0 \leq t \leq T$, the output of this circuit has the *same waveform* as that appearing at the output of the matched filter; the difference in the notations used to describe their peak values is of no practical significance. ◀

4.3 Error Rate Due to Noise

In Section 3.8 we presented a qualitative discussion of the effect of channel noise on the performance of a binary PCM system. Now that we are equipped with the matched filter as the optimum detector of a known pulse in additive white noise, we are ready to derive a formula for the error rate in such a system due to noise.

To proceed with the analysis, consider a binary PCM system based on *polar non-return-to-zero (NRZ) signaling*. In this form of signaling, symbols 1 and 0 are represented by positive and negative rectangular pulses of equal amplitude and equal duration. The channel noise is modeled as *additive white Gaussian noise* $w(t)$ of zero mean and power spectral density $N_0/2$; the Gaussian assumption is needed for later calculations. In the signaling interval $0 \leq t \leq T_b$, the received signal is thus written as follows:

$$x(t) = \begin{cases} +A + w(t), & \text{symbol 1 was sent} \\ -A + w(t), & \text{symbol 0 was sent} \end{cases} \quad (4.21)$$

where T_b is the *bit duration*, and A is the *transmitted pulse amplitude*. It is assumed that the receiver has acquired knowledge of the starting and ending times of each transmitted pulse; in other words, the receiver has prior knowledge of the pulse shape, but not its polarity. Given the noisy signal $x(t)$, the receiver is required to make a decision in each signaling interval as to whether the transmitted symbol is a 1 or a 0.

The structure of the receiver used to perform this decision-making process is shown in Figure 4.4. It consists of a matched filter followed by a sampler, and then finally a

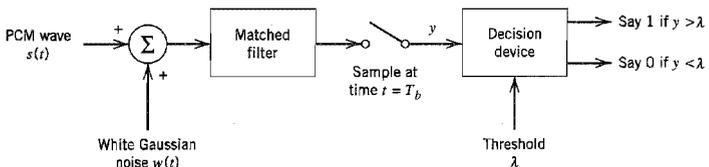


FIGURE 4.4 Receiver for baseband transmission of binary-encoded PCM wave using polar NRZ signaling.

decision device. The filter is matched to a rectangular pulse of amplitude A and duration T_b , exploiting the bit-timing information available to the receiver. The resulting matched filter output is sampled at the end of each signaling interval. The presence of channel noise $w(t)$ adds randomness to the matched filter output.

Let y denote the sample value obtained at the end of a signaling interval. The sample value y is compared to a preset *threshold* λ in the decision device. If the threshold is exceeded, the receiver makes a decision in favor of symbol 1; if not, a decision is made in favor of symbol 0. We adopt the convention that when the sample value y is exactly equal to the threshold λ , the receiver just makes a guess as to which symbol was transmitted; such a decision is the same as that obtained by flipping a fair coin, the outcome of which will not alter the average probability of error.

There are two possible kinds of error to be considered:

1. Symbol 1 is chosen when a 0 was actually transmitted; we refer to this error as an *error of the first kind*.
2. Symbol 0 is chosen when a 1 was actually transmitted; we refer to this error as an *error of the second kind*.

To determine the average probability of error, we consider these two situations separately.

Suppose that symbol 0 was sent. Then, according to Equation (4.21), the received signal is

$$x(t) = -A + w(t), \quad 0 \leq t \leq T_b \quad (4.22)$$

Correspondingly, the matched filter output, sampled at time $t = T_b$, is given by (in light of Example 4.1 with kAT_b set equal to unity for convenience of presentation)

$$\begin{aligned} y &= \int_0^{T_b} x(t) dt \\ &= -A + \frac{1}{T_b} \int_0^{T_b} w(t) dt \end{aligned} \quad (4.23)$$

which represents the sample value of a random variable Y . By virtue of the fact that the noise $w(t)$ is white and Gaussian, we may characterize the random variable Y as follows:

- ▶ The random variable Y is Gaussian distributed with a mean of $-A$.
- ▶ The variance of the random variable Y is

$$\begin{aligned} \sigma_Y^2 &= E[(Y + A)^2] \\ &= \frac{1}{T_b^2} E \left[\int_0^{T_b} \int_0^{T_b} w(t)w(u) dt du \right] \\ &= \frac{1}{T_b^2} \int_0^{T_b} \int_0^{T_b} E[w(t)w(u)] dt du \\ &= \frac{1}{T_b^2} \int_0^{T_b} \int_0^{T_b} R_w(t, u) dt du \end{aligned} \quad (4.24)$$

where $R_w(t, u)$ is the autocorrelation function of the white noise $w(t)$. Since $w(t)$ is white with a power spectral density $N_0/2$, we have

$$R_w(t, u) = \frac{N_0}{2} \delta(t - u) \quad (4.25)$$

where $\delta(t - u)$ is a time-shifted delta function. Hence, substituting Equation (4.25) into (4.24) yields

$$\begin{aligned}\sigma_Y^2 &= \frac{1}{T_b^2} \int_0^{T_b} \int_0^{T_b} \frac{N_0}{2} \delta(t - u) dt du \\ &= \frac{N_0}{2T_b}\end{aligned}\quad (4.26)$$

where we have used the sifting property of the delta function and the fact that its area is unity. The conditional probability density function of the random variable Y , given that symbol 0 was sent, is therefore

$$f_Y(y|0) = \frac{1}{\sqrt{\pi N_0/T_b}} \exp\left(-\frac{(y + A)^2}{N_0/T_b}\right) \quad (4.27)$$

This function is plotted in Figure 4.5(a). Let p_{10} denote the *conditional probability of error, given that symbol 0 was sent*. This probability is defined by the shaded area under the curve of $f_Y(y|0)$ from the threshold λ to infinity, which corresponds to the range of values assumed by y for a decision in favor of symbol 1. In the absence of noise, the matched filter output y sampled at time $t = T_b$ is equal to $-A$. When noise is present, y occasionally assumes a value greater than λ , in which case an error is made. The probability of this error, conditional on sending symbol 0, is defined by

$$\begin{aligned}p_{10} &= P(y > \lambda | \text{symbol 0 was sent}) \\ &= \int_{\lambda}^{\infty} f_Y(y|0) dy \\ &= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{\lambda}^{\infty} \exp\left(-\frac{(y + A)^2}{N_0/T_b}\right) dy\end{aligned}\quad (4.28)$$

At this point in the discussion we digress briefly and introduce the definition of the so-called *complementary error function*:³

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2) dz \quad (4.29)$$

which is closely related to the Gaussian distribution. For large positive values of u , we have the following *upper bound* on the complementary error function:

$$\operatorname{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi u}} \quad (4.30)$$

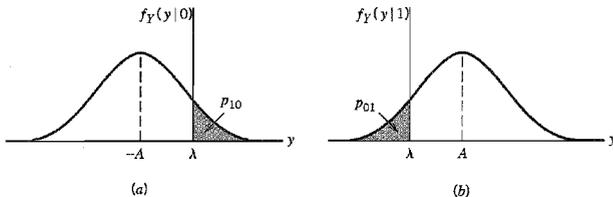


FIGURE 4.5 Noise analysis of PCM system. (a) Probability density function of random variable Y at matched filter output when 0 is transmitted. (b) Probability density function of Y when 1 is transmitted.

To reformulate the conditional probability of error p_{10} in terms of the complementary error function, we first define a new variable

$$z = \frac{y + A}{\sqrt{N_0/T_b}}$$

Accordingly, we may rewrite Equation (4.28) in the compact form

$$\begin{aligned} p_{10} &= \frac{1}{\sqrt{\pi}} \int_{(A+\lambda)/\sqrt{N_0/T_b}}^{\infty} \exp(-z^2) dz \\ &= \frac{1}{2} \operatorname{erfc}\left(\frac{A + \lambda}{\sqrt{N_0/T_b}}\right) \end{aligned} \quad (4.31)$$

Assume next that symbol 1 was transmitted. This time the Gaussian random variable Y represented by the sample value y of the matched filter output has a mean $+A$ and variance $N_0/2T_b$. Note that, compared to the situation when symbol 0 was sent, the mean of the random variable Y has changed, but its variance is exactly the same as before. The conditional probability density function of Y , given that symbol 1 was sent, is therefore

$$f_Y(y|1) = \frac{1}{\sqrt{\pi N_0/T_b}} \exp\left(-\frac{(y - A)^2}{N_0/T_b}\right) \quad (4.32)$$

which is plotted in Figure 4.5*b*. Let p_{01} denote the *conditional probability of error, given that symbol 1 was sent*. This probability is defined by the shaded area under the curve of $f_Y(y|1)$ extending from $-\infty$ to the threshold λ , which corresponds to the range of values assumed by y for a decision in favor of symbol 0. In the absence of noise, the matched filter output y sampled at time $t = T_b$ is equal to $+A$. When noise is present, y occasionally assumes a value less than λ , and an error is then made. The probability of this error, conditional on sending symbol 1, is defined by

$$\begin{aligned} p_{01} &= P(y < \lambda | \text{symbol 1 was sent}) \\ &= \int_{-\infty}^{\lambda} f_Y(y|1) dy \\ &= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{-\infty}^{\lambda} \exp\left(-\frac{(y - A)^2}{N_0/T_b}\right) dy \end{aligned} \quad (4.33)$$

To express p_{01} in terms of the complementary error function, this time we define a new variable

$$z = \frac{A - y}{\sqrt{N_0/T_b}}$$

Accordingly, we may reformulate Equation (4.33) in the compact form

$$\begin{aligned} p_{01} &= \frac{1}{\sqrt{\pi}} \int_{(A-\lambda)/\sqrt{N_0/T_b}}^{\infty} \exp(-z^2) dz \\ &= \frac{1}{2} \operatorname{erfc}\left(\frac{A - \lambda}{\sqrt{N_0/T_b}}\right) \end{aligned} \quad (4.34)$$

Having determined the conditional probabilities of error, p_{10} and p_{01} , our next task is to derive the formula for the *average probability of symbol error*, denoted by P_e . Here we note that these two possible kinds of error are mutually exclusive events in that if the receiver, at a particular sampling instant, chooses symbol 1, then symbol 0 is excluded

from appearing, and vice versa. Let p_0 and p_1 denote the *a priori* probabilities of transmitting symbols 0 and 1, respectively. Hence, the average probability of symbol error P_e in the receiver is given by

$$\begin{aligned} P_e &= p_0 p_{10} + p_1 p_{01} \\ &= \frac{p_0}{2} \operatorname{erfc}\left(\frac{A + \lambda}{\sqrt{N_0/T_b}}\right) + \frac{p_1}{2} \operatorname{erfc}\left(\frac{A - \lambda}{\sqrt{N_0/T_b}}\right) \end{aligned} \quad (4.35)$$

From Equation (4.35) we see that P_e is in fact a function of the threshold λ , which immediately suggests the need for formulating an optimum threshold that minimizes P_e . For this optimization we use *Leibniz's rule*.

Consider the integral

$$\int_{a(u)}^{b(u)} f(z, u) dz$$

Leibniz's rule states that the derivative of this integral with respect to u is

$$\frac{d}{du} \int_{a(u)}^{b(u)} f(z, u) dz = f(b(u), u) \frac{db(u)}{du} - f(a(u), u) \frac{da(u)}{du} + \int_{a(u)}^{b(u)} \frac{\partial f(z, u)}{\partial u} dz$$

For the problem at hand, we note from the definition of the complementary error function in Equation (4.29) that

$$\begin{aligned} f(z, u) &= \frac{2}{\sqrt{\pi}} \exp(-z^2) \\ a(u) &= u \\ b(u) &= \infty \end{aligned}$$

The application of Leibniz's rule to the complementary error function thus yields

$$\frac{d}{du} \operatorname{erfc}(u) = -\frac{1}{\sqrt{\pi}} \exp(-u^2) \quad (4.36)$$

Hence, differentiating Equation (4.35) with respect to λ by making use of the formula in Equation (4.36), then setting the result equal to zero and simplifying terms, we obtain the optimum threshold as

$$\lambda_{\text{opt}} = \frac{N_0}{4AT_b} \log\left(\frac{p_0}{p_1}\right) \quad (4.37)$$

For the special case when symbols 1 and 0 are equiprobable, we have

$$p_1 = p_0 = \frac{1}{2}$$

in which case Equation (4.37) reduces to

$$\lambda_{\text{opt}} = 0$$

This result is intuitively satisfying as it states that, for the transmission of equiprobable binary symbols, we should choose the threshold at the midpoint between the pulse heights $-A$ and $+A$ representing the two symbols 0 and 1. Note that for this special case we also have

$$p_{01} = p_{10}$$

A channel for which the conditional probabilities of error p_{01} and p_{10} are equal is said to be *binary symmetric*. Correspondingly, the average probability of symbol error in Equation (4.35) reduces to

$$P_e = \frac{1}{2} \operatorname{erfc} \left(\frac{A}{\sqrt{N_0/T_b}} \right) \quad (4.38)$$

Now the *transmitted signal energy per bit* is defined by

$$E_b = A^2 T_b \quad (4.39)$$

Accordingly, we may finally formulate the average probability of symbol error for the receiver in Figure 4.4 as

$$P_e = \frac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{E_b}{N_0}} \right) \quad (4.40)$$

which shows that *the average probability of symbol error in a binary symmetric channel depends solely on E_b/N_0 , the ratio of the transmitted signal energy per bit to the noise spectral density.*

Using the upper bound of Equation (4.30) on the complementary error function, we may correspondingly bound the average probability of symbol error for the PCM receiver as

$$P_e < \frac{\exp(-E_b/N_0)}{2\sqrt{\pi E_b/N_0}} \quad (4.41)$$

The PCM receiver of Figure 4.4 therefore exhibits an *exponential* improvement in the average probability of symbol error with increase in E_b/N_0 .

This important result is further illustrated in Figure 4.6 where the average probability of symbol error P_e is plotted versus the dimensionless ratio E_b/N_0 . In particular, we see that P_e decreases very rapidly as the ratio E_b/N_0 is increased, so that eventually a very “small increase” in transmitted signal energy will make the reception of binary pulses almost error free, as discussed previously in Section 3.8. Note, however, that in practical terms the increase in signal energy has to be viewed in the context of the bias; for example,

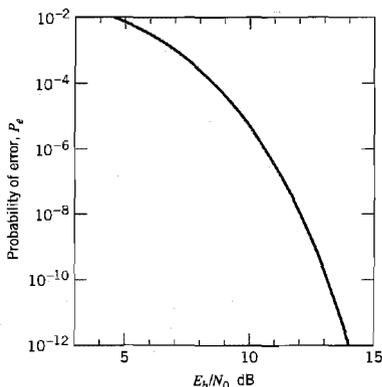


FIGURE 4.6 Probability of error in a PCM receiver.

a 3-dB increase in E_b/N_0 is much easier to implement when E_b has a small value than when its value is orders of magnitude larger.

4.4 Intersymbol Interference

The next source of bit errors in a baseband-pulse transmission system that we wish to study is intersymbol interference (ISI), which arises when the communication channel is *dispersive*. First of all, however, we need to address a key question: Given a pulse shape of interest, how do we use it to transmit data in M -ary form? The answer lies in the use of *discrete pulse modulation*, in which the amplitude, duration, or position of the transmitted pulses is varied in a discrete manner in accordance with the given data stream. However, for the baseband transmission of digital data, the use of *discrete pulse-amplitude modulation* (PAM) is one of the most efficient schemes in terms of power and bandwidth utilization. Accordingly, we confine our attention to discrete PAM systems. We begin the study by first considering the case of binary data; later in the chapter, we consider the more general case of M -ary data.

Consider then a *baseband binary PAM system*, a generic form of which is shown in Figure 4.7. The incoming binary sequence $\{b_k\}$ consists of symbols 1 and 0, each of duration T_b . The *pulse-amplitude modulator* modifies this binary sequence into a new sequence of short pulses (approximating a unit impulse), whose amplitude a_k is represented in the polar form

$$a_k = \begin{cases} +1 & \text{if symbol } b_k \text{ is 1} \\ -1 & \text{if symbol } b_k \text{ is 0} \end{cases} \quad (4.42)$$

The sequence of short pulses so produced is applied to a *transmit filter* of impulse response $g(t)$, producing the transmitted signal

$$s(t) = \sum_k a_k g(t - kT_b) \quad (4.43)$$

The signal $s(t)$ is modified as a result of transmission through the *channel* of impulse response $h(t)$. In addition, the channel adds random noise to the signal at the receiver input. The noisy signal $x(t)$ is then passed through a *receive filter* of impulse response $c(t)$. The resulting filter output $y(t)$ is sampled *synchronously* with the transmitter, with the sampling instants being determined by a *clock* or *timing signal* that is usually extracted from the receive filter output. Finally, the sequence of samples thus obtained is used to reconstruct the original data sequence by means of a *decision device*. Specifically, the amplitude of each sample is compared to a *threshold* λ . If the threshold λ is exceeded, a decision is made in favor of symbol 1. If the threshold λ is not exceeded, a decision is made in favor of symbol 0. If the sample amplitude equals the threshold exactly, the flip of a

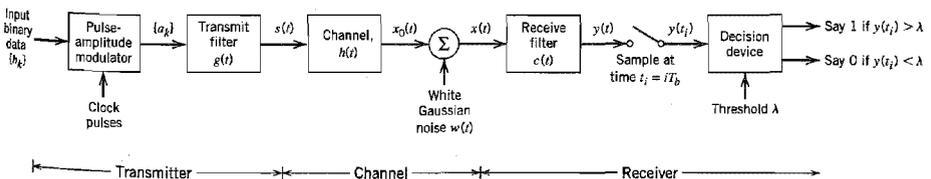


FIGURE 4.7 Baseband binary data transmission system.

fair coin will determine which symbol was transmitted (i.e., the receiver simply makes a random guess).

The receive filter output is written as

$$y(t) = \mu \sum_k a_k p(t - kT_b) + n(t) \quad (4.44)$$

where μ is a scaling factor, and the pulse $p(t)$ is to be defined. To be precise, an arbitrary time delay t_0 should be included in the argument of the pulse $p(t - kT_b)$ in Equation (4.44) to represent the effect of transmission delay through the system. To simplify the exposition, we have put this delay equal to zero in Equation (4.44) without loss of generality.

The scaled pulse $\mu p(t)$ is obtained by a double convolution involving the impulse response $g(t)$ of the transmit filter, the impulse response $h(t)$ of the channel, and the impulse response $c(t)$ of the receive filter, as shown by

$$\mu p(t) = g(t) \star h(t) \star c(t) \quad (4.45)$$

where the star denotes convolution. We assume that the pulse $p(t)$ is *normalized* by setting

$$p(0) = 1 \quad (4.46)$$

which justifies the use of μ as a scaling factor to account for amplitude changes incurred in the course of signal transmission through the system.

Since convolution in the time domain is transformed into multiplication in the frequency domain, we may use the Fourier transform to change Equation (4.45) into the equivalent form

$$\mu P(f) = G(f)H(f)C(f) \quad (4.47)$$

where $P(f)$, $G(f)$, $H(f)$, and $C(f)$ are the Fourier transforms of $p(t)$, $g(t)$, $h(t)$, and $c(t)$, respectively.

Finally, the term $n(t)$ in Equation (4.44) is the noise produced at the output of the receive filter due to the channel noise $w(t)$. It is customary to model $w(t)$ as a white Gaussian noise of zero mean.

The receive filter output $y(t)$ is sampled at time $t_i = iT_b$ (with i taking on integer values), yielding [in light of Equation (4.46)]

$$\begin{aligned} y(t_i) &= \mu \sum_{k=-\infty}^{\infty} a_k p[(i - k)T_b] + n(t_i) \\ &= \mu a_i + \mu \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k p[(i - k)T_b] + n(t_i) \end{aligned} \quad (4.48)$$

In Equation (4.48), the first term μa_i represents the contribution of the i th transmitted bit. The second term represents the residual effect of all other transmitted bits on the decoding of the i th bit; this residual effect due to the occurrence of pulses before and after the sampling instant t_i is called intersymbol interference (ISI). The last term $n(t_i)$ represents the noise sample at time t_i .

In the absence of both ISI and noise, we observe from Equation (4.48) that

$$y(t_i) = \mu a_i$$

which shows that, under these ideal conditions, the i th transmitted bit is decoded correctly. The unavoidable presence of ISI and noise in the system, however, introduces errors in the decision device at the receiver output. Therefore, in the design of the transmit and receive filters, the objective is to minimize the effects of noise and ISI and thereby deliver the digital data to their destination with the smallest error rate possible.

When the signal-to-noise ratio is high, as is the case in a telephone system, for example, the operation of the system is largely limited by ISI rather than noise; in other words, we may ignore $n(t_i)$. In the next couple of sections, we assume that this condition holds so that we may focus our attention on ISI and the techniques for its control. In particular, the issue we wish to consider is to determine the pulse waveform $p(t)$ for which the ISI is completely eliminated.

4.5 Nyquist's Criterion for Distortionless Baseband Binary Transmission

Typically, the frequency response of the channel and the transmitted pulse shape are specified, and the problem is to determine the frequency responses of the transmit and receive filters so as to reconstruct the original binary data sequence $\{b_k\}$. The receiver does this by *extracting* and then *decoding* the corresponding sequence of coefficients, $\{a_k\}$, from the output $y(t)$. The *extraction* involves sampling the output $y(t)$ at time $t = iT_b$. The *decoding* requires that the weighted pulse contribution $a_k p(iT_b - kT_b)$ for $k = i$ be *free* from ISI due to the overlapping tails of all other weighted pulse contributions represented by $k \neq i$. This, in turn, requires that we *control* the overall pulse $p(t)$, as shown by

$$p(iT_b - kT_b) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \quad (4.49)$$

where $p(0) = 1$, by normalization. If $p(t)$ satisfies the conditions of Equation (4.49), the receiver output $y(t_i)$ given in Equation (4.48) simplifies to (ignoring the noise term)

$$y(t_i) = \mu a_i \quad \text{for all } i$$

which implies zero intersymbol interference. Hence, the two conditions of Equation (4.49) ensure *perfect reception in the absence of noise*.

From a design point of view, it is informative to transform the conditions of Equation (4.49) into the frequency domain. Consider then the sequence of samples $\{p(nT_b)\}$, where $n = 0, \pm 1, \pm 2, \dots$. From the discussion presented in Chapter 3 on the sampling process, we recall that sampling in the time domain produces periodicity in the frequency domain. In particular, we may write

$$P_\delta(f) = R_b \sum_{n=-\infty}^{\infty} P(f - nR_b) \quad (4.50)$$

where $R_b = 1/T_b$ is the *bit rate* in bits per second (b/s); $P_\delta(f)$ is the Fourier transform of an infinite periodic sequence of delta functions of period T_b , whose individual areas are weighted by the respective sample values of $p(t)$. That is, $P_\delta(f)$ is given by

$$P_\delta(f) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [p(mT_b) \delta(t - mT_b)] \exp(-j2\pi ft) dt \quad (4.51)$$

Let the integer $m = i - k$. Then, $i = k$ corresponds to $m = 0$, and likewise $i \neq k$ corresponds to $m \neq 0$. Accordingly, imposing the conditions of Equation (4.49) on the sample values of $p(t)$ in the integral of Equation (4.51), we get

$$\begin{aligned} P_\delta(f) &= \int_{-\infty}^{\infty} p(0) \delta(t) \exp(-j2\pi ft) dt \\ &= p(0) \end{aligned} \quad (4.52)$$

where we have made use of the sifting property of the delta function. Since from Equation (4.46) we have $p(0) = 1$, it follows from Equations (4.50) and (4.52) that the condition for zero intersymbol interference is satisfied if

$$\sum_{n=-\infty}^{\infty} P(f - nR_b) = T_b \quad (4.53)$$

We may now state the Nyquist criterion⁴ for distortionless baseband transmission in the absence of noise: *The frequency function $P(f)$ eliminates intersymbol interference for samples taken at intervals T_b provided that it satisfies Equation (4.53)*. Note that $P(f)$ refers to the overall system, incorporating the transmit filter, the channel, and the receive filter in accordance with Equation (4.47).

■ IDEAL NYQUIST CHANNEL

The simplest way of satisfying Equation (4.53) is to specify the frequency function $P(f)$ to be in the form of a *rectangular function*, as shown by

$$\begin{aligned} P(f) &= \begin{cases} \frac{1}{2W}, & -W < f < W \\ 0, & |f| > W \end{cases} \\ &= \frac{1}{2W} \operatorname{rect}\left(\frac{f}{2W}\right) \end{aligned} \quad (4.54)$$

where $\operatorname{rect}(f)$ stands for a *rectangular function* of unit amplitude and unit support centered on $f = 0$, and the overall system bandwidth W is defined by

$$W = \frac{R_b}{2} = \frac{1}{2T_b} \quad (4.55)$$

According to the solution described by Equations (4.54) and (4.55), no frequencies of absolute value exceeding half the bit rate are needed. Hence, from Fourier-transform pair 2 of Table A6.3 we find that a signal waveform that produces zero intersymbol interference is defined by the *sinc function*:

$$\begin{aligned} p(t) &= \frac{\sin(2\pi Wt)}{2\pi Wt} \\ &= \operatorname{sinc}(2Wt) \end{aligned} \quad (4.56)$$

The special value of the bit rate $R_b = 2W$ is called the *Nyquist rate*, and W is itself called the *Nyquist bandwidth*. Correspondingly, the ideal baseband pulse transmission system described by Equation (4.54) in the frequency domain or, equivalently, Equation (4.56) in the time domain, is called the *ideal Nyquist channel*.

Figures 4.8a and 4.8b show plots of $P(f)$ and $p(t)$, respectively. In Figure 4.8a, the normalized form of the frequency function $P(f)$ is plotted for positive and negative frequencies. In Figure 4.8b, we have also included the signaling intervals and the corresponding centered sampling instants. The function $p(t)$ can be regarded as the impulse response of an ideal low-pass filter with passband magnitude response $1/2W$ and bandwidth W . The function $p(t)$ has its peak value at the origin and goes through zero at integer multiples of the bit duration T_b . It is apparent that if the received waveform $y(t)$ is sampled at the

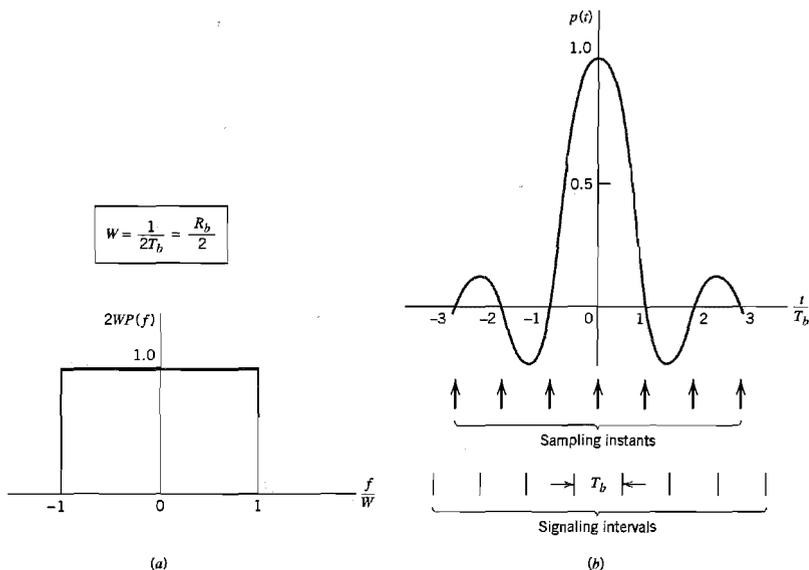


FIGURE 4.8 (a) Ideal magnitude response. (b) Ideal basic pulse shape.

instants of time $t = 0, \pm T_b, \pm 2T_b, \dots$, then the pulses defined by $\mu p(t - iT_b)$ with arbitrary amplitude μ and index $i = 0, \pm 1, \pm 2, \dots$, will not interfere with each other. This condition is illustrated in Figure 4.9 for the binary sequence 1011010.

Although the use of the ideal Nyquist channel does indeed achieve economy in bandwidth in that it solves the problem of zero intersymbol interference with the minimum

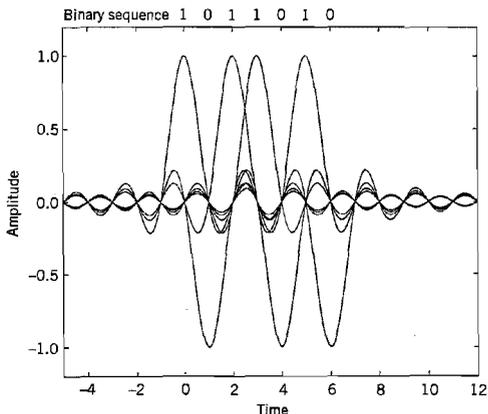


FIGURE 4.9 A series of sinc pulses corresponding to the sequence 1011010.

bandwidth possible, there are two practical difficulties that make it an undesirable objective for system design:

1. It requires that the magnitude characteristic of $P(f)$ be flat from $-W$ to W , and zero elsewhere. This is physically unrealizable because of the abrupt transitions at the band edges $\pm W$.
2. The function $p(t)$ decreases as $1/|t|$ for large $|t|$, resulting in a slow rate of decay. This is also caused by the discontinuity of $P(f)$ at $\pm W$. Accordingly, there is practically no margin of error in sampling times in the receiver.

To evaluate the effect of this *timing error*, consider the sample of $y(t)$ at $t = \Delta t$, where Δt is the timing error. To simplify the exposition, we may put the correct sampling time t_i equal to zero. In the absence of noise, we thus have (from Equation (4.48))

$$\begin{aligned} y(\Delta t) &= \mu \sum_k a_k p(\Delta t - kT_b) \\ &= \mu \sum_k a_k \frac{\sin[2\pi W(\Delta t - kT_b)]}{2\pi W(\Delta t - kT_b)} \end{aligned} \quad (4.57)$$

Since $2WT_b = 1$, by definition, we may rewrite Equation (4.57) as

$$y(\Delta t) = \mu a_0 \operatorname{sinc}(2W \Delta t) + \frac{\mu \sin(2\pi W \Delta t)}{\pi} \sum_{k \neq 0} \frac{(-1)^k a_k}{(2W \Delta t - k)} \quad (4.58)$$

The first term on the right-hand side of Equation (4.58) defines the desired symbol, whereas the remaining series represents the intersymbol interference caused by the timing error Δt in sampling the output $y(t)$. Unfortunately, it is possible for this series to diverge, thereby causing erroneous decisions in the receiver.

■ RAISED COSINE SPECTRUM

We may overcome the practical difficulties encountered with the ideal Nyquist channel by extending the bandwidth from the minimum value $W = R_b/2$ to an adjustable value between W and $2W$. We now specify the overall frequency response $P(f)$ to satisfy a condition more elaborate than that for the ideal Nyquist channel; specifically, we retain three terms of Equation (4.53) and restrict the frequency band of interest to $[-W, W]$, as shown by

$$P(f) + P(f - 2W) + P(f + 2W) = \frac{1}{2W}, \quad -W \leq f \leq W \quad (4.59)$$

We may devise several band-limited functions that satisfy Equation (4.59). A particular form of $P(f)$ that embodies many desirable features is provided by a *raised cosine spectrum*. This frequency response consists of a *flat* portion and a *rolloff* portion that has a sinusoidal form, as follows:

$$P(f) = \begin{cases} \frac{1}{2W}, & 0 \leq |f| < f_1 \\ \frac{1}{4W} \left\{ 1 - \sin \left[\frac{\pi(|f| - W)}{2W - 2f_1} \right] \right\}, & f_1 \leq |f| < 2W - f_1 \\ 0, & |f| \geq 2W - f_1 \end{cases} \quad (4.60)$$

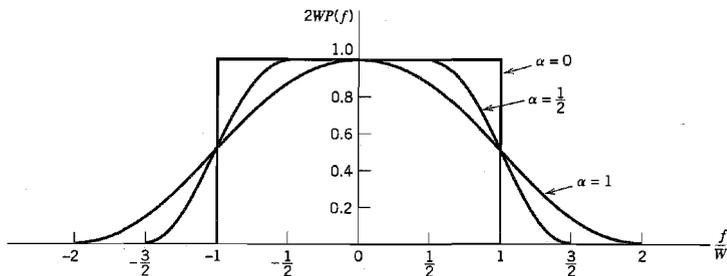
The frequency parameter f_1 and bandwidth W are related by

$$\alpha = 1 - \frac{f_1}{W} \quad (4.61)$$

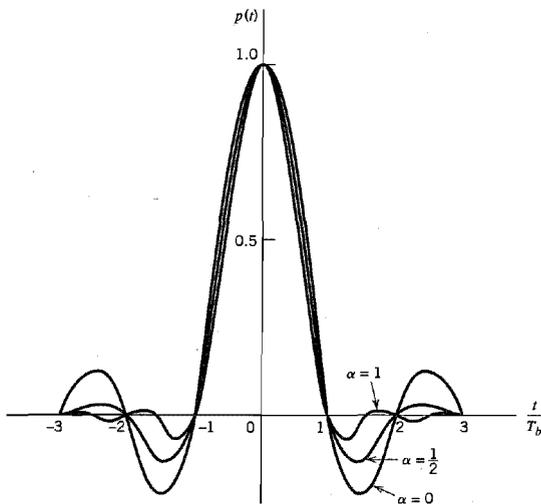
The parameter α is called the *rolloff factor*; it indicates the *excess bandwidth* over the ideal solution, W . Specifically, the transmission bandwidth B_T is defined by

$$\begin{aligned} B_T &= 2W - f_1 \\ &= W(1 + \alpha) \end{aligned}$$

The frequency response $P(f)$, normalized by multiplying it by $2W$, is plotted in Figure 4.10a for three values of α , namely, 0, 0.5, and 1. We see that for $\alpha = 0.5$ or 1, the



(a)



(b)

FIGURE 4.10 Responses for different rolloff factors. (a) Frequency response. (b) Time response.

function $P(f)$ cuts off gradually as compared with the ideal Nyquist channel (i.e., $\alpha = 0$) and is therefore easier to implement in practice. Also the function $P(f)$ exhibits odd symmetry with respect to the Nyquist bandwidth W , making it possible to satisfy the condition of Equation (4.59).

The time response $p(t)$ is the inverse Fourier transform of the frequency response $P(f)$. Hence, using the $P(f)$ defined in Equation (4.60), we obtain the result (see Problem 4.13)

$$p(t) = (\text{sinc}(2Wt)) \left(\frac{\cos(2\pi\alpha Wt)}{1 - 16\alpha^2 W^2 t^2} \right) \quad (4.62)$$

which is plotted in Figure 4.10*b* for $\alpha = 0, 0.5$, and 1 .

The time response $p(t)$ consists of the product of two factors: the factor $\text{sinc}(2Wt)$ characterizing the ideal Nyquist channel and a second factor that decreases as $1/|t|^2$ for large $|t|$. The first factor ensures zero crossings of $p(t)$ at the desired sampling instants of time $t = iT$ with i an integer (positive and negative). The second factor reduces the tails of the pulse considerably below that obtained from the ideal Nyquist channel, so that the transmission of binary waves using such pulses is relatively insensitive to sampling time errors. In fact, for $\alpha = 1$ we have the most gradual rolloff in that the amplitudes of the oscillatory tails of $p(t)$ are smallest. Thus the amount of intersymbol interference resulting from timing error decreases as the rolloff factor α is increased from zero to unity.

The special case with $\alpha = 1$ (i.e., $f_1 = 0$) is known as the *full-cosine rolloff* characteristic, for which the frequency response of Equation (4.60) simplifies to

$$P(f) = \begin{cases} \frac{1}{4W} \left[1 + \cos\left(\frac{\pi f}{2W}\right) \right], & 0 < |f| < 2W \\ 0, & |f| \geq 2W \end{cases} \quad (4.63)$$

Correspondingly, the time response $p(t)$ simplifies to

$$p(t) = \frac{\text{sinc}(4Wt)}{1 - 16W^2 t^2} \quad (4.64)$$

This time response exhibits two interesting properties:

1. At $t = \pm T_b/2 = \pm 1/4W$, we have $p(t) = 0.5$; that is, the pulse width measured at half amplitude is exactly equal to the bit duration T_b .
2. There are zero crossings at $t = \pm 3T_b/2, \pm 5T_b/2, \dots$ in addition to the usual zero crossings at the sampling times $t = \pm T_b, \pm 2T_b, \dots$.

These two properties are extremely useful in extracting a timing signal from the received signal for the purpose of synchronization. However, the price paid for this desirable property is the use of a channel bandwidth double that required for the ideal Nyquist channel corresponding to $\alpha = 0$.

► EXAMPLE 4.2 Bandwidth Requirement of the T1 System

In Example 3.2 of Chapter 3, we described the signal format for the T1 carrier system that is used to multiplex 24 independent voice inputs, based on an 8-bit PCM word. It was shown

that the bit duration of the resulting time-division multiplexed signal (including a framing bit) is

$$T_b = 0.647 \mu\text{s}$$

Assuming the use of an ideal Nyquist channel, it follows that the minimum transmission bandwidth B_T of the T1 system is (for $\alpha = 0$)

$$B_T = W = \frac{1}{2T_b} = 772 \text{ kHz}$$

However, a more realistic value for the necessary transmission bandwidth is obtained by using a full-cosine rolloff characteristic with $\alpha = 1$. In this case, we find that

$$B_T = W(1 + \alpha) = 2W = \frac{1}{T_b} = 1.544 \text{ MHz} \quad \blacktriangleleft$$

4.6 Correlative-Level Coding

Thus far we have treated intersymbol interference as an undesirable phenomenon that produces a degradation in system performance. Indeed, its very name connotes a nuisance effect. Nevertheless, by adding intersymbol interference to the transmitted signal in a controlled manner, it is possible to achieve a signaling rate equal to the Nyquist rate of $2W$ symbols per second in a channel of bandwidth W Hertz. Such schemes are called *correlative-level coding* or *partial-response signaling* schemes.⁵ The design of these schemes is based on the following premise: Since intersymbol interference introduced into the transmitted signal is known, its effect can be interpreted at the receiver in a deterministic way. Thus correlative-level coding may be regarded as a practical method of achieving the theoretical maximum signaling rate of $2W$ symbols per second in a bandwidth of W Hertz, as postulated by Nyquist, using realizable and perturbation-tolerant filters.

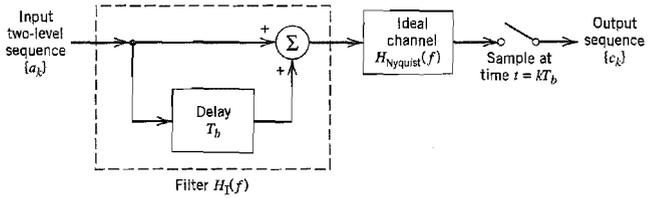
■ DUOBINARY SIGNALING

The basic idea of correlative-level coding will now be illustrated by considering the specific example of *duobinary signaling*, where “duo” implies doubling of the transmission capacity of a straight binary system. This particular form of correlative-level coding is also called *class 1 partial response*.

Consider a binary input sequence $\{b_k\}$ consisting of uncorrelated binary symbols 1 and 0, each having duration T_b . As before, this sequence is applied to a pulse-amplitude modulator producing a two-level sequence of short pulses (approximating a unit impulse), whose amplitude a_k is defined by

$$a_k = \begin{cases} +1 & \text{if symbol } b_k \text{ is 1} \\ -1 & \text{if symbol } b_k \text{ is 0} \end{cases} \quad (4.65)$$

When this sequence is applied to a *duobinary encoder*, it is converted into a *three-level output*, namely, -2 , 0 , and $+2$. To produce this transformation, we may use the scheme shown in Figure 4.11. The two-level sequence $\{a_k\}$ is first passed through a simple filter involving a single delay element and summer. For every unit impulse applied to the


FIGURE 4.11 Duobinary signaling scheme.

input of this filter, we get two unit impulses spaced T_b seconds apart at the filter output. We may therefore express the duobinary coder output c_k as the sum of the present input pulse a_k and its previous value a_{k-1} , as shown by

$$c_k = a_k + a_{k-1} \quad (4.66)$$

One of the effects of the transformation described by Equation (4.66) is to change the input sequence $\{a_k\}$ of uncorrelated two-level pulses into a sequence $\{c_k\}$ of correlated three-level pulses. This correlation between the adjacent pulses may be viewed as introducing intersymbol interference into the transmitted signal in an artificial manner. However, the intersymbol interference so introduced is under the designer's control, which is the basis of correlative coding.

An ideal delay element, producing a delay of T_b seconds, has the frequency response $\exp(-j2\pi f T_b)$, so that the frequency response of the simple delay-line filter in Figure 4.11 is $1 + \exp(-j2\pi f T_b)$. Hence, the overall frequency response of this filter connected in cascade with an ideal Nyquist channel is

$$\begin{aligned} H_I(f) &= H_{\text{Nyquist}}(f)[1 + \exp(-j2\pi f T_b)] \\ &= H_{\text{Nyquist}}(f)[\exp(j\pi f T_b) + \exp(-j\pi f T_b)] \exp(-j\pi f T_b) \quad (4.67) \\ &= 2H_{\text{Nyquist}}(f) \cos(\pi f T_b) \exp(-j\pi f T_b) \end{aligned}$$

where the subscript I in $H_I(f)$ indicates the pertinent class of partial response. For an ideal Nyquist channel of bandwidth $W = 1/2T_b$, we have (ignoring the scaling factor T_b)

$$H_{\text{Nyquist}}(f) = \begin{cases} 1, & |f| \leq 1/2T_b \\ 0, & \text{otherwise} \end{cases} \quad (4.68)$$

Thus the overall frequency response of the duobinary signaling scheme has the form of a half-cycle cosine function, as shown by

$$H_I(f) = \begin{cases} 2 \cos(\pi f T_b) \exp(-j\pi f T_b), & |f| \leq 1/2T_b \\ 0, & \text{otherwise} \end{cases} \quad (4.69)$$

for which the magnitude response and phase response are as shown in Figures 4.12a and 4.12b, respectively. An advantage of this frequency response is that it can be easily approximated, in practice, by virtue of the fact that there is continuity at the band edges.

From the first line in Equation (4.67) and the definition of $H_{\text{Nyquist}}(f)$ in Equation (4.68), we find that the impulse response corresponding to the frequency response $H_I(f)$

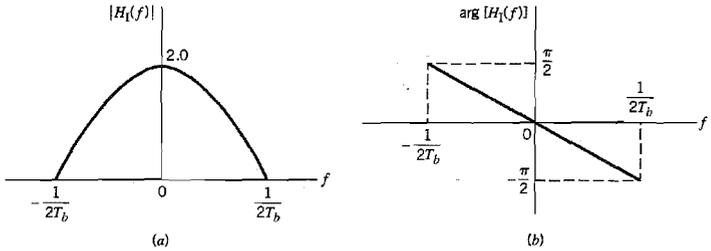


FIGURE 4.12 Frequency response of the duobinary conversion filter. (a) Magnitude response. (b) Phase response.

consists of two sinc (Nyquist) pulses that are time-displaced by T_b seconds with respect to each other, as shown by (except for a scaling factor)

$$\begin{aligned}
 h_1(t) &= \frac{\sin(\pi t/T_b)}{\pi t/T_b} + \frac{\sin[\pi(t - T_b)/T_b]}{\pi(t - T_b)/T_b} \\
 &= \frac{\sin(\pi t/T_b)}{\pi t/T_b} - \frac{\sin(\pi t/T_b)}{\pi(t - T_b)/T_b} \\
 &= \frac{T_b^2 \sin(\pi t/T_b)}{\pi t(T_b - t)}
 \end{aligned} \tag{4.70}$$

The impulse response $h_1(t)$ is plotted in Figure 4.13, where we see that it has only *two* distinguishable values at the sampling instants. The form of $h_1(t)$ shown here explains why we also refer to this type of correlative coding as partial-response signaling. The response to an input pulse is spread over more than one signaling interval; stated in another way, the response in any signaling interval is “partial.” Note also that the tails of $h_1(t)$ decay as $1/|t|^2$, which is a faster rate of decay than the $1/|t|$ encountered in the ideal Nyquist channel.

The original two-level sequence $\{a_k\}$ may be detected from the duobinary-coded sequence $\{c_k\}$ by invoking the use of Equation (4.66). Specifically, let \hat{a}_k represent the *estimate* of the original pulse a_k as conceived by the receiver at time $t = kt_b$. Then, subtracting the previous estimate \hat{a}_{k-1} from c_k , we get

$$\hat{a}_k = c_k - \hat{a}_{k-1} \tag{4.71}$$

It is apparent that if c_k is received without error and if also the previous estimate \hat{a}_{k-1} at time $t = (k - 1)T_b$ corresponds to a correct decision, then the current estimate \hat{a}_k will be

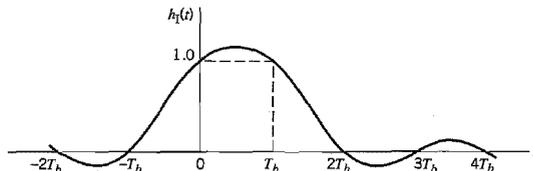


FIGURE 4.13 Impulse response of the duobinary conversion filter.

correct too. The technique of using a stored estimate of the previous symbol is called *decision feedback*.

We observe that the detection procedure just described is essentially an inverse of the operation of the simple delay-line filter at the transmitter. However, a major drawback of this detection procedure is that once errors are made, they tend to *propagate* through the output because a decision on the current input a_k depends on the correctness of the decision made on the previous input a_{k-1} .

A practical means of avoiding the error-propagation phenomenon is to use *precoding* before the duobinary coding, as shown in Figure 4.14. The precoding operation performed on the binary data sequence $\{b_k\}$ converts it into another binary sequence $\{d_k\}$ defined by

$$d_k = b_k \oplus d_{k-1} \quad (4.72)$$

where the symbol \oplus denotes *modulo-two addition* of the binary digits b_k and d_{k-1} . This addition is equivalent to a two-input EXCLUSIVE OR operation, which is performed as follows:

$$d_k = \begin{cases} \text{symbol 1} & \text{if either symbol } b_k \text{ or symbol } d_{k-1} \text{ (but not both) is 1} \\ \text{symbol 0} & \text{otherwise} \end{cases} \quad (4.73)$$

The precoded binary sequence $\{d_k\}$ is applied to a pulse-amplitude modulator, producing a corresponding two-level sequence of short pulses $\{a_k\}$, where $a_k = \pm 1$ as before. This sequence of short pulses is next applied to the duobinary coder, thereby producing the sequence $\{c_k\}$ that is related to $\{a_k\}$ as follows:

$$c_k = a_k + a_{k-1} \quad (4.74)$$

Note that unlike the linear operation of duobinary coding, the precoding described by Equation (4.72) is a *nonlinear* operation.

The combined use of Equations (4.72) and (4.74) yields

$$c_k = \begin{cases} 0 & \text{if data symbol } b_k \text{ is 1} \\ \pm 2 & \text{if data symbol } b_k \text{ is 0} \end{cases} \quad (4.75)$$

which is illustrated in Example 4.3. From Equation (4.75) we deduce the following decision rule for detecting the original binary sequence $\{b_k\}$ from $\{c_k\}$:

$$\begin{aligned} \text{If } |c_k| < 1, & \quad \text{say symbol } b_k \text{ is 1} \\ \text{If } |c_k| > 1, & \quad \text{say symbol } b_k \text{ is 0} \end{aligned} \quad (4.76)$$

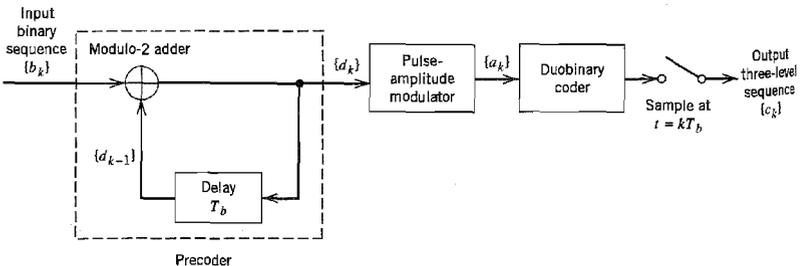


FIGURE 4.14 A precoded duobinary scheme; details of the duobinary coder are given in Figure 4.11.

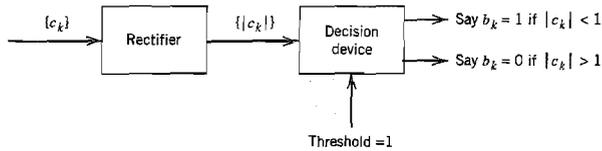


FIGURE 4.15 Detector for recovering original binary sequence from the precoded duobinary coder output.

When $|c_k| = 1$, the receiver simply makes a random guess in favor of symbol 1 or 0. According to this decision rule, the detector consists of a rectifier, the output of which is compared in a decision device to a threshold of 1. A block diagram of the detector is shown in Figure 4.15. A useful feature of this detector is that no knowledge of any input sample other than the present one is required. Hence, error propagation cannot occur in the detector of Figure 4.15.

► EXAMPLE 4.3 Duobinary Coding with Precoding

Consider the binary data sequence 0010110. To proceed with the precoding of this sequence, which involves feeding the precoder output back to the input, we add an extra bit to the precoder output. This extra bit is chosen arbitrarily to be 1. Hence, using Equation (4.73), we find that the sequence $\{d_k\}$ at the precoder output is as shown in row 2 of Table 4.1. The polar representation of the precoded sequence $\{d_k\}$ is shown in row 3 of Table 4.1. Finally, using Equation (4.74), we find that the duobinary coder output has the amplitude levels given in row 4 of Table 4.1.

To detect the original binary sequence, we apply the decision rule of Equation (4.76), and so obtain the binary sequence given in row 5 of Table 4.1. This latter result shows that, in the absence of noise, the original binary sequence is detected correctly. ▲

■ MODIFIED DUOBINARY SIGNALING

In the duobinary signaling technique the frequency response $H(f)$, and consequently the power spectral density of the transmitted pulse, is nonzero at the origin. This is considered to be an undesirable feature in some applications, since many communications channels cannot transmit a DC component. We may correct for this deficiency by using the *class IV partial response* or *modified duobinary* technique, which involves a correlation span of two binary digits. This special form of correlation is achieved by subtracting amplitude-modulated pulses spaced $2T_b$ seconds apart, as indicated in the block diagram of Figure

TABLE 4.1 Illustrating Example 4.3 on duobinary coding

Binary sequence $\{b_k\}$	0	0	1	0	1	1	0
Precoded sequence $\{d_k\}$	1	1	1	0	0	1	0
Two-level sequence $\{a_k\}$	+1	+1	+1	-1	-1	+1	-1
Duobinary coder output $\{c_k\}$	+2	+2	0	-2	0	0	-2
Binary sequence obtained by applying decision rule of Eq. (4.76)	0	0	1	0	1	1	0

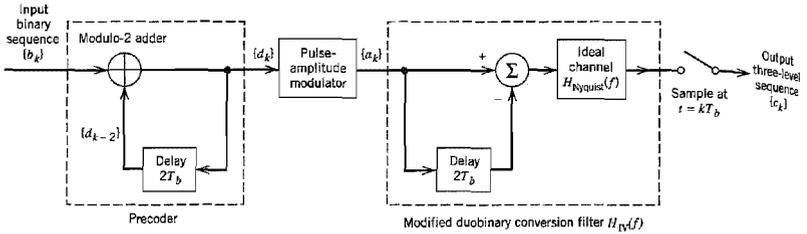


FIGURE 4.16 Modified duobinary signaling scheme.

4.16. The precoder involves a delay of $2T_b$ seconds. The output of the modified duobinary conversion filter is related to the input two-level sequence $\{a_k\}$ at the pulse-amplitude modulator output as follows:

$$c_k = a_k - a_{k-2} \tag{4.77}$$

Here, again, we find that a three-level signal is generated. With $a_k = \pm 1$, we find that c_k takes on one of three values: $+2, 0$, and -2 .

The overall frequency response of the delay-line filter connected in cascade with an ideal Nyquist channel, as in Figure 4.16, is given by

$$\begin{aligned} H_{IV}(f) &= H_{Nyquist}(f)[1 - \exp(-j4\pi fT_b)] \\ &= 2jH_{Nyquist}(f)\sin(2\pi fT_b) \exp(-j2\pi fT_b) \end{aligned} \tag{4.78}$$

where the subscript IV in $H_{IV}(f)$ indicates the pertinent class of partial response and $H_{Nyquist}(f)$ is as defined in Equation (4.68). We therefore have an overall frequency response in the form of a half-cycle sine function, as shown by

$$H_{IV}(f) = \begin{cases} 2j \sin(2\pi fT_b) \exp(-j2\pi fT_b), & |f| \leq 1/2T_b \\ 0, & \text{elsewhere} \end{cases} \tag{4.79}$$

The corresponding magnitude response and phase response of the modified duobinary coder are shown in Figures 4.17a and 4.17b, respectively. A useful feature of the modified duobinary coder is the fact that its output has no DC component. Note also that this

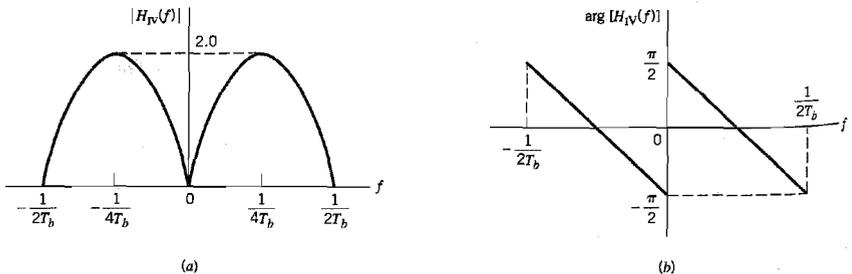


FIGURE 4.17 Frequency response of the modified duobinary conversion filter. (a) Magnitude response. (b) Phase response.

second form of correlative-level coding exhibits the same continuity at the band edges as in duobinary signaling.

From the first line of Equation (4.78) and the definition of $H_{\text{Nyquist}}(f)$ in Equation (4.68), we find that the impulse response of the modified duobinary coder consists of two sinc (Nyquist) pulses that are time-displaced by $2T_b$ seconds with respect to each other, as shown by (except for a scaling factor)

$$\begin{aligned} h_{\text{IV}}(t) &= \frac{\sin(\pi t/T_b)}{\pi t/T_b} - \frac{\sin[\pi(t - 2T_b)/T_b]}{\pi(t - 2T_b)/T_b} \\ &= \frac{\sin(\pi t/T_b)}{\pi t/T_b} - \frac{\sin(\pi t/T_b)}{\pi(t - 2T_b)/T_b} \\ &= \frac{2T_b^2 \sin(\pi t/T_b)}{\pi t(2T_b - t)} \end{aligned} \quad (4.80)$$

This impulse response is plotted in Figure 4.18, which shows that it has *three* distinguishable levels at the sampling instants. Note also that, as with duobinary signaling, the tails of $h_{\text{IV}}(t)$ for the modified duobinary signaling decay as $1/|t|^2$.

To eliminate the possibility of error propagation in the modified duobinary system, we use a precoding procedure similar to that used for the duobinary case. Specifically, prior to the generation of the modified duobinary signal, a modulo-two logical addition is used on signals $2T_b$ seconds apart, as shown by (see the front end of Figure 4.16)

$$d_k = b_k \oplus d_{k-2} = \begin{cases} \text{symbol 1} & \text{if either symbol } b_k \text{ or symbol } d_{k-2} \text{ (but not both) is 1} \\ \text{symbol 0} & \text{otherwise} \end{cases} \quad (4.81)$$

where $\{b_k\}$ is the incoming binary data sequence and $\{d_k\}$ is the sequence at the precoder output. The precoded sequence $\{d_k\}$ thus produced is then applied to a pulse-amplitude modulator and then to the modified duobinary conversion filter.

In Figure 4.16, the output digit c_k equals -2 , 0 , or $+2$, assuming that the pulse-amplitude modulator uses a polar representation for the precoded sequence $\{d_k\}$. Also we find that the detected digit \hat{b}_k at the receiver output may be extracted from c_k by disregarding the polarity of c_k . Specifically, we may formulate the following decision rule:

$$\begin{aligned} \text{If } |c_k| > 1, & \quad \text{say symbol } b_k \text{ is 1} \\ \text{If } |c_k| < 1, & \quad \text{say symbol } b_k \text{ is 0} \end{aligned} \quad (4.82)$$

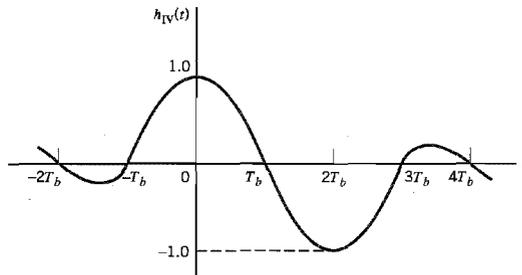


FIGURE 4.18 Impulse response of the modified duobinary conversion filter.

When $|c_k| = 1$, the receiver makes a random guess in favor of symbol 1 or 0. As with the duobinary signaling, we may note the following:

- ▶ In the absence of channel noise, the detected binary sequence $\{\hat{b}_k\}$ is exactly the same as the original binary sequence $\{b_k\}$ at the transmitter input.
- ▶ The use of Equation (4.81) requires the addition of two extra bits to the precoded sequence $\{a_k\}$. The composition of the decoded sequence $\{b_k\}$ using Equation (4.82) is invariant to the selection made for these two bits.

■ **GENERALIZED FORM OF CORRELATIVE-LEVEL CODING (PARTIAL-RESPONSE SIGNALING)**

The duobinary and modified duobinary techniques have correlation spans of 1 binary digit and 2 binary digits, respectively. It is a straightforward matter to generalize these two techniques to other schemes, which are known collectively as *correlative-level coding* or *partial-response signaling* schemes. This generalization is shown in Figure 4.19, where $H_{Nyquist}(f)$ is defined in Equation (4.68). It involves the use of a tapped-delay-line filter with tap-weights w_0, w_1, \dots, w_{N-1} . Specifically, different classes of partial-response sig-

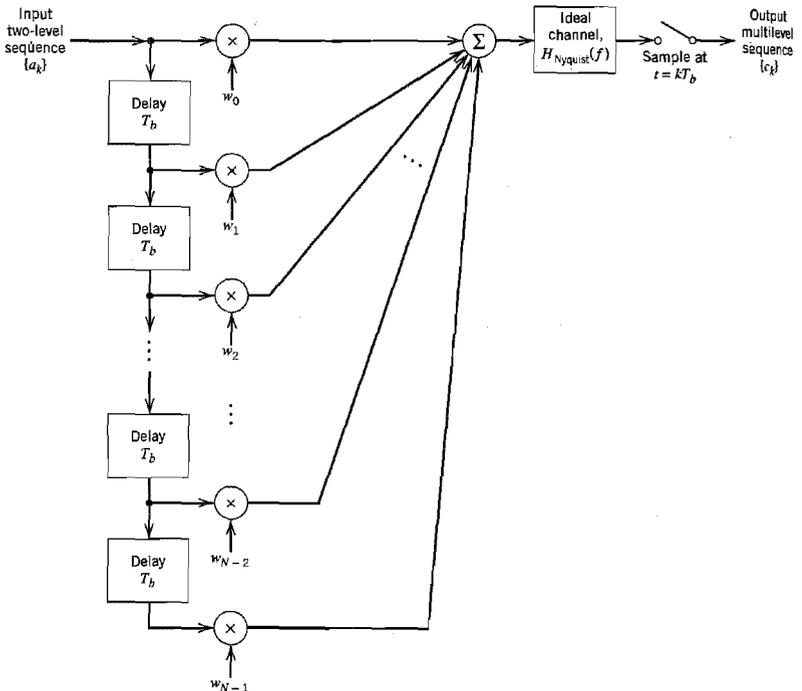


FIGURE 4.19 Generalized correlative coding scheme.

TABLE 4.2 Different classes of partial-response signaling schemes referring to Figure 4.19

Type of Class	N	w_0	w_1	w_2	w_3	w_4	Comments
I	2	1	1				Duobinary coding
II	3	1	2	1			
III	3	2	1	-1			
IV	3	1	0	-1			Modified duobinary coding
V	5	-1	0	2	0	-1	

nalng schemes may be achieved by using a weighted linear combination of N ideal Nyquist (sinc) pulses, as shown by

$$h(t) = \sum_{n=0}^{N-1} w_n \operatorname{sinc}\left(\frac{t}{T_b} - n\right) \quad (4.83)$$

An appropriate choice of the tap-weights in Equation (4.83) results in a variety of spectral shapes designed to suit individual applications. Table 4.2 presents the specific details of five different classes of partial-response signaling schemes. For example, in the duobinary case (class I partial response), we have

$$w_0 = +1$$

$$w_1 = +1$$

and $w_n = 0$ for $n \geq 2$. In the modified duobinary case (class IV partial response), we have

$$w_0 = +1$$

$$w_1 = 0$$

$$w_2 = -1$$

and $w_n = 0$ for $n \geq 3$.

The useful characteristics of partial-response signaling schemes may now be summarized as follows:

- ▶ Binary data transmission over a physical baseband channel can be accomplished at a rate close to the Nyquist rate, using realizable filters with gradual cutoff characteristics.
- ▶ Different spectral shapes can be produced, appropriate for the application at hand.

However, these desirable characteristics are achieved at a price: A larger signal-to-noise ratio is required to yield the same average probability of symbol error in the presence of noise as in the corresponding binary PAM systems because of an increase in the number of signal levels used.

4.7 Baseband M-ary PAM Transmission

In the baseband binary PAM system of Figure 4.7, the pulse-amplitude modulator produces binary pulses, that is, pulses with one of two possible amplitude levels. On the other

hand, in a *baseband M-ary PAM system*, the pulse-amplitude modulator produces one of M possible amplitude levels with $M > 2$. This form of pulse modulation is illustrated in Figure 4.20a for the case of a *quaternary* ($M = 4$) system and the binary data sequence 0010110111. The waveform shown in Figure 4.20a is based on the electrical representation for each of the four possible *dibits* (pairs of bits) given in Figure 4.20b. Note that this representation is *Gray encoded*, which means that any dibit in the quaternary alphabet differs from an adjacent dibit in a single bit position.

In an M -ary system, the information source emits a sequence of symbols from an alphabet that consists of M symbols. Each amplitude level at the pulse-amplitude modulator output corresponds to a distinct symbol, so that there are M distinct amplitude levels to be transmitted. Consider then an M -ary PAM system with a signal alphabet that contains M equally likely and statistically independent symbols, with the symbol duration denoted by T seconds. We refer to $1/T$ as the *signaling rate* of the system, which is expressed in *symbols per second*, or *bauds*. It is informative to relate the signaling rate of this system to that of an equivalent binary PAM system for which the value of M is 2 and the successive binary symbols 1 and 0 are equally likely and statistically independent, with the duration of either symbol denoted by T_b seconds. Under the conditions described here, the binary PAM system produces information at the rate of $1/T_b$ bits per seconds. We also observe that in the case of a quaternary PAM system, for example, the four possible symbols may be identified with the dibits 00, 01, 10, and 11. We thus see that each symbol represents 2 bits of information, and 1 baud is equal to 2 bits per second. We may generalize this result by stating that in an M -ary PAM system, 1 baud is equal to $\log_2 M$ bits per second, and the symbol duration T of the M -ary PAM system is related to the bit duration T_b of the equivalent binary PAM system as

$$T = T_b \log_2 M \quad (4.84)$$

Therefore, in a given channel bandwidth, we find that by using an M -ary PAM system, we are able to transmit information at a rate that is $\log_2 M$ faster than the corresponding binary PAM system. However, to realize the same average probability of symbol error, an M -ary PAM system requires more transmitted power. Specifically, we find that for M much larger than 2 and an average probability of symbol error small compared to 1, the trans-

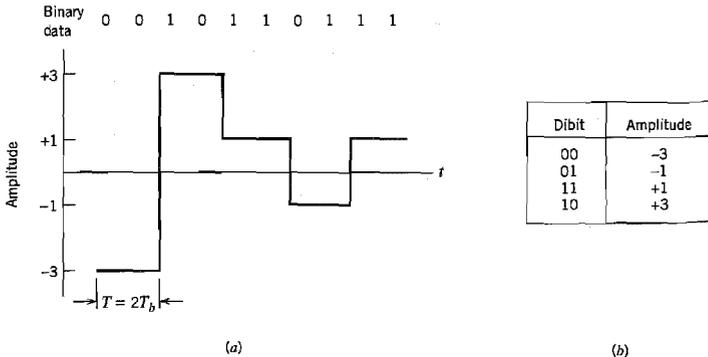


FIGURE 4.20 Output of a quaternary system. (a) Waveform. (b) Representation of the 4 possible dibits, based on Gray encoding.

mitted power must be increased by the factor $M^2/\log_2 M$, compared to a binary PAM system.

In a baseband M -ary system, first of all, the sequence of symbols emitted by the information source is converted into an M -level PAM pulse train by a pulse-amplitude modulator at the transmitter input. Next, as with the binary PAM system, this pulse train is shaped by a transmit filter and then transmitted over the communication channel, which corrupts the signal waveform with both noise and distortion. The received signal is passed through a receive filter and then sampled at an appropriate rate in synchronism with the transmitter. Each sample is compared with preset *threshold* values (also called *slicing* levels), and a decision is made as to which symbol was transmitted. We therefore find that the designs of the pulse-amplitude modulator and the decision-making device in an M -ary PAM are more complex than those in a binary PAM system. Intersymbol interference, noise, and imperfect synchronization cause errors to appear at the receiver output. The transmit and receive filters are designed to minimize these errors. Procedures used for the design of these filters are similar to those discussed in Sections 4.5 and 4.6 for baseband binary PAM systems.

4.8 Digital Subscriber Lines

At this point in our study of baseband data transmission it is rather appropriate that we digress from theoretical aspects of the study and consider a fast-growing application: *digital subscriber lines*.⁶ A digital subscriber line (DSL) operates over a local loop (less than 1.5 km) that provides a direct connection between a user terminal (e.g., computer) and a telephone company's *central office* (CO), as illustrated in Figure 4.21. Through the CO, a DSL user is connected to a broadband backbone data network, which is based on technologies such as the asynchronous transfer mode (ATM) and Internet protocol (IP); these technologies and related network resources (i.e., optical fibers, SONET) are discussed in the Background and Preview chapter. Accordingly, the information-bearing signal is kept in the digital domain all the way from the user terminal to an Internet service provider, with the signal being switched or routed at regular intervals in the course of its transmission through the data network.

In the interest of an inexpensive implementation, digital subscriber lines use twisted pairs configured to provide a high data-rate, full duplex, digital transmission capability. (Twisted pairs are also used for ordinary telephonic communication, as discussed in the Background and Preview chapter.) To achieve full-duplex, two-wire transmission, we may use one of two possible modes of operation:

1. *Time compression (TC) multiplexing*, where data transmission in the two opposite directions on the common line are separated in time. Specifically, blocks of bits of data are sent in bursts in each direction on an alternate basis, as illustrated in

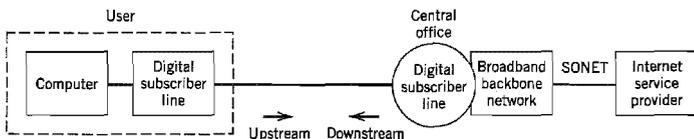


FIGURE 4.21 Block diagram depicting the operational environment of digital subscriber lines.

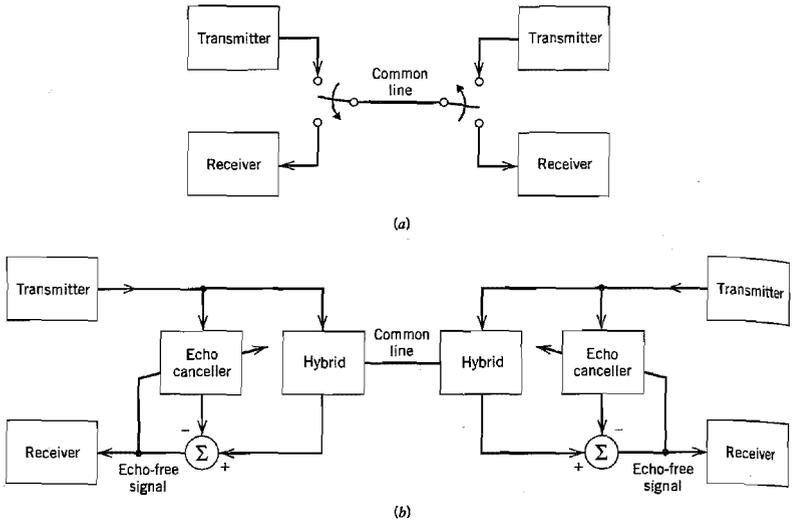


FIGURE 4.22 Full-duplex operation using (a) time compression multiplexing, and (b) echo-cancellation.

Figure 4.22a. To account for propagation time across the line, a guard time is inserted between individual bursts of data. Accordingly, the line rate is slightly greater than twice the data rate.

2. *Echo-cancellation mode*, which supports the simultaneous flow of data along the common line in both directions. For this form of transmission to be feasible, each *transceiver* (transmitter/receiver) includes a hybrid for two purposes: the separation of the transmitted signal from the received signal and the two-to-four-wire conversion, as shown in Figure 4.22b. The *hybrid*, or more precisely, the *hybrid transformer*, is basically a bridge circuit with three ports (terminal pairs), as depicted in Figure 4.23. If the bridge is not perfectly balanced, the transmitter port of the hybrid

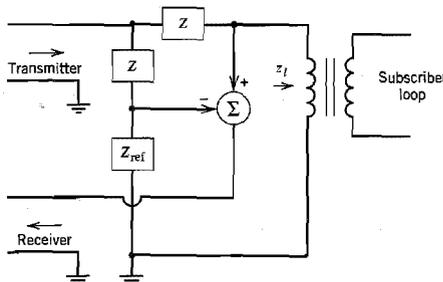


FIGURE 4.23 Simplified circuit of hybrid transformer. For the bridge to be balanced, the reference impedance Z_{ref} should equal the line impedance z_l .

becomes coupled to the receiver port, thereby giving rise to an *echo* due to leakage of the near-end (local) transmitted signal to the near-end (local) receiver. To cancel the unwanted echo, each transceiver includes an *echo canceller*, as shown in Figure 4.22*b*. Since data can flow through the line simultaneously in both directions, the line rate is the same as the data rate.

From this discussion, it is apparent that the echo-cancellation scheme offers a much better data-transmission performance than the time compression multiplexing scheme, but at the expense of increased complexity. However, by implementing the entire transceiver in a single very large-scale integrated (VLSI) chip, the cost is made affordable despite the increased complexity. In North America, the echo-cancellation scheme has been adopted as the basis for designing the transceivers. An adaptive implementation of the echo canceller is discussed in Problem 4.31.

In addition to echo, there are other impairments of the transmission medium that need to be considered. The two dominant impairments are intersymbol interference and crosstalk, which are discussed in what follows in that order.

To a first-order approximation, the squared magnitude response of a twisted pair is given by

$$|H(f)|^2 = \exp(-\alpha\sqrt{f}) \quad (4.85)$$

where

$$\alpha = k \frac{l}{l_0} \quad (4.86)$$

In Equation (4.85), the frequency f is measured in kHz, k is a physical constant of the twisted pair, l_0 is a reference length (e.g., kilometers), and l is the actual length of the twisted pair. Equation (4.85) points to a major impairment in the use of a twisted pair for baseband data transmission: the gradual falloff in the frequency response, which, in turn, gives rise to intersymbol interference.

Turning next to crosstalk, the primary cause for its occurrence is the capacitive coupling that exists between adjacent twisted pairs in a cable. Typically, the nearest five to seven twisted pairs in the cable cause most of the crosstalk. In any event, two kinds of crosstalk can be observed in a receiver of interest:

1. *Near-end crosstalk (NEXT)*, which is generated by transmitters located at the same end of the cable as the receiver, as illustrated in Figure 4.24*a*.

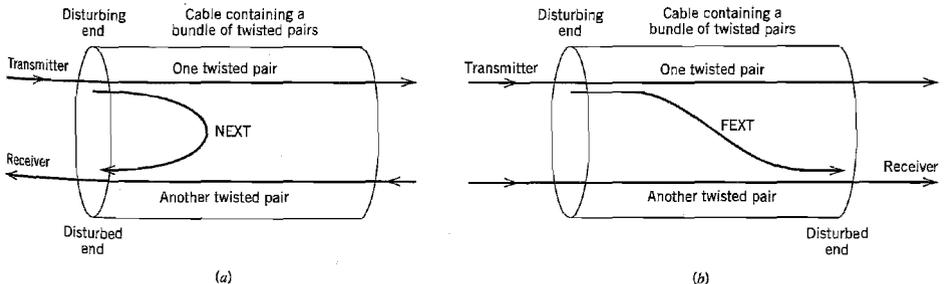


FIGURE 4.24 (a) Near-end crosstalk (NEXT). (b) Far-end crosstalk (FEXT).

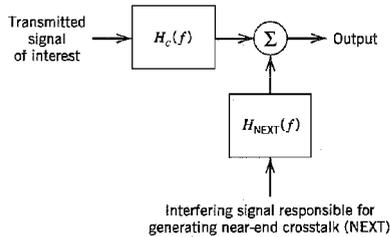


FIGURE 4.25 Model of twisted-pair channel.

2. *Far-end crosstalk (FEXT)*, which is generated by transmitters located further away from the receiver, as illustrated in Figure 4.24*b*.

FEXT naturally suffers the same line loss as the signal, whereas NEXT does not. Accordingly, in the echo-cancellation scheme of Figure 4.22*b* where signals travel in both directions in the cable, NEXT will be much stronger than FEXT. Henceforth, we ignore the effect of FEXT.

Indeed, near-end crosstalk and intersymbol interference are the two most important factors in determining the performance of a digital subscriber loop. Figure 4.25 shows the model of a twisted-pair channel dominated by these two impairments. Since all twisted pairs are usually transmitting similar signals, we may model the NEXT as a signal with the same power spectral density as the transmitted signal passing through a *crosstalk* frequency response $H_{\text{NEXT}}(f)$, which is approximated by

$$H_{\text{NEXT}}(f) = \beta f^{3/2} \quad (4.87)$$

where β is a constant of the cable. The interesting point to note from Figure 4.25 is that both the transmitted signal and the interfering signal have the same power spectral density; they differ from each other merely in their associated frequency responses, as shown in Equations (4.85) and (4.87), respectively. When the model described herein is used for simulation study, the transmitted signal is represented by a random data sequence, while the interference is represented by a Gaussian noise sequence.

■ LINE CODES FOR DIGITAL SUBSCRIBER LINES

Now that we have identified the major transmission impairments, we may describe the desirable features the spectrum of a transmitted signal should exhibit:

1. The power spectral density of the transmitted signal should be zero at zero frequency, since no DC transmission through a hybrid transformer is possible.
2. The power spectral density of the transmitted signal should be low at high frequencies for the following reasons:
 - ▶ Transmission attenuation in a twisted pair is most severe at high frequencies.
 - ▶ Crosstalk between adjacent twisted pairs increases dramatically at high frequencies because of increased capacitive coupling. In this regard, recall that the impedance of a capacitor is inversely proportional to frequency.

To satisfy these desirable properties, we have to be careful in choosing the *line code* that maps the incoming stream of data bits into electrical pulses for transmission on the line. Various possibilities, each with its own advantages and disadvantages, exist

for such a choice. The list of potential candidates for line codes includes the following:

- ▶ *Manchester code*, which is simple and has zero DC component. Its disadvantage is the occupation of a large spectrum, which makes it vulnerable to near-end crosstalk and intersymbol interference. (The Manchester code was discussed in Section 3.7.)
- ▶ *Modified duobinary code*, which has zero DC, is moderately spectrally efficient, and causes minimal intersymbol interference. However, simulation studies of the crosstalk performance of the modified duobinary code have shown that its immunity to near-end crosstalk and intersymbol interference is about 2 to 3 dB poorer than that of block codes on worst-case subscriber lines. (The modified duobinary code was discussed in Section 4.6.)
- ▶ *Bipolar code*, in which successive 1s are represented alternately by positive and negative but equal levels, and symbol 0 is represented by a zero level. Bipolar signaling has zero DC. Computer simulations have shown that its near-end crosstalk and intersymbol interference performance is slightly inferior to the modified duobinary code on all digital subscriber loops. (The bipolar code, also known as the *alternate mark inversion (AMI) codes*, was discussed in Section 3.7.)
- ▶ *2B1Q code*, which stands for two binary digits encoded into one quaternary symbol. This code is a block code representing a four-level PAM signal, as illustrated in Figure 4.20. Assuming that symbols 1 and 0 are equiprobable, the 2B1Q code has zero DC on the average. Moreover, among all the line codes considered herein, it offers the greatest baud reduction, and the best performance with respect to near-end crosstalk and intersymbol interference.

It is because of the desirable properties of the 2B1Q code compared to the Manchester code, modified duobinary code, the bipolar code, and other line codes not mentioned here,⁷ that the 2B1Q code has been adopted as the North American standard for digital subscriber loops.

Using the 2B1Q as the line code and VLSI implementation of a transceiver that incorporates adaptive equalizers and echo cancellers, it is possible to achieve a bit error rate of 10^{-7} operating full duplex at 160 kb/s on the vast majority of twisted-pair subscriber lines. A bit error rate of 10^{-7} with 12 dB noise margin, when 1 percent worst-case NEXT is present, is an accepted performance criterion for digital subscriber lines. *Noise margin* is the amount of receiver noise (including uncancelled echo) that can be tolerated without exceeding the 10^{-7} error rate.

■ ASYMMETRIC DIGITAL SUBSCRIBER LINES

Another important type of DSL is the *asymmetric digital subscriber line (ADSL)*, which is a local transmission system designed to simultaneously support three services on a single twisted-wire pair:

1. Data transmission *downstream* (toward the subscriber) at bit rates of up to 9 Mb/s.
2. Data transmission *upstream* (away from the subscriber) at bit rates of up to 1 Mb/s.
3. Plain old telephone service (POTS).

The downstream and upstream bit rates depend on the length of the twisted pair used to do the transmission. The DSL is said to be “asymmetric” because the downstream bit rate is much higher than the upstream bit rate. Analog voice is transmitted at baseband frequencies and combined with the passband transmissions of downstream and upstream

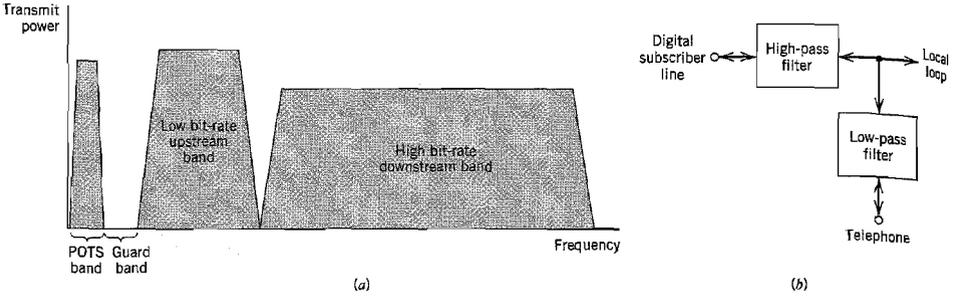


FIGURE 4.26 (a) Illustrating the different band allocations for an FDM-based ADSL system. (b) Block diagram of splitter performing the function of a multiplexer or demultiplexer. Note: both filters in the splitter are bidirectional filters.

data using frequency-division multiplexing (FDM). As illustrated in Figure 4.26a, the upstream data transmission is placed in a frequency band different from the downstream data transmission to avoid crosstalk. Moreover, a guard band is inserted between the POTS band and the upstream transmission band. The coexistence of ADSL and POTS signals on the local loop is made possible through the use of a pair of *splitters*; one splitter is placed at the CO end of the local loop and the other one is placed at the user end. In functional terms, a splitter is divided into two *bidirectional* filters, as shown in Figure 4.26b:

- ▶ A low-pass filter for the baseband transmission or extraction of voice signals.
- ▶ A high-pass filter for the passband transmission or extraction of ADSL data.

In effect, the splitter performs the role of a frequency-division multiplexer or demultiplexer, depending on the direction of signal transmission.

The motivation for making DSL asymmetric is to accommodate “video-on-demand.” In such applications, a subscriber needs a high-throughput channel to download high-bandwidth video data from a central office on demand. In the reverse direction, a much lower throughput channel is adequate to send order information as well as real-time control commands. For example, an ADSL for Internet providing downstream transmission at the DS1 rate of 1.544 Mb/s and upstream transmission of about 160 kb/s would meet the requirements of this application. The approximate 10:1 *asymmetry ratio* realized by such a system prevents the flow of acknowledgment packets in the IP from becoming a bottleneck to the faster direction of data transmission.

It is very difficult to transmit data over a twisted pair at the DS1 rate and higher, and doing so requires the use of sophisticated modulation techniques. The treatment of this subject is deferred to Chapter 6.

4.9 Optimum Linear Receiver

Resuming our study of the baseband data transmission system depicted in Figure 4.7, we have thus far treated the following two channel conditions separately:

- ▶ *Channel noise acting alone*, which led to formulation of the matched filter receiver.
- ▶ *Intersymbol interference acting alone*, which led to formulation of the pulse-shaping transmit filter so as to realize the Nyquist channel.

In a real-life situation, however, channel noise and intersymbol interference act together, affecting the behavior of a data transmission system in a combined manner. In this section, we formulate the basis for designing a linear receiver optimized for the general case of a linear channel that is both dispersive and noisy.

In one approach to the design of a linear receiver, the receiver is viewed as a *zero-forcing equalizer* followed by a decision-making device. The objective of this form of equalization is to have the “intersymbol interference forced to zero” at all the instants $t = kT$ at which the channel output is sampled, except for $k = 0$ where the symbol of interest is assumed to occur. Under this condition, symbol-to-symbol detection is assured to be optimal in accordance with the Nyquist criterion, provided that the channel noise $w(t)$ is zero.

The zero-forcing equalizer is relatively easy to design because it ignores the effect of the channel noise $w(t)$. A serious consequence of this oversight, however, is that it leads to overall performance degradation due to *noise enhancement*, a phenomenon that is an inherent feature of zero-forcing equalization; see Problem 4.32. A more refined approach for the receiver design is to use the *mean-square error criterion*, which provides a balanced solution to the problem of reducing the effects of both channel noise and intersymbol interference. Indeed, for a prescribed computational complexity, an equalizer designed on this latter basis always performs as well as, and often better than, its zero-forcing counterpart. Henceforth, we concentrate on the mean-square error criterion for receiver design.

Referring back to the baseband binary data transmission system of Figure 4.7, the receive filter characterized by the impulse response $c(t)$ produces the following response due to the channel output $x(t)$:

$$y(t) = \int_{-\infty}^{\infty} c(\tau)x(t - \tau) d\tau \quad (4.88)$$

The channel output $x(t)$ is itself defined by

$$x(t) = \sum_k a_k q(t - kT_b) + w(t) \quad (4.89)$$

where a_k is the symbol transmitted at time $t = kT_b$ and $w(t)$ is the channel noise. The time function $q(t)$ is the convolution of two impulse responses: $g(t)$ pertaining to the pulse-shaping transmit filter, and $h(t)$ pertaining to the channel. Substituting Equation (4.89) into (4.88) and sampling the resulting output $y(t)$ at time $t = iT_b$, we may write

$$y(iT_b) = \xi_i + n_i \quad (4.90)$$

where ξ_i is the signal component defined by

$$\xi_i = \sum_k a_k \int_{-\infty}^{\infty} c(\tau)q(iT_b - kT_b - \tau) d\tau \quad (4.91)$$

and n_i is the noise component defined by

$$n_i = \int_{-\infty}^{\infty} c(\tau)w(iT_b - \tau) d\tau \quad (4.92)$$

The condition for perfect operation of the receiver is to have $y(iT_b) = a_i$, where a_i is the transmitted symbol. Deviation from this condition results in the error signal

$$\begin{aligned} e_i &= y(iT_b) - a_i \\ &= \xi_i + n_i - a_i \end{aligned} \quad (4.93)$$

Accordingly, we may formally define the mean-square error as

$$J = \frac{1}{2} E[e_i^2] \quad (4.94)$$

where E is the statistical expectation operator, and the factor $1/2$ is introduced for convenience of presentation. Substituting Equation (4.93) into (4.94) and then expanding terms, we get

$$J = \frac{1}{2} E[\xi_i^2] + \frac{1}{2} E[n_i^2] + \frac{1}{2} E[a_i^2] + E[\xi_i n_i] - E[n_i a_i] - E[\xi_i a_i] \quad (4.95)$$

We now evaluate the six expectation terms in this equation in the order they appear here:

1. In a *stationary* environment the mean-square term $E[\xi_i^2]$ is independent of the instant of time $t = iT_b$, at which the receive filter output is sampled. Hence, we may simplify the expression of this term by writing

$$E[\xi_i^2] = \sum_l \sum_k E[a_l a_k] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(\tau_1) c(\tau_2) q(lT_b - \tau_1) q(kT_b - \tau_2) d\tau_1 d\tau_2$$

Assuming that, first, the binary symbols $a_k = \pm 1$ as in Equation (4.42) and, second, the transmitted symbols are statistically independent, that is,

$$E[a_l a_k] = \begin{cases} 1 & \text{for } l = k \\ 0 & \text{otherwise} \end{cases} \quad (4.96)$$

we may further reduce the expression for the mean-square term $E[\xi_i^2]$ to

$$E[\xi_i^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_q(\tau_1, \tau_2) c(\tau_1) c(\tau_2) d\tau_1 d\tau_2 \quad (4.97)$$

where

$$R_q(\tau_1, \tau_2) = \sum_k q(kT_b - \tau_1) q(kT_b - \tau_2) \quad (4.98)$$

The factor $R_q(\tau_1, \tau_2)$ is the *temporal autocorrelation function* of the sequence $\{q(kT_b)\}$. Stationarity of this sequence means that (see Section 1.5)

$$R_q(\tau_1, \tau_2) = R_q(\tau_2 - \tau_1) = R_q(\tau_1 - \tau_2)$$

2. The mean-square term $E[n_i^2]$ due to channel noise is given by (using Equation (4.92))

$$\begin{aligned} E[n_i^2] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(\tau_1) c(\tau_2) E[w(iT_b - \tau_1) w(iT_b - \tau_2)] d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(\tau_1) c(\tau_2) R_w(\tau_2 - \tau_1) d\tau_1 d\tau_2 \end{aligned} \quad (4.99)$$

where $R_w(\tau_2 - \tau_1)$ is the *ensemble-averaged autocorrelation function* of the channel noise $w(t)$. With $w(t)$ assumed to be white with power spectral density $N_0/2$, we have

$$R_w(\tau_2 - \tau_1) = \frac{N_0}{2} \delta(\tau_2 - \tau_1) \quad (4.100)$$

Hence, the expression for $E[n_i^2]$ simplifies to

$$E[n_i^2] = \frac{N_0}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c(\tau_1) c(\tau_2) \delta(\tau_2 - \tau_1) d\tau_1 d\tau_2 \quad (4.101)$$

3. The mean-square term $E[a_i^2]$ due to the transmitted symbol a_i is unity by virtue of Equation (4.96); that is,

$$E[a_i^2] = 1 \quad \text{for all } i \quad (4.102)$$

4. The expectation of the cross-product term $\xi_i n_i$ is zero for two reasons: first ξ_i and n_i are independent and, second, the channel noise $w(t)$, and therefore n_i , has zero mean; that is,

$$E[\xi_i n_i] = 0 \quad \text{for all } i \quad (4.103)$$

5. For similar reasons, the expectation of the cross-product term $n_i a_i$ is also zero; that is,

$$E[n_i a_i] = 0 \quad \text{for all } i \quad (4.104)$$

6. Finally, the expectation of the cross-product term $\xi_i a_i$ is given by (using Equation (4.91))

$$E[\xi_i a_i] = \sum_k E[a_k a_i] \int_{-\infty}^{\infty} c(\tau) q(iT_b - kT_b - \tau) d\tau \quad (4.105)$$

By virtue of the statistical independence of the transmitted symbols described in Equation (4.96), this expectation reduces to

$$E[\xi_i a_i] = \int_{-\infty}^{\infty} c(\tau) q(-\tau) d\tau \quad (4.106)$$

Thus substituting Equations (4.97), (4.101) to (4.104) and (4.106) into (4.95), we may express the mean-square error J for the binary data transmission system of Figure 4.7 as

$$J = \frac{1}{2} + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(R_q(t - \tau) + \frac{N_0}{2} \delta(t - \tau) \right) c(t) c(\tau) dt d\tau - \int_{-\infty}^{\infty} c(t) q(-t) dt \quad (4.107)$$

For convenience of presentation, we have made the following changes in variables: τ_1 and τ_2 in the first integral are replaced by t and τ , respectively, and τ is replaced with t in the second integral. Note also that this expression for the mean-square error J is in actual fact normalized with respect to the variance of the transmitted symbols a_k by virtue of the assumption made in Equation (4.96).

With the formula of Equation (4.107) for the mean-square error J at hand, we are now ready to specify the design of the receive filter in Figure 4.7. Differentiating Equation (4.107) with respect to the impulse response $c(t)$ of the receive filter, and then setting the result equal to zero, we get

$$\int_{-\infty}^{\infty} \left(R_q(t - \tau) + \frac{N_0}{2} \delta(t - \tau) \right) c(\tau) d\tau = q(-t) \quad (4.108)$$

Equation (4.108) is the formula for finding the impulse response $c(t)$ of the equalizer optimized in the mean-square error sense. An equalizer so designed is referred to as the *minimum-mean square error (mmse) equalizer*.

Taking the Fourier transform of both sides of Equation (4.108), we obtain

$$\left(S_q(f) + \frac{N_0}{2} \right) C(f) = Q^*(f) \quad (4.109)$$

where $c(t) \rightleftharpoons C(f)$, $q(t) \rightleftharpoons Q(f)$, and $R_q \rightleftharpoons S_q(f)$. Solving Equation (4.109) for $C(f)$, we get

$$C(f) = \frac{Q^*(f)}{S_q(f) + \frac{N_0}{2}} \tag{4.110}$$

In Problem 4.33 it is shown that the power spectral density of the sequence $\{q(kT_b)\}$ can be expressed as

$$S_q(f) = \frac{1}{T_b} \sum_k \left| Q\left(f + \frac{k}{T_b}\right) \right|^2 \tag{4.111}$$

which means that the frequency response $C(f)$ of the optimum linear receiver is *periodic* with period $1/T_b$. Equation (4.110) suggests the interpretation of the optimum linear receiver as the cascade connection of two basic components:³

- ▶ A *matched filter* whose impulse response is $q(-t)$, where $q(t) = g(t) \star b(t)$.
- ▶ A *transversal (tapped-delay-line) equalizer* whose frequency response is the inverse of the periodic function $S_q(f) + (N_0/2)$.

To implement Equation (4.110) exactly we need an equalizer of infinite length. In practice, we may approximate the optimum solution by using an equalizer with a finite set of coefficients $\{c_k\}_{k=-N}^N$, provided N is large enough. Thus the receiver takes the form shown in Figure 4.27. Note that the block labeled z^{-1} in Figure 4.27 introduces a delay equal to T_b , which means that the tap spacing of the equalizer is exactly the same as the bit duration T_b . An equalizer so configured is said to be *synchronous* with the transmitter.

■ PRACTICAL CONSIDERATIONS

The mmse receiver of Figure 4.27 works well in the laboratory, where we have access to the system to be equalized, in which case we may determine a transversal equalizer characterized by the set of coefficients $\{c_k\}_{k=-N}^N$, which provides an adequate approximation to the frequency response $C(f)$ of Equation (4.110). In a real-life telecommunications environment, however, the channel is usually time varying. For example, in a public

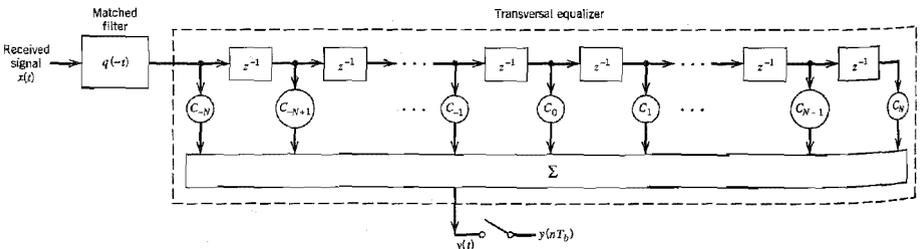


FIGURE 4.27 Optimum linear receiver consisting of the cascade connection of matched filter and transversal equalizer.

switched telephone network, we find that two factors contribute to the distribution of pulse distortion on different link connections:

- ▶ Differences in the transmission characteristics of the individual links that may be switched together.
- ▶ Differences in the number of links in a connection.

The result is that the telephone channel is random in the sense of being one of an ensemble of possible physical realizations. Consequently, the use of a fixed pair of matched filter and equalizer designed on the basis of average channel characteristics may not adequately reduce the effects of intersymbol interference and channel noise. To realize the full transmission capability of the telephone channel, we need an *adaptive receiver*⁹ that provides for the *adaptive implementation of both the matched filter and the equalizer in a combined manner*. The receiver is adaptive in the sense that the equalizer coefficients are adjusted automatically in accordance with a built-in algorithm.

Another point of interest is that it may be desirable to have the taps of the equalizer spaced by an amount closer than the symbol period; typically, the spacing between adjacent taps is set equal to $T/2$. The resulting structure is known as a *fractionally spaced equalizer (FSE)*. An FSE has the capability of compensating for delay distortion much more effectively than a conventional synchronous equalizer. Another advantage of the FSE is the fact that data transmission may begin with an arbitrary sampling phase. However, mathematical analysis of the FSE is more complicated than for a synchronous equalizer and will therefore not be pursued here.¹⁰

4.10 Adaptive Equalization

In this section we develop a simple and yet effective algorithm for the adaptive equalization of a linear channel of unknown characteristics. Figure 4.28 shows the structure of an adaptive synchronous equalizer, which incorporates the matched filtering action. The algorithm used to adjust the equalizer coefficients assumes the availability of a desired response. One's first reaction to the availability of a replica of the transmitted signal is: If such a signal is available at the receiver, why do we need adaptive equalization? To answer this question, we first note that a typical telephone channel changes little during an average data call. Accordingly, prior to data transmission, the equalizer is adjusted under the guid-

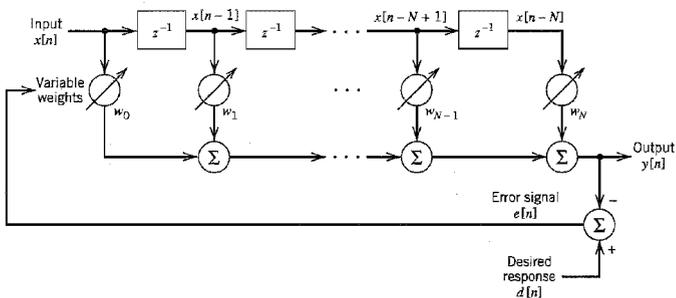


FIGURE 4.28 Block diagram of adaptive equalizer.

ance of a *training sequence* transmitted through the channel. A synchronized version of this training sequence is generated at the receiver, where (after a time shift equal to the transmission delay through the channel) it is applied to the equalizer as the desired response. A training sequence commonly used in practice is the *pseudonoise (PN) sequence*, which consists of a deterministic periodic sequence with noise-like characteristics. Two identical PN sequence generators are used, one at the transmitter and the other at the receiver. When the training process is completed, the PN sequence generator is switched off, and the adaptive equalizer is ready for normal data transmission. Detailed description of PN sequence generators is presented in Chapter 7.

■ LEAST-MEAN-SQUARE ALGORITHM (REVISITED)

To simplify notational matters, we let

$$\begin{aligned}x[n] &= x(nT) \\y[n] &= y(nT)\end{aligned}$$

Then, the output $y[n]$ of the tapped-delay-line equalizer in response to the input sequence $\{x[n]\}$ is defined by the discrete convolution sum (see Figure 4.28)

$$y[n] = \sum_{k=0}^N w_k x[n - k] \quad (4.112)$$

where w_k is the weight at the k th tap, and $N + 1$ is the total number of taps. The tap-weights constitute the adaptive filter coefficients. We assume that the input sequence $\{x[n]\}$ has finite energy. We have used a notation for the equalizer weights in Figure 4.28 that is different from the corresponding notation in Figure 4.27 to emphasize the fact that the equalizer in Figure 4.28 also incorporates matched filtering.

The adaptation may be achieved by observing the error between the desired pulse shape and the actual pulse shape at the filter output, measured at the sampling instants, and then using this error to estimate the direction in which the tap-weights of the filter should be changed so as to approach an optimum set of values. For the adaptation, we may use a criterion based on minimizing the *peak distortion*, defined as the worst-case intersymbol interference at the output of the equalizer. The development of an adaptive equalizer using such a criterion builds on the zero-forcing concept described briefly in Section 4.9. However, the equalizer is optimum only when the peak distortion at its input is less than 100 percent (i.e., the intersymbol interference is not too severe). A better approach is to use a mean-square error criterion, which is more general in application; also an adaptive equalizer based on the mean-square error criterion appears to be less sensitive to timing perturbations than one based on the peak distortion criterion. Accordingly, in what follows we use the mean-square error criterion to derive the adaptive equalization algorithm.

Let $a[n]$ denote the *desired response* defined as the polar representation of the n th transmitted binary symbol. Let $e[n]$ denote the *error signal* defined as the difference between the desired response $a[n]$ and the actual response $y[n]$ of the equalizer, as shown by

$$e[n] = a[n] - y[n] \quad (4.113)$$

In the *least-mean-square (LMS) algorithm*¹¹ for adaptive equalization, the error signal $e[n]$ actuates the adjustments applied to the individual tap weights of the equalizer as the algorithm proceeds from one iteration to the next. A derivation of the LMS algorithm for

adaptive prediction was presented in Section 3.13. Recasting Equation (3.72) into its most general form, we may state the formula for the LMS algorithm in words as follows:

$$\left(\begin{array}{c} \text{Updated value} \\ \text{of } k\text{th tap-} \\ \text{weight} \end{array} \right) = \left(\begin{array}{c} \text{Old value} \\ \text{of } k\text{th tap-} \\ \text{weight} \end{array} \right) + \left(\begin{array}{c} \text{Step-size} \\ \text{parameter} \end{array} \right) \cdot \left(\begin{array}{c} \text{Input signal} \\ \text{applied to} \\ k\text{th tap-} \\ \text{weight} \end{array} \right) \left(\begin{array}{c} \text{Error} \\ \text{signal} \end{array} \right) \quad (4.114)$$

Let μ denote the step-size parameter. From Figure 4.28 we see that the input signal applied to the k th tap-weight at time step n is $x[n - k]$. Hence, using $\hat{w}_k(n)$ as the old value of the k th tap-weight at time step n , the updated value of this tap-weight at time step $n + 1$ is, in light of Equation (4.114), defined by

$$\hat{w}_k[n + 1] = \hat{w}_k[n] + \mu x[n - k]e[n], \quad k = 0, 1, \dots, N \quad (4.115)$$

where

$$e[n] = a[n] - \sum_{k=0}^N \hat{w}_k[n]x[n - k] \quad (4.116)$$

These two equations constitute the *LMS algorithm for adaptive equalization*. Note that the length of the adaptive equalizer in Figure 4.28 is not to be confused with the length of the equalizer in Figure 4.27.

We may simplify the formulation of the LMS algorithm using matrix notation. Let the $(N + 1)$ -by-1 vector $\mathbf{x}[n]$ denote the tap-inputs of the equalizer:

$$\mathbf{x}[n] = [x[n], \dots, x[n - N + 1], x[n - N]]^T \quad (4.117)$$

where the superscript T denotes matrix transposition. Correspondingly, let the $(N + 1)$ -by-1 vector $\hat{\mathbf{w}}[n]$ denote the tap-weights of the equalizer:

$$\hat{\mathbf{w}}[n] = [\hat{w}_0[n], \hat{w}_1[n], \dots, \hat{w}_N[n]]^T \quad (4.118)$$

We may then use matrix notation to recast the convolution sum of Equation (4.112) in the compact form

$$y[n] = \mathbf{x}^T[n]\hat{\mathbf{w}}[n] \quad (4.119)$$

where $\mathbf{x}^T[n]\hat{\mathbf{w}}[n]$ is referred to as the *inner product* of the vectors $\mathbf{x}[n]$ and $\hat{\mathbf{w}}[n]$. We may now summarize the LMS algorithm for adaptive equalization as follows:

1. Initialize the algorithm by setting $\hat{\mathbf{w}}[1] = 0$ (i.e., set all the tap-weights of the equalizer to zero at $n = 1$, which corresponds to time $t = T$).
2. For $n = 1, 2, \dots$, compute

$$\begin{aligned} y[n] &= \mathbf{x}^T[n]\hat{\mathbf{w}}[n] \\ e[n] &= a[n] - y[n] \\ \hat{\mathbf{w}}[n + 1] &= \hat{\mathbf{w}}[n] + \mu e[n]\mathbf{x}[n] \end{aligned}$$

where μ is the step-size parameter.

3. Continue the iterative computation until the equalizer reaches a “steady state,” by which we mean that the actual mean-square error of the equalizer essentially reaches a constant value.

The LMS algorithm is an example of a feedback system, as illustrated in the block diagram of Figure 4.29, which pertains to the k th filter coefficient. It is therefore possible

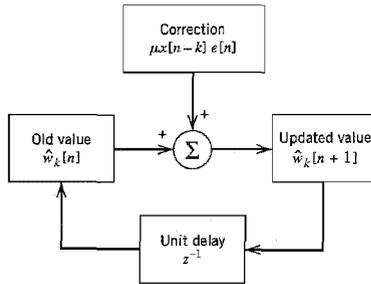


FIGURE 4.29 Signal-flow graph representation of the LMS algorithm involving the k th tap weight.

for the algorithm to diverge (i.e., for the adaptive equalizer to become unstable). Unfortunately, the convergence behavior of the LMS algorithm is difficult to analyze. Nevertheless, provided that the step-size parameter μ is assigned a small value, we find that after a large number of iterations the behavior of the LMS algorithm is roughly similar to that of the *steepest-descent algorithm*, which uses the actual gradient rather than a noisy estimate for the computation of the tap-weights. (The steepest-descent algorithm was discussed in Section 3.13.)

■ OPERATION OF THE EQUALIZER

There are two modes of operation for an adaptive equalizer, namely, the training mode and decision-directed mode, as shown in Figure 4.30. During the *training mode*, as explained previously, a known PN sequence is transmitted and a synchronized version of it is generated in the receiver, where (after a time shift equal to the transmission delay) it is applied to the adaptive equalizer as the desired response; the tap-weights of the equalizer are thereby adjusted in accordance with the LMS algorithm.

When the training process is completed, the adaptive equalizer is switched to its second mode of operation: the *decision-directed mode*. In this mode of operation, the error signal is defined by

$$e[n] = \hat{d}[n] - y[n] \quad (4.120)$$

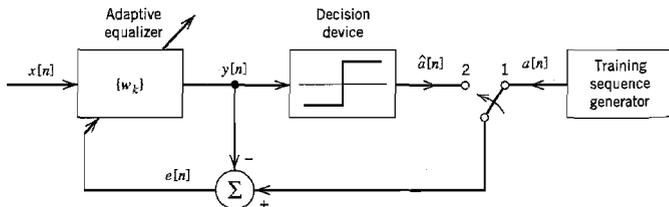


FIGURE 4.30 Illustrating the two operating modes of an adaptive equalizer: For the training mode, the switch is in position 1; and for the tracking mode, it is moved to position 2.

where $y[n]$ is the equalizer output at time $t = nT$, and $\hat{a}[n]$ is the final (not necessarily) correct estimate of the transmitted symbol $a[n]$. Now, in normal operation the decisions made by the receiver are correct with high probability. This means that the error estimates are correct most of the time, thereby permitting the adaptive equalizer to operate satisfactorily. Furthermore, an adaptive equalizer operating in a decision-directed mode is able to track relatively slow variations in channel characteristics.

It turns out that the larger the step-size parameter μ , the faster the tracking capability of the adaptive equalizer. However, a large step-size parameter μ may result in an unacceptably high *excess mean-square error*, defined as that part of the mean-square value of the error signal in excess of the minimum attainable value J_{\min} (which results when the tap-weights are at their optimum settings). We therefore find that in practice the choice of a suitable value for the step-size parameter μ involves making a compromise between fast tracking and reducing the excess mean-square error.

▣ DECISION-FEEDBACK EQUALIZATION

To develop further insight into adaptive equalization, consider a baseband channel with impulse response denoted in its sampled form by the sequence $\{b[n]\}$ where $b[n] = b(nT)$. The response of this channel to an input sequence $\{x[n]\}$, in the absence of noise, is given by the discrete convolution sum

$$\begin{aligned} y[n] &= \sum_k h[k]x[n-k] \\ &= b[0]x[n] + \sum_{k<0} h[k]x[n-k] + \sum_{k>0} h[k]x[n-k] \end{aligned} \quad (4.121)$$

The first term of Equation (4.121) represents the desired data symbol. The second term is due to the *precursors* of the channel impulse response that occur before the main sample $b[0]$ associated with the desired data symbol. The third term is due to the *postcursors* of the channel impulse response that occur after the main sample $b[0]$. The precursors and postcursors of a channel impulse response are illustrated in Figure 4.31. The idea of *decision-feedback equalization*¹² is to use data decisions made on the basis of precursors of the channel impulse response to take care of the postcursors; for the idea to work, however, the decisions would obviously have to be correct. Provided that this condition is satisfied,

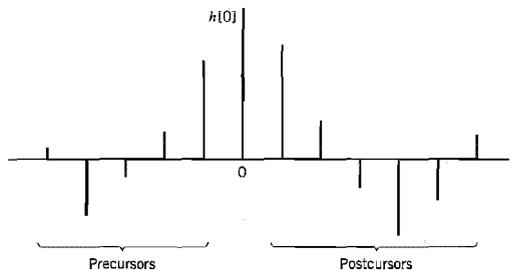


FIGURE 4.31 Impulse response of a discrete-time channel, depicting the precursors and postcursors.

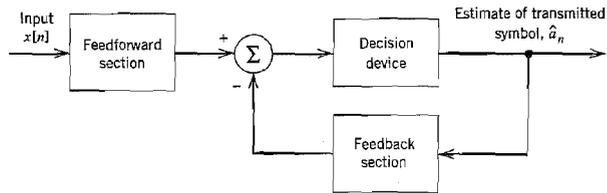


FIGURE 4.32 Block diagram of decision-feedback equalizer.

a decision-feedback equalizer is able to provide an improvement over the performance of the tapped-delay-line equalizer.

A *decision-feedback equalizer* (DFE) consists of a feedforward section, a feedback section, and a decision device connected together as shown in Figure 4.32. The feedforward section consists of a tapped-delay-line filter whose taps are spaced at the reciprocal of the signaling rate. The data sequence to be equalized is applied to this section. The feedback section consists of another tapped-delay-line filter whose taps are also spaced at the reciprocal of the signaling rate. The input applied to the feedback section consists of the decisions made on previously detected symbols of the input sequence. The function of the feedback section is to subtract out that portion of the intersymbol interference produced by previously detected symbols from the estimates of future samples.

Note that the inclusion of the decision device in the feedback loop makes the equalizer intrinsically *nonlinear* and therefore more difficult to analyze than an ordinary tapped-delay-line equalizer. Nevertheless, the mean-square error criterion can be used to obtain a mathematically tractable optimization of a decision-feedback equalizer. Indeed, the LMS algorithm can be used to jointly adapt both the feedforward tap-weights and the feedback tap-weights based on a *common* error signal; see Problem 4.37.

On the basis of extensive comparative evaluations of a linear equalizer and decision-feedback equalizer reported in the literature,¹³ we may report that when the frequency response of a linear channel is characterized by severe amplitude distortion or relatively sharp amplitude cutoff, the decision-feedback equalizer offers a significant improvement in performance over a linear equalizer for an equal number of taps. It is presupposed here that the feedback decisions in the DFE are all correct. For an example of sharp amplitude cutoff, see the frequency response of a telephone channel depicted in Figure 8 in the Background and Preview chapter.

Unlike a linear equalizer, a decision-feedback equalizer suffers from *error propagation*. However, despite the fact that the DFE is a feedback system, error propagation will not persist indefinitely. Rather, decision errors tend to occur in *bursts*. To justify this kind of behavior, we offer the following intuitive reasoning:¹⁴

- ▶ Let L denote the number of taps in the feedback section of a DFE. After a sequence of L consecutive correct decisions, all decision errors in the feedback section will be flushed out. This points to an error propagation of finite duration.
- ▶ When a decision error is made, the probability of the next decision being erroneous too is clearly no worse than $1/2$.
- ▶ Let K denote the duration of error propagation, that is, the number of symbols needed to make L consecutive correct decisions. Then the average error rate is $(K/2)P_0$, where $K/2$ is the average number of errors produced by a single decision error, and P_0 is the probability of error given that the past L decisions are all correct.

- ▶ In a fair-coin tossing experiment, the average number of coin tosses, K , needed to get L successive heads (representing no errors) turns out to be $2(2^L - 1)$.

It follows therefore that the effect of error propagation in a decision-feedback equalizer is to increase the average error rate by a factor approximately equal to 2^L , compared to the probability of making the first error. For example, for $L = 3$ the average error rate is increased by less than an order of magnitude due to error propagation.

4.11 Computer Experiments: Eye Patterns

In previous sections of this chapter we have discussed various techniques for dealing with the effects of channel noise and intersymbol interference on the performance of a baseband pulse-transmission system. In the final analysis, what really matters is how to evaluate the combined effect of these impairments on overall system performance in an operational environment. An experimental tool for such an evaluation in an insightful manner is the so-called *eye pattern*, which is defined as the synchronized superposition of all possible realizations of the signal of interest (e.g., received signal, receiver output) viewed within a particular signaling interval. The eye pattern derives its name from the fact that it resembles the human eye for binary waves. The interior region of the eye pattern is called the *eye opening*.

An eye pattern provides a great deal of useful information about the performance of a data transmission system, as described in Figure 4.33. Specifically, we make the following statements:

- ▶ The width of the eye opening defines the *time interval over which the received signal can be sampled without error from intersymbol interference*; it is apparent that the preferred time for sampling is the instant of time at which the eye is open the widest.
- ▶ The *sensitivity of the system to timing errors* is determined by the rate of closure of the eye as the sampling time is varied.
- ▶ The height of the eye opening, at a specified sampling time, defines the *noise margin* of the system.

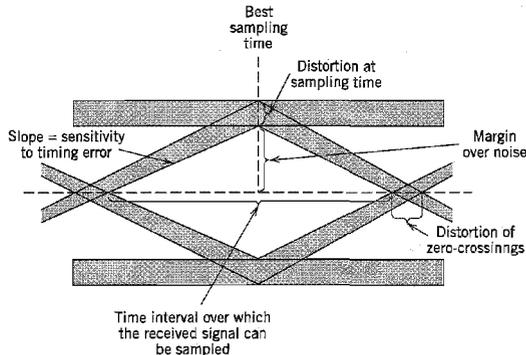


FIGURE 4.33 Interpretation of the eye pattern.

When the effect of intersymbol interference is severe, traces from the upper portion of the eye pattern cross traces from the lower portion, with the result that the eye is completely closed. In such a situation, it is impossible to avoid errors due to the combined presence of intersymbol interference and noise in the system.

In the case of an M -ary system, the eye pattern contains $(M - 1)$ eye openings stacked up vertically one on the other, where M is the number of discrete amplitude levels used to construct the transmitted signal. In a strictly linear system with truly random data, all these eye openings would be identical.

In the next two experiments, we use computer simulations to study the eye patterns for a quaternary ($M = 4$) baseband PAM transmission system under noiseless, noisy, and band-limited conditions. The effect of channel nonlinearity on eye patterns is discussed in Problem 4.38.

Experiment 1: Effect of Channel Noise

Figure 4.34a shows the eye diagram of the system under idealized conditions: no channel noise and no bandwidth limitation. The source symbols used are randomly generated on a computer, with raised cosine pulse-shaping. The system parameters used for the generation of the eye diagram are as follows: Nyquist bandwidth $W = 0.5$ Hz, rolloff factor $\alpha = 0.5$, and symbol duration $T = T_b \log_2 M = 2T_b$. The openings in Figure 4.34a are perfect, indicating reliable operation of the system. Note that this figure has $M - 1 = 3$ openings.

Figures 4.34b and 4.34c show the eye diagrams for the system, but this time with channel noise corrupting the received signal. These two figures were simulated for signal-to-noise ratio $\text{SNR} = 20$ dB and 10 dB, respectively, with the SNR being measured at the channel output. When $\text{SNR} = 20$ dB the effect of channel noise is hardly discernible in Figure 4.34b, but when $\text{SNR} = 10$ dB the openings of the eye diagram in Figure 4.34c are barely visible.

Experiment 2: Effect of Bandwidth Limitation

Figures 4.35a and 4.35b show the eye diagrams for the quaternary system using the same parameters as before, but this time under a bandwidth-limited condition and a noiseless channel. Specifically, the channel is now modeled by a low-pass *Butterworth filter*, whose squared magnitude response is defined by

$$|H(f)|^2 = \frac{1}{1 + (f/f_0)^{2N}}$$

where N is the *order* of the filter, and f_0 is its 3-dB cutoff frequency. For the computer experiment described in Figure 4.35a, the following values are used:

$$N = 25 \text{ and } f_0 = 0.975 \text{ Hz}$$

The bandwidth required by the PAM transmission system is computed to be

$$B_T = W(1 + \alpha) = 0.75 \text{ Hz}$$

Although the channel bandwidth (i.e., cutoff frequency) is greater than absolutely necessary, its effect on the passband is observed as a decrease in the size of the eye openings compared to those in Figure 4.34a. Instead of the distinct values at time $t = 1$ s (as shown in Figure 4.34a), now there is a blurred region.

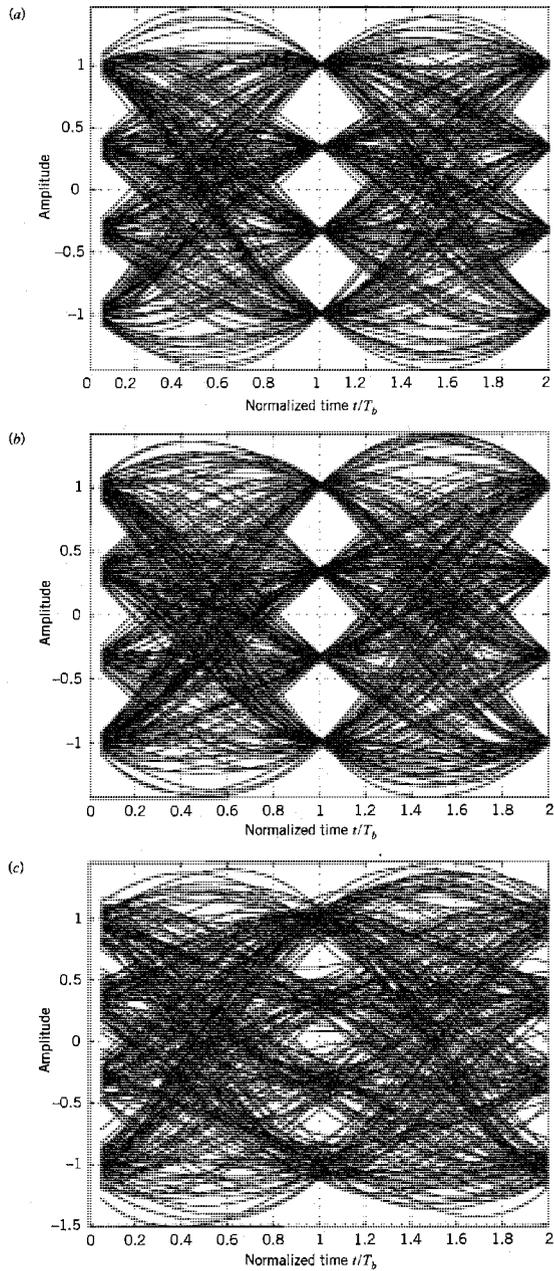


FIGURE 4.34 (a) Eye diagram for noiseless quaternary system. (b) Eye diagram for quaternary system with SNR = 20 dB. (c) Eye diagram for quaternary system with SNR = 10 dB.

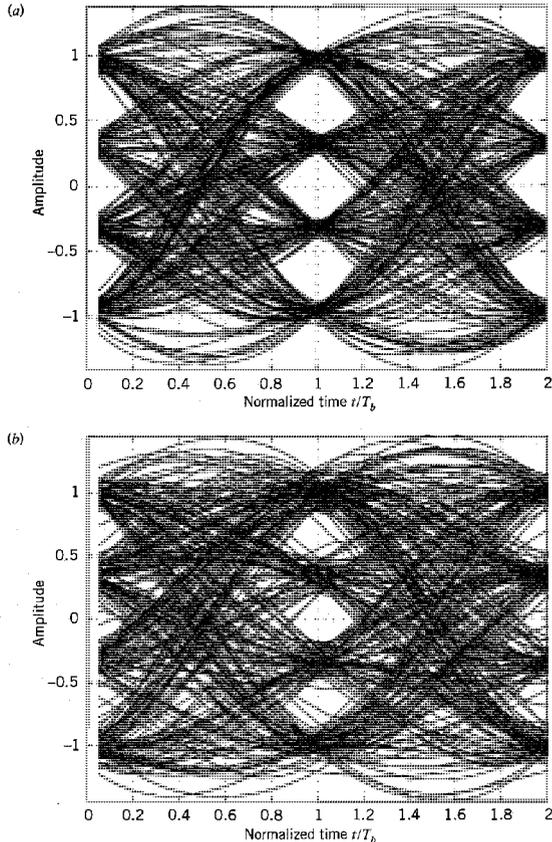


FIGURE 4.35 (a) Eye diagram for noiseless band-limited quaternary system: cutoff frequency $f_0 = 0.975$ Hz. (b) Eye diagram for noiseless band-limited quaternary system: cutoff frequency $f_0 = 0.5$ Hz.

In Figure 4.35b the channel bandwidth is reduced further by modeling the channel as a low-pass Butterworth filter with $N = 25$ and $f_0 = 0.5$ Hz. The effect of reduced channel bandwidth is to further reduce the extent to which the eyes are open.

4.12 Summary and Discussion

In this chapter, we studied the effects of channel noise and intersymbol interference on the performance of baseband-pulse transmission systems. Intersymbol interference (ISI) is different from noise in that it is a *signal-dependent* form of interference that arises because of deviations in the frequency response of a channel from the ideal low-pass filter (Nyquist channel); it disappears when the transmitted signal is switched off. The result of these deviations is that the received pulse corresponding to a particular data symbol is affected

by the tail ends of the pulses representing the previous symbols and the front ends of the pulses representing the subsequent symbols.

Depending on the received signal-to-noise ratio, we may distinguish three different situations that can arise in baseband-pulse transmission systems for channels with fixed characteristics:

1. *The effect of ISI is negligible in comparison to that of channel noise.*

The proper procedure in this case is to use a matched filter, which is the optimum linear time-invariant filter for maximizing the peak pulse signal-to-noise ratio.

2. *The received signal-to-noise ratio is high enough to ignore the effect of channel noise.*

In this case, we need to guard against the effects of ISI on the reconstruction of the transmitted data at the receiver. In particular, control must be exercised over the shape of the received pulse. This design objective can be achieved in one of two different ways:

- ▶ Using a raised cosine spectrum for the overall frequency response of the baseband-pulse transmission system.
- ▶ Using correlative-level coding or partial-response signaling that adds ISI to the transmitted signal in a controlled manner.

3. *The ISI and noise are both significant.*

For a mathematically tractable solution to this more difficult situation, we may use the mean-square error criterion. The resulting optimum linear receiver is called the minimum mean-square error (mmse) receiver. It consists of the cascade connection of a matched filter and linear transversal (tapped-delay-line) equalizer.

When, however, the channel is random in the sense of being one of an ensemble of possible physical realizations, which is frequently the case in a telecommunications environment, the use of fixed filter designs based on average channel characteristics may not be adequate. In situations of this kind, the preferred approach is to use an adaptive equalizer, the purpose of which is to compensate for variations in the frequency response of the channel automatically during the course of data transmission. The combined use of a tapped-delay-line filter and the least-mean-square (LMS) algorithm for adjusting the tap-weights provides the basis of a simple and yet highly effective method for implementing the adaptive equalizer. Such a device is capable of dealing with the combined effects of ISI and receiver noise in a nonstationary environment. Its practical value lies in the fact that almost every modem (modulator-demodulator) in commercial use today for the transmission of digital data over a voice-grade telephone channel uses an adaptive equalizer as an integral part.

Another important application of adaptive filtering is in the design of echo cancellers that constitute a critical component of transceivers for digital subscriber lines. Typically, a digital subscriber line uses a twisted pair as the transmission medium, the very same one used in ordinary telephone channels. However, unlike telephone channels, digital subscriber lines are designed to provide a high data-rate digital transmission capability between a digital network and subscriber plants, with a data rate of 64 kb/s and up.

NOTES AND REFERENCES

1. The classic books on baseband-pulse transmission are Lucky, Salz, and Weldon (1968) and Sunde (1969). For detailed treatment of different aspects of the subject, see Gitlin, Hayes, and Weinstein (1992). Proakis (1995), and Benedetto, Biglieri, and Castellani (1987).

2. The characterization of a matched filter was first derived by North in a classified report (RCA Laboratories Report PTR-6C, June 1943), which was published 20 years later: see the paper by North (1963). A similar result was obtained independently by Van Vleck and Middleton, who coined the term *matched filter*: see the paper by Van Vleck and Middleton (1946). For review material on the matched filter and its properties, see the papers by Turin (1960, 1976).
3. The *error function* denoted by $\text{erf}(u)$, is defined in a number of different ways in the literature. We shall use the following definition:

$$\text{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-z^2) dz$$

The error function has two useful properties:

(i) $\text{erf}(-u) = -\text{erf}(u)$

This is known as the *symmetry relation*.

(ii) As u approaches infinity, $\text{erf}(u)$ approaches unity; that is,

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \exp(-z^2) dz = 1$$

The *complementary error function* is defined by

$$\text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2) dz$$

which is related to the error function as follows:

$$\text{erfc}(u) = 1 - \text{erf}(u)$$

Table A6.6 gives values of the error function $\text{erf}(u)$ for u in the range 0 to 3.3.

For large positive values of u , we have two simple bounds on $\text{erfc}(u)$, one lower and the other upper, as shown by

$$\frac{\exp(-u^2)}{\sqrt{\pi}u} \left(1 - \frac{1}{2u^2}\right) < \text{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi}u}$$

The complementary error function provides the basis for a compact formulation of the probability of symbol error, as explained in Section 4.3. Another function that is also commonly used in the literature for this purpose is the Q -function. Consider a *standardized* Gaussian random variable X of zero mean and unit variance. The probability that an observed value of the random variable X will be greater than v is given by the Q -function:

$$Q(v) = \frac{1}{\sqrt{2\pi}} \int_v^{\infty} \exp\left(-\frac{x^2}{2}\right) dx$$

The Q -function defines the area under the standardized Gaussian tail. The Q -function is related to the complementary error function as

$$Q(v) = \frac{1}{2} \text{erfc}\left(\frac{v}{\sqrt{2}}\right)$$

Conversely, putting $u = v/\sqrt{2}$, we have

$$\text{erfc}(u) = 2Q(\sqrt{2}u)$$

4. The criterion described in Equation (4.49) or Equation (4.53) was first formulated by Nyquist in the study of telegraph transmission theory; the 1928 paper by Nyquist is a classic. In the literature, this criterion is referred to as *Nyquist's first criterion*. In his 1928 paper, Nyquist described another method, referred to in the literature as *Nyquist's second*

criterion. The second method makes use of the instants of transition between unlike symbols in the received signal rather than centered samples. A discussion of the first and second criteria is presented in Bennett (1970, pp. 78–92) and in the paper by Gibby and Smith (1965). A third criterion attributed to Nyquist is discussed in Sunde (1969); see also the papers by Pasupathy (1974) and Sayar and Pasupathy (1987).

5. Correlative-level coding and partial-response signaling are synonymous; both terms are used in the literature. The idea of correlative coding was originated by Lender (1963). Lender's work was generalized for binary data transmission by Kretzmer (1966). For further details on correlative coding techniques, see the book by Gitlin, Hayes, and Weinstein (1992); see also the papers by Pasupathy (1977), Kabal and Pasupathy (1975), and Sousa and Pasupathy (1983).
6. The material on digital subscriber lines presented in Section 4.8 is based on the two papers by Lin and Tzeng (1988), and Lechleider (1989), and the books by Starr, Cioffi, and Silverman (1999) and Chen (1998).
7. For a discussion of line codes for digital subscriber loops, see Gitlin et al. (1992).
8. In Ericson (1971) it is shown that for every "reasonable" performance criterion, the optimum receiver can be realized as a matched filter followed by a tapped-delay-line equalizer, as shown in Figure 4.27. In addition to the mean-square error criterion considered in Section 4.9, reasonable performance criteria of interest include the following:
 - (i) Minimization of the probability of symbol error.
 - (ii) Zero-forcing equalization (to reduce the intersymbol interference to zero), followed by minimization of the probability of symbol error subject to this constraint.
 - (iii) Minimization of signal-to-noise ratio at the sampling instants.

Criterion (i) is the most natural approach to the optimization of a linear receiver; this approach, pursued in Aaron and Tufts (1966), is, unfortunately, complicated. Criterion (ii), due to Lucky et al. (1968), is a much simpler approach. Criterion (iii) is due to George (1965).

9. Adaptive equalization of telephone channels was pioneered by Lucky (1965, 1966). Since that time, numerous adaptive equalization schemes have been published in the literature, which provide equalization for specific synchronous data-transmission systems. For review papers on adaptive equalization, see Proakis (1975) and Qureshi (1982, 1985). Adaptive equalization is also discussed in detail in the books by Gitlin, Hayes, and Weinstein (1992, Chapter 8) and Proakis (1995, Chapter 6).
10. It appears that early work on fractionally spaced equalizers was initiated by Brady (1970). Other contributions to the subject include subsequent work by Ungerboeck (1976) and Gitlin and Weinstein (1981). A detailed discussion of fractionally spaced equalizers is also presented in Gitlin et al. (1992).
11. The LMS algorithm was originated by Widrow and Hoff, Jr. (1960). For a detailed convergence analysis of the LMS algorithm, see Haykin (1996, Chapter 9), and Widrow and Stearns (1985, Chapter 6).
12. Decision-feedback equalization was first described in a report by Austin (1967). The optimization of the decision-feedback equalizer for minimum mean-square error was first accomplished by Monsen (1971). A readable account of decision-feedback equalization is presented in the book by Gitlin, Hayes, and Weinstein (1992, pp. 500–510).

Tomlinson (1971) and Harashima and Miyakawa (1972) describe a device for eliminating error propagation in a decision-feedback equalizer. The device, known as the *Tomlinson-Harashima precoder*, appears in the transmitter as a preprocessor to the modulator. The basic idea of this precoder is to move the feedback section in the decision feedback equalizer to the transmitter where it is impossible to make decision errors. However, this

modification may result in a significant increase in transmit power; modulo arithmetic is used to overcome most of this power increase.

13. For performance comparison between linear equalizers and decision-feedback equalizers, see Gitlin et al. (1992) and Proakis (1995).
14. The intuitive discussion on error propagation in decision-feedback equalizers presented in Section 4.10 follows Gitlin et al. (1992).

For a rigorous evaluation of the probability of symbol error P_e in a decision-feedback equalizer with error propagation, see Duttweiler et al. (1974). In this paper it is shown that in the worst-case intersymbol interference, P_e is multiplied by a factor of 2^L relative to the probability of error that results in the absence of decision errors at high signal-to-noise ratios, where L is the number of taps in the feedback section. The result derived by Duttweiler et al. provides theoretical justification for the intuitive arguments presented in Section 4.10.

PROBLEMS

Matched Filters

- 4.1 Consider the signal $s(t)$ shown in Figure P4.1.
 - (a) Determine the impulse response of a filter matched to this signal and sketch it as a function of time.
 - (b) Plot the matched filter output as a function of time.
 - (c) What is the peak value of the output?

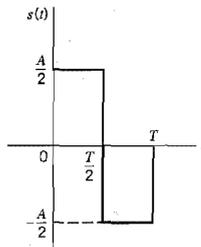


FIGURE P4.1

- 4.2 Figure P4.2a shows a pair of pulses that are orthogonal to each other over the interval $[0, T]$. In this problem we investigate the use of this pulse-pair to study a *two-dimensional matched filter*.
 - (a) Determine the matched filters for the pulses $s_1(t)$ and $s_2(t)$ considered individually; for $s_1(t)$ the filter is the same as that considered in Problem 4.1.
 - (b) Form a two-dimensional matched filter by connecting the two matched filters of Part (a) in parallel, as shown in Figure P4.2b. Hence, demonstrate the following:
 - (i) When the pulse $s_1(t)$ is applied to this two-dimensional filter, the response of the lower matched filter is zero.
 - (ii) When the pulse $s_2(t)$ is applied to the two-dimensional filter, the response of the upper matched filter is zero.
 Generalize the results of your investigation.

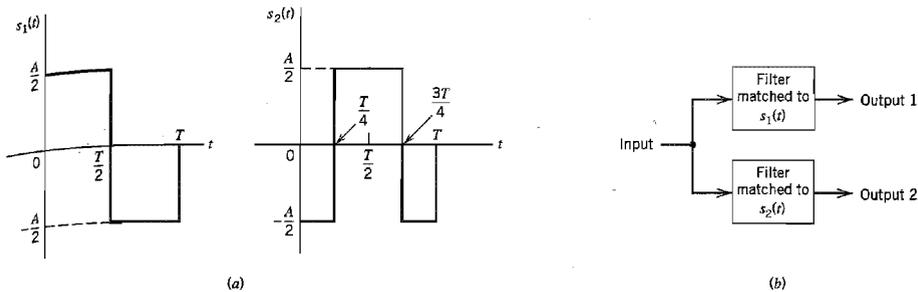


FIGURE P4.2

4.3 Consider a rectangular pulse defined by

$$g(t) = \begin{cases} A, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

It is proposed to approximate the matched filter for $g(t)$ by an ideal low-pass filter of bandwidth B ; maximization of the peak pulse signal-to-noise ratio is the primary objective.

- (a) Determine the optimum value of B for which the ideal low-pass filter provides the best approximation to the matched filter.
 - (b) By how many decibels is the ideal low-pass filter worse off than the matched filter?
- 4.4 In this problem we explore another method for the approximate realization of a matched filter, this time using the simple resistance-capacitance (RC) low-pass filter shown in Figure P4.4. The frequency response of this filter is

$$H(f) = \frac{1}{1 + jf/f_0}$$

where $f_0 = 1/2\pi RC$. The input signal $g(t)$ is a rectangular pulse of amplitude A and duration T . The requirement is to optimize the selection of the 3-dB cutoff frequency f_0 of the filter so that the peak pulse signal-to-noise ratio at the filter output is maximized. With this objective in mind, show that the optimum value of f_0 is $0.2/T$, for which the loss in signal-to-noise ratio compared to the matched filter is about 1 dB.

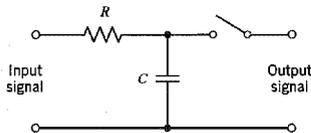


FIGURE P4.4

Probability of Error Calculation

4.5 The formula for the optimum threshold in the receiver of Figure 4.4 is, in general, given by Equation (4.37). Discuss, in graphical terms, how this optimum choice affects the

contributions of the two terms in Equation (4.35) for the average probability of symbol error P_e by considering the following two cases:

- (a) $p_0 > p_1$
- (b) $p_1 < p_0$

where p_0 and p_1 are the *a priori* probabilities of symbols 0 and 1, respectively.

- 4.6 In a binary PCM system, symbols 0 and 1 have *a priori probabilities* p_0 and p_1 , respectively. The conditional probability density function of the random variable Y (with sample value y) obtained by sampling the matched filter output in the receiver of Figure 4.4 at the end of a signaling interval, given that symbol 0 was transmitted, is denoted by $f_Y(y|0)$. Similarly, $f_Y(y|1)$ denotes the conditional probability density function of Y , given that symbol 1 was transmitted. Let λ denote the threshold used in the receiver, so that if the sample value y exceeds λ , the receiver decides in favor of symbol 1; otherwise, it decides in favor of symbol 0. Show that the optimum threshold λ_{opt} , for which the average probability of error is a minimum, is given by the solution of

$$\frac{f_Y(\lambda_{opt}|1)}{f_Y(\lambda_{opt}|0)} = \frac{p_0}{p_1}$$

- 4.7 A binary PCM system using polar NRZ signaling operates just above the error threshold with an average probability of error equal to 10^{-6} . Suppose that the signaling rate is doubled. Find the new value of the average probability of error. You may use Table A6.6 to evaluate the complementary error function.
- 4.8 A continuous-time signal is sampled and then transmitted as a PCM signal. The random variable at the input of the decision device in the receiver has a variance of 0.01 volts².
- (a) Assuming the use of polar NRZ signaling, determine the pulse amplitude that must be transmitted for the average error rate not to exceed 1 bit in 10^8 bits.
 - (b) If the added presence of interference causes the error rate to increase to 1 bit in 10^6 bits, what is the variance of the interference?
- 4.9 A binary PCM wave uses unipolar NRZ signaling to transmit symbols 1 and 0; symbol 1 is represented by a rectangular pulse of amplitude A and duration T_b . The channel noise is modeled as additive, white and Gaussian, with zero mean and power spectral density $N_0/2$. Assuming that symbols 1 and 0 occur with equal probability, find an expression for the average probability of error at the receiver output, using a matched filter as described in Section 4.3.
- 4.10 Repeat Problem 4.9 for the case of unipolar return-to-zero signaling, in which case symbol 1 is represented by a pulse of amplitude A and duration $T_b/2$ and symbol 0 is represented by transmitting no pulse.

Hence show that this unipolar type of signaling requires twice the average power of unipolar nonreturn-to-zero (i.e., on-off) signaling for the same average probability of symbol error.

- 4.11 In this problem, we revisit the PCM receiver of Figure 4.4, but this time we consider the use of bipolar nonreturn-to-zero signaling, in which case the transmitted signal $s(t)$ is defined by
- Binary symbol 1: $s(t) = \pm A$ for $0 < t \leq T$
 - Binary symbol 0: $s(t) = 0, 0 < t \leq T$

Determine the average probability of symbol error P_e for this receiver assuming that the binary symbols 0 and 1 are equiprobable.

Raised Cosine Spectrum

- 4.12 The nonreturn-to-zero pulse of Figure P4.12 may be viewed as a very crude form of a Nyquist pulse. Compare the spectral characteristics of these two pulses.

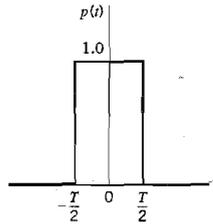


FIGURE P4.12

- 4.13 Determine the inverse Fourier transform of the frequency function $P(f)$ defined in Equation (4.60).
- 4.14 An analog signal is sampled, quantized, and encoded into a binary PCM wave. The specifications of the PCM system include the following:
 Sampling rate = 8 kHz
 Number of representation levels = 64
 The PCM wave is transmitted over a baseband channel using discrete pulse-amplitude modulation. Determine the minimum bandwidth required for transmitting the PCM wave if each pulse is allowed to take on the following number of amplitude levels: 2, 4, or 8.
- 4.15 Consider a baseband binary PAM system that is designed to have a raised-cosine spectrum $P(f)$. The resulting pulse $p(t)$ is defined in Equation (4.62). How would this pulse be modified if the system was designed to have a linear phase response?
- 4.16 A computer puts out binary data at the rate of 56 kb/s. The computer output is transmitted using a baseband binary PAM system that is designed to have a raised-cosine spectrum. Determine the transmission bandwidth required for each of the following rolloff factors: $\alpha = 0.25, 0.5, 0.75, 1.0$.
- 4.17 Repeat Problem 4.16, given that each set of three successive binary digits in the computer output are coded into one of eight possible amplitude levels, and the resulting signal is transmitted using an eight-level PAM system designed to have a raised-cosine spectrum.
- 4.18 An analog signal is sampled, quantized, and encoded into a binary PCM wave. The number of representation levels used is 128. A synchronizing pulse is added at the end of each code word representing a sample of the analog signal. The resulting PCM wave is transmitted over a channel of bandwidth 12 kHz using a quaternary PAM system with raised-cosine spectrum. The rolloff factor is unity.
 (a) Find the rate (b/s) at which information is transmitted through the channel.
 (b) Find the rate at which the analog signal is sampled. What is the maximum possible value for the highest frequency component of the analog signal?
- 4.19 A binary PAM wave is to be transmitted over a baseband channel with an absolute maximum bandwidth of 75 kHz. The bit duration is $10 \mu\text{s}$. Find a raised-cosine spectrum that satisfies these requirements.

Correlative-Level Coding

- 4.20 The duobinary, ternary, and bipolar signaling techniques have one common feature: They all employ three amplitude levels. In what way does the duobinary technique differ from the other two?
- 4.21 The binary data stream 001101001 is applied to the input of a duobinary system.
 (a) Construct the duobinary coder output and corresponding receiver output, without a precoder.

- (b) Suppose that owing to error during transmission, the level at the receiver input produced by the second digit is reduced to zero. Construct the new receiver output.
- 4.22 Repeat Problem 4.21, assuming the use of a precoder in the transmitter.
- 4.23 The scheme shown in Figure P4.23 may be viewed as a differential encoder (consisting of the modulo-2 adder and the 1-unit delay element) connected in cascade with a special form of correlative coder (consisting of the 1-unit delay element and summer). A single-delay element is shown in Figure P4.23 since it is common to both the differential encoder and the correlative coder. In this differential encoder, a transition is represented by symbol 0 and no transition by symbol 1.
- (a) Find the frequency response and impulse response of the correlative coder part of the scheme shown in Figure P4.23.
- (b) Show that this scheme may be used to convert the on-off representation of a binary sequence (applied to the input) into the bipolar representation of the sequence at the output. You may illustrate this conversion by considering the sequence 010001101.
- For descriptions of on-off, bipolar, and differential encoding of binary sequences, see Section 3.7.

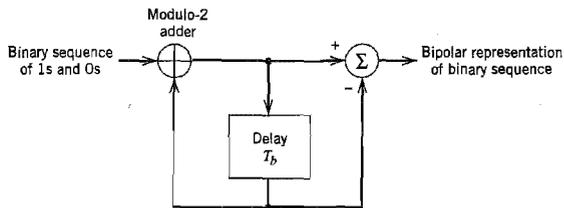


FIGURE P4.23

- 4.24 Consider a random binary wave $x(t)$ in which the 1s and 0s occur with equal probability, the symbols in adjacent time slots are statistically independent, and symbol 1 is represented by A volts and symbol 0 by zero volts. This on-off binary wave is applied to the circuit of Figure P4.23.
- (a) Using the result of Problem 4.23, show that the power spectral density of the bipolar wave $y(t)$ appearing at the output of the circuit equals
- $$S_X(f) = T_b A^2 \sin^2(\pi f T_b) \operatorname{sinc}^2(f T_b)$$
- (b) Plot the power spectral densities of the on-off and bipolar binary waves, and compare them.
- 4.25 The binary data stream 011100101 is applied to the input of a modified duobinary system.
- (a) Construct the modified duobinary coder output and corresponding receiver output, without a precoder.
- (b) Suppose that due to error during transmission, the level produced by the third digit is reduced to zero. Construct the new receiver output.
- 4.26 Repeat Problem 4.25 assuming the use of a precoder in the transmitter.

M-ary PAM Systems

- 4.27 Consider a baseband M -ary system using M discrete amplitude levels. The receiver model is as shown in Figure P4.27, the operation of which is governed by the following assumptions:

(a) The signal component in the received wave is

$$m(t) = \sum_n a_n \operatorname{sinc}\left(\frac{t}{T} - n\right)$$

where $1/T$ is the signaling rate in bauds.

(b) The amplitude levels are $a_n = \pm A/2, \pm 3A/2, \dots, \pm(M-1)A/2$ if M is even, and $a_n = 0, \pm A, \dots, \pm(M-1)A/2$ if M is odd.

(c) The M levels are equiprobable, and the symbols transmitted in adjacent time slots are statistically independent.

(d) The channel noise $w(t)$ is white and Gaussian with zero mean and power spectral density $N_0/2$.

(e) The low-pass filter is ideal with bandwidth $B = 1/2T$.

(f) The threshold levels used in the decision device are $0, \pm A, \dots, \pm(M-2)A/2$ if M is even, and $\pm A/2, \pm 3A/2, \dots, \pm(M-2)A/2$ if M is odd.

The average probability of symbol error in this system is defined by

$$P_e = \left(1 - \frac{1}{M}\right) \operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)$$

where σ is the standard deviation of the noise at the input of the decision device. Demonstrate the validity of this general formula by determining P_e for the following three cases: $M = 2, 3, 4$.

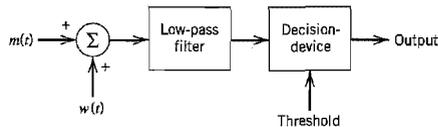


FIGURE P4.27

4.28 Suppose that in a baseband M -ary PAM system with M equally likely amplitude levels, as described in Problem 4.27, the average probability of symbol error P_e is less than 10^{-6} so as to make the occurrence of decoding errors negligible. Show that the minimum value of received signal-to-noise ratio in such a system is approximately given by

$$(\operatorname{SNR})_{\min} \approx 7.8(M^2 - 1)$$

Digital Subscriber Lines

4.29 The amplitude distribution of cross-talk in a digital subscriber line may be modeled as Gaussian. Justify the validity of such a model. *Hint:* Typically, a cable contains many twisted pairs.

4.30 (a) Derive the formula for the power spectral density of a transmitted signal using the 2B1Q line code.

(b) Plot the power spectrum of the following line codes:

- ▶ Manchester code
- ▶ Modified duobinary code
- ▶ Bipolar return-to-zero code
- ▶ 2B1Q code

Hence compare the relative merits of these line codes for their suitability in a digital subscriber loop.

4.31 In this problem we use the LMS algorithm to formulate an adaptive echo canceller for use in a digital subscriber line. The basic principle of adaptive echo cancellation is to synthesize a replica of the echo and subtract it from the returned signal in an adaptive manner, as illustrated in Figure P4.31. The synthesized echo, denoted by $\hat{r}[n]$, is generated by passing the transmitted signal through an adaptive filter that ideally matches the transfer function of the echo path. The returned signal, consisting of the sum of actual echo $r[n]$ and the received signal $x[n]$, may be viewed as the desired response for the adaptive filtering process.

Using the LMS algorithm, formulate the equations that define the operation of the adaptive echo canceller in Figure P4.31.

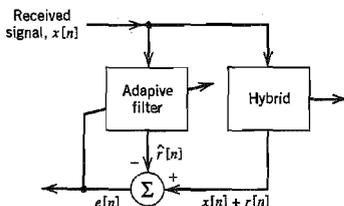


FIGURE P4.31

Equalization

4.32 Figure P4.32 shows the cascade connection of a linear channel and a synchronous tapped-delay-line equalizer. The impulse response of the channel is denoted by $c(t)$, and that of the equalizer is denoted by $b(t)$. The $b(t)$ is defined by

$$b(t) = \sum_{k=-N}^N w_k \delta(t - kT)$$

where T is the spacing between adjacent taps of the equalizer, and the w_k are its tap-weights (coefficients). The impulse response of the cascaded system of Figure P4.32 is denoted by $p(t)$. The $p(t)$ is sampled uniformly at the rate $1/T$. To eliminate intersymbol interference, we require that the Nyquist criterion for distortionless transmission be satisfied, as shown by

$$p(nT) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

(a) By imposing this condition, show that the $(2N + 1)$ tap-weights of the resulting zero-forcing equalizer satisfy the following set of $(2N + 1)$ simultaneous equations:

$$\sum_{k=-N}^N w_k c_{n-k} = \begin{cases} 1, & n = 0 \\ 0, & n \neq \pm 1, \pm 2, \dots, \pm N \end{cases}$$

where $c_n = c(nT)$. Hence, show that the zero-forcing equalizer is an *inverse filter* in that its transfer function is equal to the reciprocal of the transfer function of the channel.

(b) A shortcoming of the zero-forcing equalizer is *noise enhancement* that can result in poor performance in the presence of channel noise. To explore this phenomenon, consider a low-pass channel with a notch at the Nyquist frequency, that is, $H(f)$ is zero at $f = 1/2T$. Assuming that the channel noise is additive and white, show that the power spectral density of the noise at the equalizer output approaches infinity at $f = 1/2T$.

Even if the channel has no notch in its frequency response, the power spectral density of the noise at the equalizer output can assume high values. Justify the validity of this general statement.

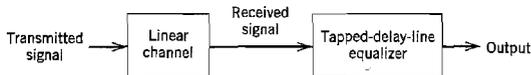


FIGURE P4.32

- 4.33 Consider Equation (4.108), which defines the impulse response of a minimum mean-square error receiver.
- Justify the validity of Equation (4.109) that is the Fourier-transformed version of Equation (4.108).
 - The power spectral density $S_q(f)$ in Equation (4.109) is the Fourier transform of the autocorrelation $R_q(\tau_1, \tau_2)$ of the time function $q(t)$. The $R_q(\tau_1, \tau_2)$ is defined by Equation (4.98). Starting with Equation (4.98), derive the formula of Equation (4.111).
- 4.34 Some radio systems suffer from *multipath distortion*, which is caused by the existence of more than one propagation path between the transmitter and the receiver. Consider a channel the output of which, in response to a signal $s(t)$, is defined by (in the absence of noise)

$$x(t) = a_1s(t - t_{01}) + a_2s(t - t_{02})$$

where a_1 and a_2 are constant, and t_{01} and t_{02} represent transmission delays. It is proposed to use the three-tap delay-line-filter of Figure P4.34 to equalize the multipath distortion produced by this channel.

- Evaluate the transfer function of the channel.
- Evaluate the parameters of the tapped-delay-line filter in terms of a_1, a_2, t_{01} , and t_{02} , assuming that $a_2 \ll a_1$ and $t_{02} > t_{01}$.

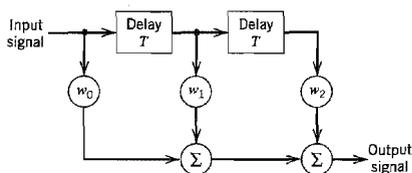


FIGURE P4.34

- 4.35 Let the sequence $\{x(nT)\}$ denote the input applied to a tapped-delay-line equalizer. Show that intersymbol interference is eliminated completely by the equalizer provided that its frequency response satisfies the condition

$$H(f) = \frac{T}{\sum_b X(f - k/T)}$$

where T is the symbol duration.

As the number of taps in the equalizer approaches infinity, the frequency response of the equalizer becomes a Fourier series with real coefficients and can therefore approx-

- imate any function in the interval $(-1/2T, 1/2T)$. Demonstrate this property of the equalizer.
- 4.36 The step-size parameter μ plays a critical role in the operation of the LMS algorithm. In this context, discuss the following two issues:
- Stability.* If μ exceeds a certain critical value, the algorithm diverges (i.e., the system becomes unstable).
 - Memory.* The reciprocal of μ may be viewed as a measure of the algorithm's memory: As we make μ smaller, more of the past samples of the input signal influence operation of the algorithm.
- 4.37 Let the vectors $\mathbf{w}^{(1)}[n]$ and $\mathbf{w}^{(2)}[n]$ denote the tap-weights of the feed-forward and feed-back sections of the decision-feedback equalizer in Figure 4.32. Formulate the LMS algorithm for adjusting the tap-weights of this equalizer.

Computer Experiments

- 4.38 In Section 4.11 we studied the eye diagram of a quaternary ($M = 4$) PAM baseband transmission system under both noisy and band-limited conditions. In that experiment, the channel was assumed linear. In a strictly linear system with truly random data, all the eye openings would be identical. In practice, however, it is often possible to discern asymmetries in the eye pattern, which are caused by nonlinearities in the communication channel.

In this experiment, we study the effect of a nonlinear channel on the openings of an eye pattern. Specifically, we repeat the computer experiment pertaining to the noiseless eye pattern of Figure 4.34a for $M = 4$, but this time assume that the channel is nonlinear with the following input-output relation:

$$x(t) = s(t) + as^2(t)$$

where $s(t)$ is the channel input and $x(t)$ is the channel output, and a is a constant.

- Do the experiment for $a = 0, 0.05, 0.1, 0.2$.
 - Hence, discuss how varying a affects the shape of the eye pattern.
- 4.39 In this experiment we study the *root raised-cosine pulse* due to Chennakeshu and Saulnier (1993). This pulse, denoted by $p(t)$, has the following properties:
- ▶ The pulse $p(t)$ is symmetric in time, that is, $p(-t) = p(t)$.
 - ▶ The squared Fourier transform of $p(t)$, namely, $P^2(f)$, satisfies the raised cosine spectrum of Equation (4.60), but the Fourier transform $P(f)$ itself does not.
 - ▶ The pulse $p(t)$ satisfies the *orthogonality constraint*:

$$\int_{-\infty}^{\infty} p(t)p(t - nT) dt = 0, \quad n = \pm 1, \pm 2, \dots$$

where T is the symbol period.

- Compute the baseband waveform of the binary data stream 101100 for rolloff factor $\alpha = 0.3$.
- Compare the waveform computed in part (a) with that obtained using the ordinary raised-cosine spectrum.

SIGNAL-SPACE ANALYSIS

This chapter discusses some basic issues that pertain to the transmission of signals over an additive white Gaussian noise (AWGN) channel. Specifically, it addresses the following topics:

- ▶ *Geometric representation of signals with finite energy, which provides a mathematically elegant and highly insightful tool for the study of data transmission.*
- ▶ *Maximum likelihood procedure for the detection of a signal in AWGN channel.*
- ▶ *Derivation of the correlation receiver that is equivalent to the matched filter receiver discussed in the previous chapter.*
- ▶ *Probability of symbol error and the union bound for its approximate calculation.*

The material presented herein naturally leads to the study of passband data transmission covered in Chapter 6.

5.1 Introduction

Consider the most basic form of a digital communication system depicted in Figure 5.1. A *message source* emits one *symbol* every T seconds, with the symbols belonging to an alphabet of M symbols denoted by m_1, m_2, \dots, m_M . Consider, for example, the remote connection of two digital computers, with one computer acting as an information source that calculates digital outputs based on observations and inputs fed into it. The resulting computer output is expressed as a sequence of 0s and 1s, which are transmitted to a second computer over a communication channel. In this case, the alphabet consists simply of two binary symbols: 0 and 1. A second example is that of a quaternary PCM encoder with an alphabet consisting of four possible symbols: 00, 01, 10, and 11. In any event, the *a priori* probabilities p_1, p_2, \dots, p_M specify the message source output. In the absence of prior information, it is customary to assume that the M symbols of the alphabet are *equally likely*. Then we may express the probability that symbol m_i is emitted by the source as

$$\begin{aligned} p_i &= P(m_i) \\ &= \frac{1}{M} \text{ for } i = 1, 2, \dots, M \end{aligned} \quad (5.1)$$

The transmitter takes the message source output m_i and codes it into a *distinct* signal $s_i(t)$ suitable for transmission over the channel. The signal $s_i(t)$ occupies the full duration T allotted to symbol m_i . Most important, $s_i(t)$ is a real-valued *energy signal* (i.e., a signal with finite energy), as shown by

$$E_i = \int_0^T s_i^2(t) dt, \quad i = 1, 2, \dots, M \quad (5.2)$$



FIGURE 5.1 Block diagram of a generic digital communication system.

The channel is assumed to have two characteristics:

1. The channel is *linear*, with a bandwidth that is wide enough to accommodate the transmission of signal $s_i(t)$ with negligible or no distortion.
2. The channel noise, $w(t)$, is the sample function of a *zero-mean white Gaussian noise process*. The reasons for this second assumption are that it makes receiver calculations tractable, and it is a reasonable description of the type of noise present in many practical communication systems.

We refer to such a channel as an *additive white Gaussian noise (AWGN) channel*. Accordingly, we may express the *received signal* $x(t)$ as

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \dots, M \end{cases} \quad (5.3)$$

and thus model the channel as in Figure 5.2.

The receiver has the task of observing the received signal $x(t)$ for a duration of T seconds and making a best *estimate* of the transmitted signal $s_i(t)$ or, equivalently, the symbol m_i . However, owing to the presence of channel noise, this decision-making process is statistical in nature, with the result that the receiver will make occasional errors. The requirement is therefore to design the receiver so as to minimize the *average probability of symbol error*, defined as

$$P_e = \sum_{i=1}^M p_i P(\hat{m} \neq m_i | m_i) \quad (5.4)$$

where m_i is the transmitted symbol, \hat{m} is the estimate produced by the receiver, and $P(\hat{m} \neq m_i | m_i)$ is the conditional error probability given that the i th symbol was sent. The resulting receiver is said to be *optimum in the minimum probability of error sense*.

This model provides a basis for the design of the optimum receiver, for which we will use geometric representation of the known set of transmitted signals, $\{s_i(t)\}$. This method, discussed in Section 5.2, provides a great deal of insight, with considerable simplification of detail.

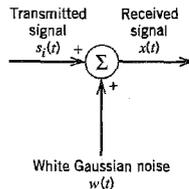


FIGURE 5.2 Additive white Gaussian noise (AWGN) model of a channel.

5.2 Geometric Representation of Signals

The essence of *geometric representation of signals*¹ is to represent any set of M energy signals $\{s_i(t)\}$ as linear combinations of N *orthonormal basis functions*, where $N \leq M$. That is to say, given a set of real-valued energy signals $s_1(t), s_2(t), \dots, s_M(t)$, each of duration T seconds, we write

$$s_i(t) = \sum_{j=1}^N s_{ij} \phi_j(t), \quad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \dots, M \end{cases} \quad (5.5)$$

where the coefficients of the expansion are defined by

$$s_{ij} = \int_0^T s_i(t) \phi_j(t) dt, \quad \begin{cases} i = 1, 2, \dots, M \\ j = 1, 2, \dots, N \end{cases} \quad (5.6)$$

The real-valued basis functions $\phi_1(t), \phi_2(t), \dots, \phi_N(t)$ are *orthonormal*, by which we mean

$$\int_0^T \phi_i(t) \phi_j(t) dt = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (5.7)$$

where δ_{ij} is the *Kronecker delta*. The first condition of Equation (5.7) states that each basis function is *normalized* to have unit energy. The second condition states that the basis functions $\phi_1(t), \phi_2(t), \dots, \phi_N(t)$ are *orthogonal* with respect to each other over the interval $0 \leq t \leq T$.

The set of coefficients $\{s_{ij}\}_{j=1}^N$ may naturally be viewed as an N -dimensional vector, denoted by s_i . The important point to note here is that the vector s_i bears a *one-to-one* relationship with the transmitted signal $s_i(t)$:

- ▶ Given the N elements of the vectors s_i (i.e., $s_{i1}, s_{i2}, \dots, s_{iN}$) operating as input, we may use the scheme shown in Figure 5.3a to generate the signal $s_i(t)$, which follows

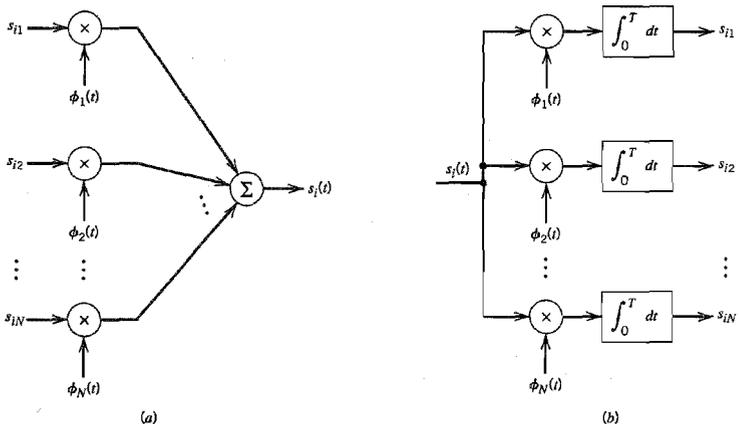


FIGURE 5.3 (a) Synthesizer for generating the signal $s_i(t)$. (b) Analyzer for generating the set of signal vectors $\{s_i\}$.

directly from Equation (5.5). It consists of a bank of N multipliers, with each multiplier having its own basis function, followed by a summer. This scheme may be viewed as a *synthesizer*.

- Conversely, given the signals $s_i(t)$, $i = 1, 2, \dots, M$, operating as input, we may use the scheme shown in Figure 5.3b to calculate the coefficients $s_{i1}, s_{i2}, \dots, s_{iN}$ which follows directly from Equation (5.6). This second scheme consists of a bank of N *product-integrators* or *correlators* with a common input, and with each one of them supplied with its own basis function. The scheme of Figure 5.3b may be viewed as an *analyzer*.

Accordingly, we may state that each signal in the set $\{s_i(t)\}$ is completely determined by the *vector* of its coefficients

$$s_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \quad i = 1, 2, \dots, M \quad (5.8)$$

The vector s_i is called a *signal vector*. Furthermore, if we conceptually extend our conventional notion of two- and three-dimensional Euclidean spaces to an N -dimensional Euclidean space, we may visualize the set of signal vectors $\{s_i | i = 1, 2, \dots, M\}$ as defining a corresponding set of M points in an N -dimensional Euclidean space, with N mutually perpendicular axes labeled $\phi_1, \phi_2, \dots, \phi_N$. This N -dimensional Euclidean space is called the *signal space*.

The idea of visualizing a set of energy signals geometrically, as just described, is of profound importance. It provides the mathematical basis for the geometric representation of energy signals, thereby paving the way for the noise analysis of digital communication systems in a conceptually satisfying manner. This form of representation is illustrated in Figure 5.4 for the case of a two-dimensional signal space with three signals, that is, $N = 2$ and $M = 3$.

In an N -dimensional Euclidean space, we may define *lengths* of vectors and *angles* between vectors. It is customary to denote the length (also called the *absolute value* or *norm*) of a signal vector s_i by the symbol $\|s_i\|$. The squared-length of any signal vector s_i is defined to be the *inner product* or *dot product* of s_i with itself, as shown by

$$\begin{aligned} \|s_i\|^2 &= s_i^T s_i \\ &= \sum_{j=1}^N s_{ij}^2, \quad i = 1, 2, \dots, M \end{aligned} \quad (5.9)$$

where s_{ij} is the j th element of s_i , and the superscript T denotes matrix transposition.

There is an interesting relationship between the energy content of a signal and its representation as a vector. By definition, the energy of a signal $s_i(t)$ of duration T seconds is

$$E_i = \int_0^T s_i^2(t) dt \quad (5.10)$$

Therefore, substituting Equation (5.5) into (5.10), we get

$$E_i = \int_0^T \left[\sum_{j=1}^N s_{ij} \phi_j(t) \right] \left[\sum_{k=1}^N s_{ik} \phi_k(t) \right] dt$$

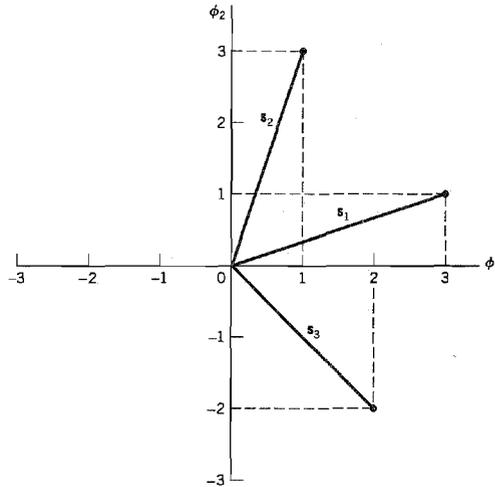


FIGURE 5.4 Illustrating the geometric representation of signals for the case when $N = 2$ and $M = 3$.

Interchanging the order of summation and integration, and then rearranging terms, we get

$$E_i = \sum_{j=1}^N \sum_{k=1}^N s_{ij}s_{ik} \int_0^T \phi_j(t)\phi_k(t)dt \quad (5.11)$$

But since the $\phi_j(t)$ form an orthonormal set, in accordance with the two conditions of Equation (5.7), we find that Equation (5.11) reduces simply to

$$\begin{aligned} E_i &= \sum_{j=1}^N s_{ij}^2 \\ &= \|s_i\|^2 \end{aligned} \quad (5.12)$$

Thus Equations (5.9) and (5.12) show that the energy of a signal $s_i(t)$ is equal to the squared length of the signal vector $s_i(t)$ representing it.

In the case of a pair of signals $s_i(t)$ and $s_k(t)$, represented by the signal vectors s_i and s_k , respectively, we may also show that

$$\int_0^T s_i(t)s_k(t) dt = s_i^T s_k \quad (5.13)$$

Equation (5.13) states that the *inner product* of the signals $s_i(t)$ and $s_k(t)$ over the interval $[0, T]$, using their time-domain representations, is equal to the inner product of their respective vector representations s_i and s_k . Note that the inner product of $s_i(t)$ and $s_k(t)$ is *invariant* to the choice of basis functions $\{\phi_j(t)\}_{j=1}^N$ in that it only depends on the components of the signals $s_i(t)$ and $s_k(t)$ projected onto each of the basis functions.

Yet another useful relation involving the vector representations of the signals $s_i(t)$ and $s_k(t)$ is described by

$$\begin{aligned}\|s_i - s_k\|^2 &= \sum_{j=1}^N (s_{ij} - s_{kj})^2 \\ &= \int_0^T (s_i(t) - s_k(t))^2 dt\end{aligned}\quad (5.14)$$

where $\|s_i - s_k\|$ is the *Euclidean distance*, d_{iks} , between the points represented by the signal vectors s_i and s_k .

To complete the geometric representation of energy signals, we need to have a representation for the angle θ_{ik} subtended between two signal vectors s_i and s_k . By definition, the *cosine of the angle* θ_{ik} is equal to the inner product of these two vectors divided by the product of their individual norms, as shown by

$$\cos \theta_{ik} = \frac{s_i^T s_k}{\|s_i\| \|s_k\|}\quad (5.15)$$

The two vectors s_i and s_k are thus *orthogonal* or *perpendicular* to each other if their inner product $s_i^T s_k$ is zero, in which case $\theta_{ik} = 90$ degrees; this condition is intuitively satisfying.

► EXAMPLE 5.1 Schwarz Inequality

Consider any pair of energy signals $s_1(t)$ and $s_2(t)$. The *Schwarz inequality* states that

$$\left(\int_{-\infty}^{\infty} s_1(t)s_2(t)dt \right)^2 \leq \left(\int_{-\infty}^{\infty} s_1^2(t)dt \right) \left(\int_{-\infty}^{\infty} s_2^2(t)dt \right)\quad (5.16)$$

The equality holds if and only if $s_2(t) = cs_1(t)$, where c is any constant.

To prove this important inequality, let $s_1(t)$ and $s_2(t)$ be expressed in terms of the pair of orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ as follows:

$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t)$$

$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t)$$

where $\phi_1(t)$ and $\phi_2(t)$ satisfy the orthonormality conditions over the entire time interval $(-\infty, \infty)$:

$$\int_{-\infty}^{\infty} \phi_i(t)\phi_j(t)dt = \delta_{ij} = \begin{cases} 1 & \text{for } j = i \\ 0 & \text{otherwise} \end{cases}$$

On this basis, we may represent the signals $s_1(t)$ and $s_2(t)$ by the following respective pair of vectors, as illustrated in Figure 5.5:

$$s_1 = \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix}$$

$$s_2 = \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix}$$

From Figure 5.5 we readily see that angle θ subtended between the vectors s_1 and s_2 is

$$\begin{aligned}\cos \theta &= \frac{s_1^T s_2}{\|s_1\| \|s_2\|} \\ &= \frac{\int_{-\infty}^{\infty} s_1(t)s_2(t)dt}{\left(\int_{-\infty}^{\infty} s_1^2(t)dt \right)^{1/2} \left(\int_{-\infty}^{\infty} s_2^2(t)dt \right)^{1/2}}\end{aligned}\quad (5.17)$$

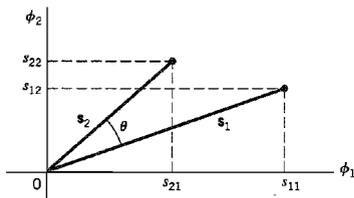


FIGURE 5.5 Vector representations of signals $s_1(t)$ and $s_2(t)$, providing the background picture for proving the Schwarz inequality.

where we have made use of Equations (5.15), (5.13) and (5.9). Recognizing that $|\cos \theta| \leq 1$, the Schwarz inequality of Equation (5.16) immediately follows from Equation (5.17). Moreover, from the first line of Equation (5.17) we note that $|\cos \theta| = 1$ if and only if $s_2 = cs_1$, that is, $s_2(t) = cs_1(t)$, where c is an arbitrary constant.

The proof of the Schwarz inequality, as presented here, applies to real-valued signals. It may be readily extended to complex-valued signals, in which case Equation (5.16) is reformulated as

$$\left| \int_{-\infty}^{\infty} s_1(t)s_2^*(t)dt \right|^2 \leq \left(\int_{-\infty}^{\infty} |s_1(t)|^2 dt \right) \left(\int_{-\infty}^{\infty} |s_2(t)|^2 dt \right) \quad (5.18)$$

where the equality holds if and only if $s_2(t) = cs_1(t)$, where c is a constant; see Problem 5.9. It is the complex form of the Schwarz inequality that was used in Chapter 4 to derive the matched filter. \blacktriangleleft

■ GRAM-SCHMIDT ORTHOGONALIZATION PROCEDURE

Having demonstrated the elegance of the geometric representation of energy signals, how do we justify it in mathematical terms? The answer lies in the *Gram-Schmidt orthogonalization procedure*, for which we need a *complete orthonormal set of basis functions*. To proceed with the formulation of this procedure, suppose we have a set of M energy signals denoted by $s_1(t)$, $s_2(t)$, \dots , $s_M(t)$. Starting with $s_1(t)$ chosen from this set arbitrarily, the first basis function is defined by

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \quad (5.19)$$

where E_1 is the energy of the signal $s_1(t)$. Then, clearly, we have

$$\begin{aligned} s_1(t) &= \sqrt{E_1}\phi_1(t) \\ &= s_{11}\phi_1(t) \end{aligned} \quad (5.20)$$

where the coefficient $s_{11} = \sqrt{E_1}$ and $\phi_1(t)$ has unit energy, as required.

Next, using the signal $s_2(t)$, we define the coefficient s_{21} as

$$s_{21} = \int_0^T s_2(t)\phi_1(t)dt \quad (5.21)$$

We may thus introduce a new intermediate function

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) \quad (5.22)$$

which is orthogonal to $\phi_1(t)$ over the interval $0 \leq t \leq T$ by virtue of Equation (5.21) and the fact that the basis function $\phi_1(t)$ has unit energy. Now, we are ready to define the second basis function as

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t) dt}} \quad (5.23)$$

Substituting Equation (5.22) into (5.23) and simplifying, we get the desired result

$$\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} \quad (5.24)$$

where E_2 is the energy of the signal $s_2(t)$. It is clear from Equation (5.23) that

$$\int_0^T \phi_2^2(t) dt = 1$$

and from Equation (5.24) that

$$\int_0^T \phi_1(t)\phi_2(t) dt = 0$$

That is to say, $\phi_1(t)$ and $\phi_2(t)$ form an orthonormal pair, as required.

Continuing in this fashion, we may in general define

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t) \quad (5.25)$$

where the coefficients s_{ij} are themselves defined by

$$s_{ij} = \int_0^T s_i(t)\phi_j(t) dt, \quad j = 1, 2, \dots, i-1 \quad (5.26)$$

Equation (5.22) is a special case of Equation (5.25) with $i = 2$. Note also that for $i = 1$, the function $g_i(t)$ reduces to $s_i(t)$.

Given the $g_i(t)$, we may now define the set of basis functions

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t) dt}}, \quad i = 1, 2, \dots, N \quad (5.27)$$

which form an orthonormal set. The dimension N is less than or equal to the number of given signals, M , depending on one of two possibilities:

- ▶ The signals $s_1(t), s_2(t), \dots, s_M(t)$ form a *linearly independent set*, in which case $N = M$.
- ▶ The signals $s_1(t), s_2(t), \dots, s_M(t)$ are *not* linearly independent, in which case $N < M$, and the intermediate function $g_i(t)$ is zero for $i > N$.

TABLE 5.1 Amplitude Levels of the 2B1Q Code

Signal		Gray code
Symbol	Amplitude	
$s_1(t)$	-3	00
$s_2(t)$	-1	01
$s_3(t)$	+1	11
$s_4(t)$	+3	10

Note that the conventional Fourier series expansion of a periodic signal is an example of a particular expansion of the type described herein. Also, the representation of a band-limited signal in terms of its samples taken at the Nyquist rate may be viewed as another sample of a particular expansion of this type. However, two important distinctions should be made:

1. The form of the basis functions $\phi_1(t)$, $\phi_2(t)$, \dots , $\phi_N(t)$ has not been specified. That is to say, unlike the Fourier series expansion of a periodic signal or the sampled representation of a band-limited signal, we have not restricted the Gram-Schmidt orthogonalization procedure to be in terms of sinusoidal functions or sinc functions of time.
2. The expansion of the signal $s_i(t)$ in terms of a finite number of terms is not an approximation wherein only the first N terms are significant but rather an *exact* expression where N and only N terms are significant.

EXAMPLE 5.2 2B1Q Code

The 2B1Q code was described in Chapter 4 as the North American line code for digital subscriber lines. It represents a quaternary PAM signal as shown in the Gray-encoded alphabet of Table 5.1. The four possible signals, $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$, are amplitude-scaled versions of a Nyquist pulse. Each signal represents a dibit. We wish to find the vector representation of the 2B1Q code.

This example is simple enough for us to solve it by inspection. Let $\phi_1(t)$ denote the Nyquist pulse, normalized to have unit energy. The $\phi_1(t)$ so defined is the only basis function for the vector representation of the 2B1Q code. Accordingly, the signal-space representation of this code is as shown in Figure 5.6. It consists of four signal vectors s_1 , s_2 , s_3 , and s_4 , which are located on the ϕ_1 -axis in a symmetric manner about the origin. In this example, we thus have $M = 4$ and $N = 1$.

We may generalize the result depicted in Figure 5.6 for the 2B1Q code as follows. The signal-space diagram of an M -ary pulse-amplitude modulated signal, in general, is one-dimensional with M signal points uniformly positioned on the only axis of the diagram. \blacktriangleleft

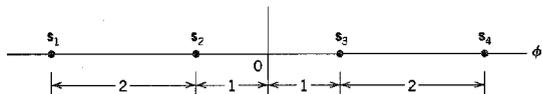


FIGURE 5.6 Signal-space representation of the 2B1Q code.

5.3 Conversion of the Continuous AWGN Channel into a Vector Channel

Suppose that the input to the bank of N product integrators or correlators in Figure 5.3b is not the transmitted signal $s_i(t)$ but rather the received signal $x(t)$ defined in accordance with the idealized AWGN channel of Figure 5.2. That is to say,

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \dots, M \end{cases} \quad (5.28)$$

where $w(t)$ is a sample function of a white Gaussian noise process $W(t)$ of zero mean and power spectral density $N_0/2$. Correspondingly, we find that the output of correlator j , say, is the sample value of a random variable X_j , as shown by

$$\begin{aligned} x_j &= \int_0^T x(t)\phi_j(t)dt \\ &= s_{ij} + w_j, \quad j = 1, 2, \dots, N \end{aligned} \quad (5.29)$$

The first component, s_{ij} , is a deterministic quantity contributed by the transmitted signal $s_i(t)$; it is defined by

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt \quad (5.30)$$

The second component, w_j , is the sample value of a random variable W_j that arises because of the presence of the channel noise $w(t)$; it is defined by

$$w_j = \int_0^T w(t)\phi_j(t)dt \quad (5.31)$$

Consider next a new random process $X'(t)$ whose sample function $x'(t)$ is related to the received signal $x(t)$ as follows:

$$x'(t) = x(t) - \sum_{j=1}^N x_j\phi_j(t) \quad (5.32)$$

Substituting Equations (5.28) and (5.29) into (5.32), and then using the expansion of Equation (5.5), we get

$$\begin{aligned} x'(t) &= s_i(t) + w(t) - \sum_{j=1}^N (s_{ij} + w_j)\phi_j(t) \\ &= w(t) - \sum_{j=1}^N w_j\phi_j(t) \\ &= w'(t) \end{aligned} \quad (5.33)$$

The sample function $x'(t)$ therefore depends solely on the channel noise $w(t)$. On the basis of Equations (5.32) and (5.33), we may thus express the received signal as

$$\begin{aligned} x(t) &= \sum_{j=1}^N x_j\phi_j(t) + x'(t) \\ &= \sum_{j=1}^N x_j\phi_j(t) + w'(t) \end{aligned} \quad (5.34)$$

Accordingly, we may view $w'(t)$ as a sort of *remainder* term that must be included on the right to preserve the equality in Equation (5.34). It is informative to contrast the expansion

of the received signal $x(t)$ given in Equation (5.34) with the corresponding expansion of the transmitted signal $s_i(t)$ given in Equation (5.5). The latter expansion is entirely deterministic, whereas that of Equation (5.34) is random (stochastic), which is to be expected.

STATISTICAL CHARACTERIZATION OF THE CORRELATOR OUTPUTS

We now wish to develop a statistical characterization of the set of N correlator outputs. Let $X(t)$ denote the random process, a sample function of which is represented by the received signal $x(t)$. Correspondingly, let X_j denote the random variable whose sample value is represented by the correlator output x_j , $j = 1, 2, \dots, N$. According to the AWGN model of Figure 5.2, the random process $X(t)$ is a Gaussian process. It follows therefore that X_j is a Gaussian random variable for all j (see Property 1 of a Gaussian process, Section 1.8). Hence, X_j is characterized completely by its mean and variance, which are determined next.

Let W_j denote the random variable represented by the sample value w_j produced by the j th correlator in response to the white Gaussian noise component $w(t)$. The random variable W_j has zero mean, because the noise process $W(t)$ represented by $w(t)$ in the AWGN model of Figure 5.2 has zero mean by definition. Consequently, the mean of X_j depends only on s_{ij} , as shown by

$$\begin{aligned}\mu_{x_j} &= E[X_j] \\ &= E[s_{ij} + W_j] \\ &= s_{ij} + E[W_j] \\ &= s_{ij}\end{aligned}\tag{5.35}$$

To find the variance of X_j , we note that

$$\begin{aligned}\sigma_{x_j}^2 &= \text{var}[X_j] \\ &= E[(X_j - s_{ij})^2] \\ &= E[W_j^2]\end{aligned}\tag{5.36}$$

where the last line follows from Equation (5.29) with x_j and w_j replaced by X_j and W_j , respectively. According to Equation (5.31), the random variable W_j is defined by

$$W_j = \int_0^T W(t)\phi_j(t)dt$$

We may therefore expand Equation (5.36) as follows:

$$\begin{aligned}\sigma_{x_j}^2 &= E\left[\int_0^T W(t)\phi_j(t)dt \int_0^T W(u)\phi_j(u)du\right] \\ &= E\left[\int_0^T \int_0^T \phi_j(t)\phi_j(u)W(t)W(u)dtdu\right]\end{aligned}\tag{5.37}$$

Interchanging the order of integration and expectation:

$$\begin{aligned}\sigma_{x_j}^2 &= \int_0^T \int_0^T \phi_j(t)\phi_j(u)E[W(t)W(u)]dtdu \\ &= \int_0^T \int_0^T \phi_j(t)\phi_j(u)R_w(t, u)dtdu\end{aligned}\tag{5.38}$$

where $R_W(t, u)$ is the autocorrelation function of the noise process $W(t)$. Since this noise is stationary, $R_W(t, u)$ depends only on the time difference $t - u$. Furthermore, since the noise $W(t)$ is white with a constant power spectral density $N_0/2$, we may express $R_W(t, u)$ as follows [see Equation (1.95)]:

$$R_W(t, u) = \frac{N_0}{2} \delta(t - u) \quad (5.39)$$

Therefore, substituting Equation (5.39) into (5.38), and then using the sifting property of the delta function $\delta(t)$, we get

$$\begin{aligned} \sigma_{X_j}^2 &= \frac{N_0}{2} \int_0^T \int_0^T \phi_j(t) \phi_j(u) \delta(t - u) dt du \\ &= \frac{N_0}{2} \int_0^T \phi_j^2(t) dt \end{aligned} \quad (5.40)$$

Since the $\phi_j(t)$ have unit energy, by definition, we finally get the simple result

$$\sigma_{X_j}^2 = \frac{N_0}{2} \quad \text{for all } j \quad (5.41)$$

This important result shows that all the correlator outputs denoted by X_j , with $j = 1, 2, \dots, N$, have a variance equal to the power spectral density $N_0/2$ of the noise process $W(t)$.

Moreover, since the $\phi_j(t)$ form an orthogonal set, we find that the X_j are mutually uncorrelated, as shown by

$$\begin{aligned} \text{cov}[X_j, X_k] &= E[(X_j - \mu_{X_j})(X_k - \mu_{X_k})] \\ &= E[(X_j - s_{\hat{y}})(X_k - s_{ik})] \\ &= E[W_j W_k] \\ &= E \left[\int_0^T W(t) \phi_j(t) dt \int_0^T W(u) \phi_k(u) du \right] \\ &= \int_0^T \int_0^T \phi_j(t) \phi_k(u) R_W(t, u) dt du \\ &= \frac{N_0}{2} \int_0^T \int_0^T \phi_j(t) \phi_k(u) \delta(t - u) dt du \\ &= \frac{N_0}{2} \int_0^T \phi_j(t) \phi_k(t) dt \\ &= 0, \quad j \neq k \end{aligned} \quad (5.42)$$

Since the X_j are Gaussian random variables, Equation (5.42) implies that they are also statistically independent (see Property 4 of a Gaussian Process, Section 1.8).

Define the vector of N random variables

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \quad (5.43)$$

whose elements are independent Gaussian random variables with mean values equal to $s_{\hat{y}}$ and variances equal to $N_0/2$. Since the elements of the vector \mathbf{X} are statistically independent

dent, we may express the conditional probability density function of the vector \mathbf{X} , given that the signal $s_i(t)$ or correspondingly the symbol m_i was transmitted, as the product of the conditional probability density functions of its individual elements as shown by

$$f_{\mathbf{X}}(\mathbf{x} | m_i) = \prod_{j=1}^N f_{X_j}(x_j | m_i), \quad i = 1, 2, \dots, M \quad (5.44)$$

where the vector \mathbf{x} and scalar x_j are sample values of the random vector \mathbf{X} and random variable X_j , respectively. The vector \mathbf{x} is called the *observation vector*; correspondingly, x_j is called an *observable element*. Any channel that satisfies Equation (5.44) is called a *memoryless channel*.

Since each X_j is a Gaussian random variable with mean s_{ij} and variance $N_0/2$, we have

$$f_{X_j}(x_j | m_i) = \frac{1}{\sqrt{\pi N_0}} \exp \left[-\frac{1}{N_0} (x_j - s_{ij})^2 \right], \quad \begin{array}{l} j = 1, 2, \dots, N \\ i = 1, 2, \dots, M \end{array} \quad (5.45)$$

Therefore, substituting Equation (5.45) into (5.44) yields

$$f_{\mathbf{X}}(\mathbf{x} | m_i) = (\pi N_0)^{-N/2} \exp \left[-\frac{1}{N_0} \sum_{j=1}^N (x_j - s_{ij})^2 \right], \quad i = 1, 2, \dots, M \quad (5.46)$$

It is now clear that the elements of the random vector \mathbf{X} completely characterize the summation term $\sum_j X_j \phi_j(t)$, whose sample value is represented by the first term in Equation (5.34). However, there remains the noise term $w'(t)$ in this equation, which depends only on the channel noise $w(t)$. Since the noise process $W(t)$ represented by $w(t)$ is Gaussian with zero mean, it follows that the noise process $W'(t)$ represented by the sample function $w'(t)$ is also a zero-mean Gaussian process. Finally, we note that any random variable $W'(t_k)$, say, derived from the noise process $W'(t)$ by sampling it at time t_k , is in fact statistically independent of the set of random variables $\{X_j\}$; that is to say (see Problem 5.10),

$$E[X_j W'(t_k)] = 0, \quad \begin{cases} j = 1, 2, \dots, N \\ 0 \leq t_k \leq T \end{cases} \quad (5.47)$$

Since any random variable based on the remainder noise process $W'(t)$ is independent of the set of random variables $\{X_j\}$ as well as the set of transmitted signals $\{s_i(t)\}$, Equation (5.47) states that the random variable $W'(t_k)$ is irrelevant to the decision as to which particular signal was actually transmitted. In other words, the correlator outputs determined by the received signal $x(t)$ are the only data that are useful for the decision-making process and, hence, represent *sufficient statistics* for the problem at hand. By definition, sufficient statistics summarize the whole of the relevant information supplied by an observation vector.

We may now summarize the results presented in this section by formulating the *theorem of irrelevance*:

Insofar as signal detection in additive white Gaussian noise is concerned, only the projections of the noise onto the basis functions of the signal set $\{s_i(t)\}_{i=1}^M$ affects the sufficient statistics of the detection problem; the remainder of the noise is irrelevant.

As a corollary to this theorem, we may state that the AWGN channel of Figure 5.2 is equivalent to an N -dimensional *vector channel* described by the observation vector

$$\mathbf{x} = \mathbf{s}_i + \mathbf{w}, \quad i = 1, 2, \dots, M \quad (5.48)$$

where the dimension N is the number of basis functions involved in formulating the signal vector \mathbf{s}_i . The individual components of the signal vector \mathbf{s}_i and noise vector \mathbf{w} are defined by Equations (5.6) and (5.31), respectively. The theorem of irrelevance and its corollary are indeed basic to the understanding of the signal detection problem as described next.

5.4 Likelihood Functions

The conditional probability density functions $f_{\mathbf{x}}(\mathbf{x}|m_i)$, $i = 1, 2, \dots, M$, are the very characterization of an AWGN channel. Their derivation leads to a functional dependence on the observation vector \mathbf{x} , given the transmitted message symbol m_i . However, at the receiver we have the exact opposite situation: We are given the observation vector \mathbf{x} and the requirement is to estimate the message symbol m_i that is responsible for generating \mathbf{x} . To emphasize this latter viewpoint, we introduce the idea of a *likelihood function*, denoted by $L(m_i)$ and defined by

$$L(m_i) = f_{\mathbf{x}}(\mathbf{x}|m_i), \quad i = 1, 2, \dots, M \quad (5.49)$$

It is important however to recognize that although the $L(m_i)$ and $f_{\mathbf{x}}(\mathbf{x}|m_i)$ have exactly the same mathematical form, their individual meanings are different.

In practice, we find it more convenient to work with the *log-likelihood function*, denoted by $l(m_i)$ and defined by

$$l(m_i) = \log L(m_i), \quad i = 1, 2, \dots, M \quad (5.50)$$

The log-likelihood function bears a one-to-one relationship to the likelihood function for two reasons:

1. By definition, a probability density function is always nonnegative. It follows therefore that the likelihood function is likewise a nonnegative quantity.
2. The logarithmic function is a monotonically increasing function of its argument.

The use of Equation (5.46) in (5.50) yields the log-likelihood functions for an AWGN channel as

$$l(m_i) = -\frac{1}{N_0} \sum_{j=1}^N (x_j - s_{ij})^2, \quad i = 1, 2, \dots, M \quad (5.51)$$

where we have ignored the constant term $-(N/2) \log(\pi N_0)$ as it bears no relation whatsoever to the message symbol m_i . Note that the s_{ij} , $j = 1, 2, \dots, N$, are the elements of the signal vector \mathbf{s}_i representing the message symbol m_i . With Equation (5.51) at our disposal, we are now ready to address the basic receiver design problem.

5.5 Coherent Detection of Signals in Noise: Maximum Likelihood Decoding

Suppose that in each time slot of duration T seconds, one of the M possible signals $s_1(t)$, $s_2(t)$, \dots , $s_M(t)$ is transmitted with equal probability, $1/M$. For geometric signal representation, the signal $s_i(t)$, $i = 1, 2, \dots, M$, is applied to a bank of correlators, with a common input and supplied with an appropriate set of N orthonormal basis functions. The resulting correlator outputs define the *signal vector* \mathbf{s}_i . Since knowledge of the signal vector \mathbf{s}_i is as good as knowing the transmitted signal $s_i(t)$ itself, and vice versa, we may represent $s_i(t)$ by a point in a Euclidean space of dimension $N \leq M$. We refer to this point as the *trans-*

mitted signal point or message point. The set of message points corresponding to the set of transmitted signals $\{s_i(t)\}_{i=1}^M$ is called a *signal constellation*.

However, the representation of the received signal $x(t)$ is complicated by the presence of additive noise $w(t)$. We note that when the received signal $x(t)$ is applied to the bank of N correlators, the correlator outputs define the observation vector \mathbf{x} . From Equation (5.48), the vector \mathbf{x} differs from the signal vector \mathbf{s}_i by the *noise vector* \mathbf{w} whose orientation is completely random. The noise vector \mathbf{w} is completely characterized by the noise $w(t)$; the converse of this statement, however, is not true. The noise vector \mathbf{w} represents that portion of the noise $w(t)$ that will interfere with the detection process; the remaining portion of this noise, denoted by $w'(t)$, is tuned out by the bank of correlators.

Now, based on the observation vector \mathbf{x} , we may represent the received signal $x(t)$ by a point in the same Euclidean space used to represent the transmitted signal. We refer to this second point as the *received signal point*. The received signal point wanders about the message point in a completely random fashion, in the sense that it may lie anywhere inside a Gaussian-distributed "cloud" centered on the message point. This is illustrated in Figure 5.7a for the case of a three-dimensional signal space. For a particular realization of the noise vector \mathbf{w} (i.e., a particular point inside the random cloud of Figure 5.7a), the relationship between the observation vector \mathbf{x} and the signal vector \mathbf{s}_i is as illustrated in Figure 5.7b.

We are now ready to state the signal detection problem:

Given the observation vector \mathbf{x} , perform a mapping from \mathbf{x} to an estimate \hat{m} of the transmitted symbol, m_i , in a way that would minimize the probability of error in the decision-making process.

Suppose that, given the observation vector \mathbf{x} , we make the decision $\hat{m} = m_i$. The probability of error in this decision, which we denote by $P_e(m_i | \mathbf{x})$, is simply

$$\begin{aligned} P_e(m_i | \mathbf{x}) &= P(m_i \text{ not sent} | \mathbf{x}) \\ &= 1 - P(m_i \text{ sent} | \mathbf{x}) \end{aligned} \quad (5.52)$$

The decision-making criterion is to minimize the probability of error in mapping each given observation vector \mathbf{x} into a decision. On the basis of Equation (5.52), we may therefore state the *optimum decision rule*:

$$\begin{aligned} &\text{Set } \hat{m} = m_i \text{ if} \\ &P(m_i \text{ sent} | \mathbf{x}) \geq P(m_k \text{ sent} | \mathbf{x}) \quad \text{for all } k \neq i \end{aligned} \quad (5.53)$$

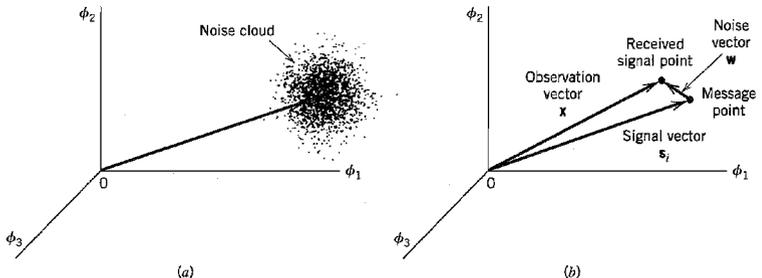


FIGURE 5.7 Illustrating the effect of noise perturbation, depicted in (a), on the location of the received signal point, depicted in (b).

where $k = 1, 2, \dots, M$. This decision rule is referred to as the *maximum a posteriori probability (MAP) rule*.

The condition of Equation (5.53) may be expressed more explicitly in terms of the *a priori* probabilities of the transmitted signals and in terms of the likelihood functions. Using Bayes' rule in Equation (5.53), and for the moment ignoring possible ties in the decision-making process, we may restate the MAP rule as follows:

$$\begin{aligned} &\text{Set } \hat{m} = m_i \text{ if} \\ &\frac{p_k f_{\mathbf{X}}(\mathbf{x} | m_k)}{f_{\mathbf{X}}(\mathbf{x})} \text{ is maximum for } k = i \end{aligned} \quad (5.54)$$

where p_k is the *a priori* probability of transmitting symbol m_k , $f_{\mathbf{X}}(\mathbf{x} | m_k)$ is the conditional probability density function of the random observation vector \mathbf{X} given the transmission of symbol m_k , and $f_{\mathbf{X}}(\mathbf{x})$ is the unconditional probability density function of \mathbf{X} . In Equation (5.54) we may note the following:

- ▶ The denominator term $f_{\mathbf{X}}(\mathbf{x})$ is independent of the transmitted symbol.
- ▶ The *a priori* probability $p_k = p_i$ when all the source symbols are transmitted with equal probability.
- ▶ The conditional probability density function $f_{\mathbf{X}}(\mathbf{x} | m_k)$ bears a one-to-one relationship to the log-likelihood function $l(m_k)$.

Accordingly, we may restate the decision rule of Equation (5.54) in terms of $l(m_k)$ simply as follows:

$$\begin{aligned} &\text{Set } \hat{m} = m_i \text{ if} \\ &l(m_k) \text{ is maximum for } k = i \end{aligned} \quad (5.55)$$

This decision rule is referred to as the *maximum likelihood rule*, and the device for its implementation is correspondingly referred to as the *maximum likelihood decoder*. According to Equation (5.55), a maximum likelihood decoder computes the log-likelihood functions as metrics for all the M possible message symbols, compares them, and then decides in favor of the maximum. Thus the maximum likelihood decoder differs from the maximum *a posteriori* decoder in that it assumes equally likely message symbols.

It is useful to have a graphical interpretation of the maximum likelihood decision rule. Let Z denote the N -dimensional space of all possible observation vectors \mathbf{x} . We refer to this space as the *observation space*. Because we have assumed that the decision rule must say $\hat{m} = m_i$, where $i = 1, 2, \dots, M$, the total observation space Z is correspondingly partitioned into M -*decision regions*, denoted by Z_1, Z_2, \dots, Z_M . Accordingly, we may restate the decision rule of Equation (5.55) as follows:

$$\begin{aligned} &\text{Observation vector } \mathbf{x} \text{ lies in region } Z_i \text{ if} \\ &l(m_k) \text{ is maximum for } k = i \end{aligned} \quad (5.56)$$

Aside from the boundaries between the decision regions Z_1, Z_2, \dots, Z_M , it is clear that this set of regions covers the entire space of possible observation vectors \mathbf{x} . We adopt the convention that all ties are resolved at random; that is, the receiver simply makes a guess. Specifically, if the observation vector \mathbf{x} falls on the boundary between any two decision regions, Z_i and Z_k , say, the choice between the two possible decisions $\hat{m} = m_i$ and $\hat{m} = m_k$ is resolved *a priori* by the flip of a fair coin. Clearly, the outcome of such an event does not affect the ultimate value of the probability of error since, on this boundary, the condition of Equation (5.53) is satisfied with the equality sign.

The maximum likelihood decision rule of Equation (5.55) or its geometric counterpart described in Equation (5.56) is of a generic kind, with the channel noise $w(t)$ being additive as the only restriction imposed on it. We next specialize this rule for the case when $w(t)$ is both white and Gaussian.

From the log-likelihood function defined in Equation (5.51) for an AWGN channel we note that $l(m_k)$ attains its maximum value when the summation term

$$\sum_{j=1}^N (x_j - s_{kj})^2$$

is minimized by the choice $k = i$. Accordingly, we may formulate the maximum likelihood decision rule for an AWGN channel as

$$\begin{aligned} \text{Observation vector } \mathbf{x} \text{ lies in region } Z_i \text{ if} \\ \sum_{j=1}^N (x_j - s_{kj})^2 \text{ is minimum for } k = i \end{aligned} \quad (5.57)$$

Next, we note from our earlier discussion that (see Equation (5.14) for comparison)

$$\sum_{j=1}^N (x_j - s_{kj})^2 = \|\mathbf{x} - \mathbf{s}_k\|^2 \quad (5.58)$$

where $\|\mathbf{x} - \mathbf{s}_k\|$ is the Euclidean distance between the received signal point and message point, represented by the vectors \mathbf{x} and \mathbf{s}_k , respectively. Accordingly, we may restate the decision rule of Equation (5.57) as follows:

$$\begin{aligned} \text{Observation vector } \mathbf{x} \text{ lies in region } Z_i \text{ if} \\ \text{the Euclidean distance } \|\mathbf{x} - \mathbf{s}_k\| \text{ is minimum for } k = i \end{aligned} \quad (5.59)$$

Equation (5.59) states that *the maximum likelihood decision rule is simply to choose the message point closest to the received signal point*, which is intuitively satisfying.

In practice, the need for squarers in the decision rule of Equation (5.59) is avoided by recognizing that

$$\sum_{j=1}^N (x_j - s_{kj})^2 = \sum_{j=1}^N x_j^2 - 2 \sum_{j=1}^N x_j s_{kj} + \sum_{j=1}^N s_{kj}^2 \quad (5.60)$$

The first summation term of this expansion is independent of the index k and may therefore be ignored. The second summation term is the inner product of the observation vector \mathbf{x} and signal vector \mathbf{s}_k . The third summation term is the energy of the transmitted signal $s_k(t)$. Accordingly, we may formulate a decision rule equivalent to that of Equation (5.59) as follows:

$$\begin{aligned} \text{Observation vector } \mathbf{x} \text{ lies in region } Z_i \text{ if} \\ \sum_{j=1}^N x_j s_{kj} - \frac{1}{2} E_k \text{ is maximum for } k = i \end{aligned} \quad (5.61)$$

where E_k is the energy of the transmitted signal $s_k(t)$:

$$E_k = \sum_{j=1}^N s_{kj}^2 \quad (5.62)$$

From Equation (5.61) we deduce that, for an AWGN channel, the decision regions are regions of the N -dimensional observation space Z , bounded by linear [($N - 1$)-dimensional hyperplane] boundaries. Figure 5.8 shows the example of decision regions for

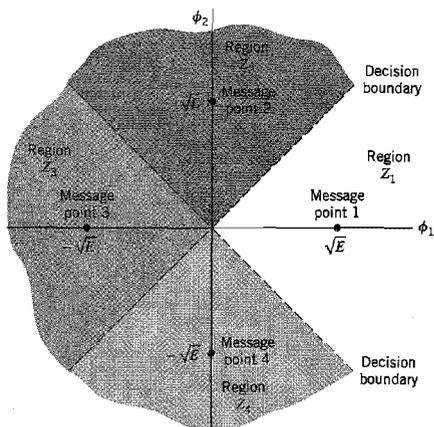


FIGURE 5.8 Illustrating the partitioning of the observation space into decision regions for the case when $N = 2$ and $M = 4$; it is assumed that the M transmitted symbols are equally likely.

$M = 4$ signals and $N = 2$ dimensions, assuming that the signals are transmitted with equal energy, E , and equal probability.

5.6 Correlation Receiver

From the material presented in the previous sections, we find that for an AWGN channel and for the case when the transmitted signals $s_1(t), s_2, \dots, s_M(t)$ are equally likely, the optimum receiver consists of two subsystems, which are detailed in Figure 5.9 and described here:

1. The *detector* part of the receiver is shown in Figure 5.9a. It consists of a bank of M *product-integrators* or *correlators*, supplied with a corresponding set of coherent reference signals or orthonormal basis functions $\phi_1(t), \phi_2(t), \dots, \phi_N(t)$ that are generated locally. This bank of correlators operates on the received signal $x(t)$, $0 \leq t \leq T$, to produce the observation vector \mathbf{x} .
2. The second part of the receiver, namely, the *signal transmission decoder* is shown in Figure 5.9b. It is implemented in the form of a maximum-likelihood decoder that operates on the observation vector \mathbf{x} to produce an estimate, \hat{m} , of the transmitted symbol $m_i, i = 1, 2, \dots, M$, in a way that would minimize the average probability of symbol error. In accordance with Equation (5.61), the N elements of the observation vector \mathbf{x} are first multiplied by the corresponding N elements of each of the M signal vectors s_1, s_2, \dots, s_M , and the resulting products are successively summed in accumulators to form the corresponding set of inner products $\{\mathbf{x}^T \mathbf{s}_k \mid k = 1, 2, \dots, M\}$. Next, the inner products are corrected for the fact that the transmitted signal energies may be unequal. Finally, the largest in the resulting set of numbers is selected, and an appropriate decision on the transmitted message is made.

The optimum receiver of Figure 5.9 is commonly referred to as a *correlation receiver*.

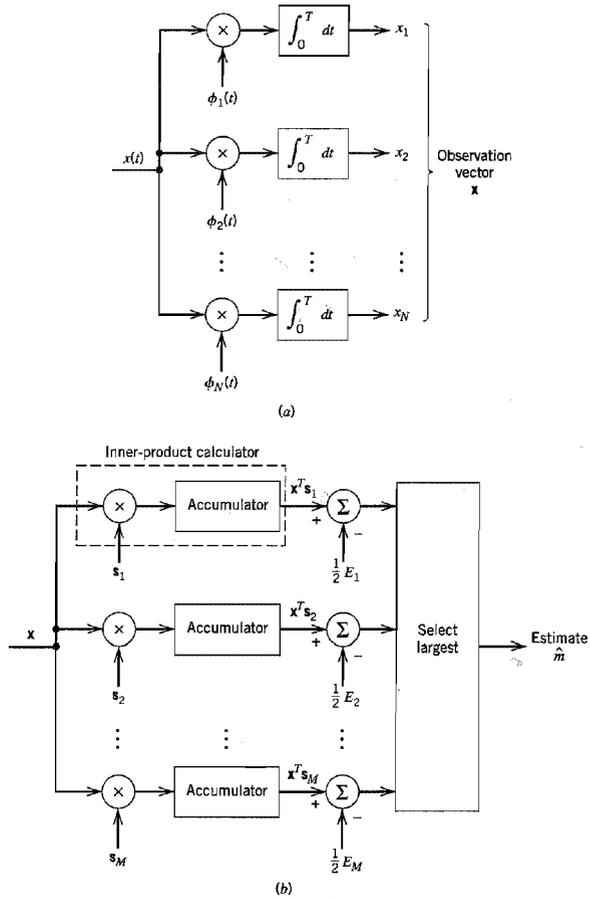


FIGURE 5.9 (a) Detector or demodulator. (b) Signal transmission decoder.

■ EQUIVALENCE OF CORRELATION AND MATCHED FILTER RECEIVERS

The detector shown in Figure 5.9a involves a set of correlators. Alternatively, we may use a corresponding set of *matched filters* to build the detector; the matched filter and its properties were considered in Section 4.2. To demonstrate the equivalence of a correlator and a matched filter, consider a linear time-invariant filter with impulse response $h_j(t)$. With the received signal $x(t)$ used as the filter input, the resulting filter output, $y_j(t)$, is defined by the convolution integral:

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau)h_j(t - \tau)d\tau \tag{5.63}$$

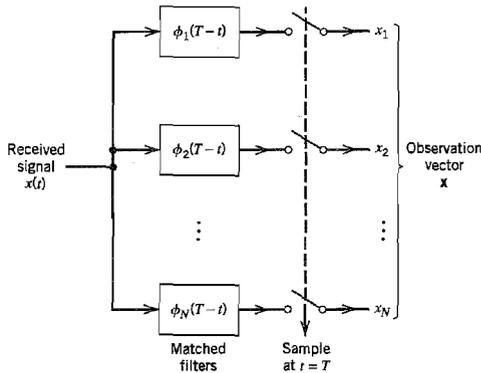


FIGURE 5.10 Detector part of matched filter receiver; the signal transmission decoder is as shown in Fig. 5.9b.

From the definition of a matched filter presented in Section 4.2, we recall that the impulse response $h_j(t)$ of a linear time-invariant filter matched to an input signal $\phi_j(t)$ is a time-reversed and delayed version of the input $\phi_j(t)$. Suppose that we set

$$h_j(t) = \phi_j(T - t) \quad (5.64)$$

Then the resulting filter output is

$$y_j(t) = \int_{-\infty}^{\infty} x(\tau) \phi_j(T - t + \tau) d\tau \quad (5.65)$$

Sampling this output at time $t = T$, we get

$$y_j(T) = \int_{-\infty}^{\infty} x(\tau) \phi_j(\tau) d\tau$$

Since, by definition, $\phi_j(t)$ is zero outside the interval $0 \leq t \leq T$, we find that $y_j(T)$ is in actual fact the j th correlator output x_j produced by the received signal $x(t)$ in Figure 5.9a, as shown by

$$y_j(T) = \int_0^T x(\tau) \phi_j(\tau) d\tau \quad (5.66)$$

Accordingly, the detector part of the optimum receiver may also be implemented using a bank of matched filters, as shown in Figure 5.10. It is important to note, however, that the output of each correlator in Figure 5.9a is equivalent to the output of a corresponding matched filter in Figure 5.10 only when that output is sampled at time $t = T$.

5.7 Probability of Error

To complete the statistical characterization of the correlation receiver depicted in Figure 5.9, we need to evaluate its noise performance. To do so, suppose that the observation space Z is partitioned, in accordance with the maximum likelihood decision rule, into a set of M regions $\{Z_i\}_{i=1}^M$. Suppose also that symbol m_i (or, equivalently, signal vector \mathbf{s}_i) is

transmitted, and an observation vector \mathbf{x} is received. Then an error occurs whenever the received signal point represented by \mathbf{x} does not fall inside region Z_i associated with the message point represented by s_i . Averaging over all possible transmitted symbols, we readily see that the *average probability of symbol error*, P_e is

$$\begin{aligned} P_e &= \sum_{i=1}^M p_i P(\mathbf{x} \text{ does not lie in } Z_i | m_i \text{ sent}) \\ &= \frac{1}{M} \sum_{i=1}^M P(\mathbf{x} \text{ does not lie in } Z_i | m_i \text{ sent}) \\ &= 1 - \frac{1}{M} \sum_{i=1}^M P(\mathbf{x} \text{ lies in } Z_i | m_i \text{ sent}) \end{aligned} \quad (5.67)$$

where we have used standard notation to denote the probability of an event and the conditional probability of an event. Since \mathbf{x} is the sample value of random vector \mathbf{X} , we may rewrite Equation (5.67) in terms of the likelihood function (when m_i is sent) as follows:

$$P_e = 1 - \frac{1}{M} \sum_{i=1}^M \int_{Z_i} f_{\mathbf{X}}(\mathbf{x} | m_i) d\mathbf{x} \quad (5.68)$$

For an N -dimensional observation vector, the integral in Equation (5.68) is likewise N -dimensional.

■ INVARIANCE OF THE PROBABILITY OF ERROR TO ROTATION AND TRANSLATION

The way in which the observation space Z is partitioned into the set of regions Z_1, Z_2, \dots, Z_M , in the maximum likelihood detection of a signal in additive white Gaussian noise, is uniquely defined by the signal constellation under study. Accordingly, changes in the orientation of the signal constellation with respect to both the coordinate axes and origin of the signal space do *not* affect the probability of symbol error P_e defined in Equation (5.68). This result is a consequence of two facts:

1. In maximum likelihood detection, the probability of symbol error P_e depends solely on the relative Euclidean distances between the message points in the constellation.
2. The additive white Gaussian noise is *spherically symmetric* in all directions in the signal space.

Consider first the invariance of P_e with respect to rotation. The effect of a rotation applied to all the message points in a constellation is equivalent to multiplying the N -dimensional signal vector s_i by an N -by- N *orthonormal matrix* denoted by \mathbf{Q} for all i . The matrix \mathbf{Q} satisfies the condition

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{I} \quad (5.69)$$

where \mathbf{I} is the *identity matrix* whose diagonal elements are all unity and its off-diagonal elements are all zero. Note that according to Equation (5.69), the inverse of a real-valued orthonormal matrix is equal to its transposed form. Thus the signal vector s_i is replaced by its rotated version

$$s_{i,\text{rotate}} = \mathbf{Q}s_i, \quad i = 1, 2, \dots, M \quad (5.70)$$

Correspondingly, the N -by-1 noise vector \mathbf{w} is replaced by its rotated version

$$\mathbf{w}_{\text{rotate}} = \mathbf{Q}\mathbf{w} \quad (5.71)$$

However, the statistical characteristics of the noise vector are unaffected by this rotation for the following reasons:

- ▶ From Chapter 1 we recall that a linear combination of Gaussian random variables is also Gaussian. Since the noise vector \mathbf{w} is Gaussian, by assumption, it follows that the rotated noise vector $\mathbf{w}_{\text{rotate}}$ is also Gaussian.
- ▶ Since the noise vector \mathbf{w} has zero mean, the rotated noise vector $\mathbf{w}_{\text{rotate}}$ also has zero mean, as shown by

$$\begin{aligned} E[\mathbf{w}_{\text{rotate}}] &= E[\mathbf{Q}\mathbf{w}] \\ &= \mathbf{Q}E[\mathbf{w}] \\ &= \mathbf{0} \end{aligned} \quad (5.72)$$

- ▶ The covariance matrix of the noise vector \mathbf{w} is equal to $(N_0/2)\mathbf{I}$, where $N_0/2$ is the power spectral density of the AWGN $w(t)$; that is,

$$E[\mathbf{w}\mathbf{w}^T] = \frac{N_0}{2} \mathbf{I} \quad (5.73)$$

Hence, the covariance matrix of the rotated noise vector $\mathbf{w}_{\text{rotate}}$ is

$$\begin{aligned} E[\mathbf{w}_{\text{rotate}}\mathbf{w}_{\text{rotate}}^T] &= E[\mathbf{Q}\mathbf{w}(\mathbf{Q}\mathbf{w})^T] \\ &= E[\mathbf{Q}\mathbf{w}\mathbf{w}^T\mathbf{Q}^T] \\ &= \mathbf{Q}E[\mathbf{w}\mathbf{w}^T]\mathbf{Q}^T \\ &= \frac{N_0}{2} \mathbf{Q}\mathbf{Q}^T \\ &= \frac{N_0}{2} \mathbf{I} \end{aligned} \quad (5.74)$$

where in the last two lines we have made use of Equations (5.73) and (5.69).

In light of these observations, we may express the observation vector for the rotated signal constellation as

$$\mathbf{x}_{\text{rotate}} = \mathbf{Q}\mathbf{s}_i + \mathbf{w}, \quad i = 1, 2, \dots, M \quad (5.75)$$

From Equation (5.59) we know that the decision rule for maximum likelihood detection is based on the Euclidean distance from the observation vector $\mathbf{x}_{\text{rotate}}$ to the rotated signal vector $\mathbf{s}_{i,\text{rotate}} = \mathbf{Q}\mathbf{s}_i$. Comparing Equation (5.75) to Equation (5.48), we readily see that

$$\|\mathbf{x}_{\text{rotate}} - \mathbf{s}_{i,\text{rotate}}\| = \|\mathbf{x} - \mathbf{s}_i\| \quad \text{for all } i \quad (5.76)$$

We may therefore formally state the *principle of rotational invariance* as follows:

If a signal constellation is rotated by an orthonormal transformation, that is,

$$\mathbf{s}_{i,\text{rotate}} = \mathbf{Q}\mathbf{s}_i, \quad i = 1, 2, \dots, M$$

where \mathbf{Q} is an orthonormal matrix, then the probability of symbol error P_e incurred in maximum likelihood signal detection over an AWGN channel is completely unchanged.

We illustrate this principle with an example. The signal constellation shown in Figure 5.11*b* is the same as that of Figure 5.11*a*, except that it has been rotated through 45 degrees. Although these two constellations do indeed look different, the principle of rotational invariance tells us immediately that the P_e is the same for both of them.

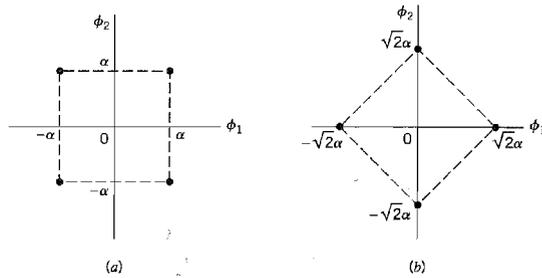


FIGURE 5.11 A pair of signal constellations for illustrating the principle of rotational invariance.

Consider next the issue of invariance to translation. Suppose all the message points in a signal constellation are translated by a constant vector amount \mathbf{a} , as shown by

$$\mathbf{s}_{i,\text{translate}} = \mathbf{s}_i - \mathbf{a}, \quad i = 1, 2, \dots, M \quad (5.77)$$

The observation vector is correspondingly translated by the same vector amount, as shown by

$$\mathbf{x}_{\text{translate}} = \mathbf{x} - \mathbf{a} \quad (5.78)$$

From Equations (5.77) and (5.78) we see that the translate \mathbf{a} is common to both the translated signal vector \mathbf{s}_i and translated observation vector \mathbf{x} . We therefore immediately deduce that

$$\|\mathbf{x}_{\text{translate}} - \mathbf{s}_{i,\text{translate}}\| = \|\mathbf{x} - \mathbf{s}_i\| \quad \text{for all } i \quad (5.79)$$

and thus formulate the *principle of translational invariance* as follows:

If a signal constellation is translated by a constant vector amount, then the probability of symbol error P_e incurred in maximum likelihood signal detection over an AWGN channel is completely unchanged.

As an example, consider the two signal constellations shown in Figure 5.12, which pertain to a pair of different 4-level PAM signals. The constellation of Figure 5.12b is the same as that of Figure 5.12a, except for a translation of $3\alpha/2$ to the right along the ϕ_1 -axis. The principle of translational invariance says that the P_e is the same for both of these constellations.

■ MINIMUM ENERGY SIGNALS

A useful application of the principle of translational invariance is in the translation of a given signal constellation in such a way that the average energy is minimized. To explore

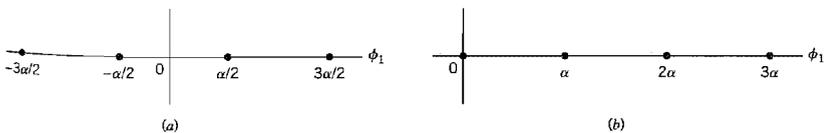


FIGURE 5.12 A pair of signal constellations for illustrating the principle of translational invariance.

this issue, consider a set of symbols m_1, m_2, \dots, m_M represented by the signal vectors s_1, s_2, \dots, s_M , respectively. The average energy of this signal constellation translated by a vector amount \mathbf{a} is

$$\mathcal{E}_{\text{translate}} = \sum_{i=1}^M \|s_i - \mathbf{a}\|^2 p_i \quad (5.80)$$

where p_i is the probability that symbol m_i is emitted by the source of information. The squared Euclidean distance between s_i and \mathbf{a} is expanded as

$$\|s_i - \mathbf{a}\|^2 = \|s_i\|^2 - 2\mathbf{a}^T s_i + \|\mathbf{a}\|^2$$

We may therefore rewrite Equation (5.80) in the expanded form

$$\begin{aligned} \mathcal{E}_{\text{translate}} &= \sum_{i=1}^M \|s_i\|^2 p_i - 2 \sum_{i=1}^M \mathbf{a}^T s_i p_i + \|\mathbf{a}\|^2 \sum_{i=1}^M p_i \\ &= \mathcal{E} - 2\mathbf{a}^T E[\mathbf{s}] + \|\mathbf{a}\|^2 \end{aligned} \quad (5.81)$$

where \mathcal{E} is the average energy of the original signal constellation, and

$$E[\mathbf{s}] = \sum_{i=1}^M s_i p_i \quad (5.82)$$

Differentiating Equation (5.81) with respect to the vector \mathbf{a} and then setting the result equal to zero, we readily find that the minimizing translate is

$$\mathbf{a}_{\min} = E[\mathbf{s}] \quad (5.83)$$

The minimum average energy of the signal constellation translated in this way is

$$\mathcal{E}_{\text{translate, min}} = \mathcal{E} - \|\mathbf{a}_{\min}\|^2 \quad (5.84)$$

We may now state the procedure for finding the *minimum energy translate*:

Given a signal constellation $\{s_i\}_{i=1}^M$, the corresponding signal constellation with minimum average energy is obtained by subtracting from each signal vector s_i in the given constellation an amount equal to the constant vector $E[\mathbf{s}]$, where $E[\mathbf{s}]$ is defined by Equation (5.82).

Recalling that the energy (or power) needed for signal transmission is a primary resource, the minimum energy translate provides a principled method for translating a signal constellation of interest so as to minimize the energy requirement. For example, the constellation of Figure 5.12a has minimum average energy, whereas that of Figure 5.12b does not.

■ UNION BOUND ON THE PROBABILITY OF ERROR²

For AWGN channels, the formulation of the average probability of symbol error, P_e , is conceptually straightforward. We simply write P_e in integral form by substituting Equation (5.46) into Equation (5.68). Unfortunately, however, numerical computation of the integral is impractical, except in a few simple (but important) cases. To overcome this computational difficulty, we may resort to the use of *bounds*, which are usually adequate to predict the signal-to-noise ratio (within a decibel or so) required to maintain a prescribed error rate. The approximation to the integral defining P_e is made by simplifying the integral or simplifying the region of integration. In the sequel, we use the latter procedure to develop a simple yet useful upper bound called the *union bound* as an approximation to the

average probability of symbol error for a set of M equally likely signals (symbols) in an AWGN channel.

Let A_{ik} , with $(i, k) = 1, 2, \dots, M$, denote the event that the observation vector \mathbf{x} is closer to the signal vector \mathbf{s}_k than to \mathbf{s}_i , when the symbol m_i (vector \mathbf{s}_i) is sent. The conditional probability of symbol error when symbol m_i is sent, $P_e(m_i)$, is equal to the probability of the union of events, $A_{i1}, A_{i2}, \dots, A_{i,i-1}, A_{i,i+1}, \dots, A_{iM}$. From probability theory we know that the probability of a finite union of events is overbounded by the sum of the probabilities of the constituent events. We may therefore write

$$P_e(m_i) \leq \sum_{\substack{k=1 \\ k \neq i}}^M P(A_{ik}), \quad i = 1, 2, \dots, M \quad (5.85)$$

This relationship is illustrated in Figure 5.13 for the case of $M = 4$. In Figure 5.13a, we show the four message points and associated decision regions, with the point \mathbf{s}_1 assumed to represent the transmitted symbol. In Figure 5.13b, we show the three constituent signal-space descriptions where, in each case, the transmitted message point \mathbf{s}_1 and one other message point are retained. According to Figure 5.13a the conditional probability of symbol error, $P_e(m_i)$, is equal to the probability that the observation vector \mathbf{x} lies in the shaded region of the two-dimensional signal-space diagram. Clearly, this probability is less than the sum of the probabilities of the three individual events that \mathbf{x} lies in the shaded regions of the three constituent signal spaces depicted in Figure 5.13b.

It is important to note that, in general, the probability $P(A_{ik})$ is different from the probability $P(\hat{m} = m_k | m_i)$. The latter is the probability that the observation vector \mathbf{x} is

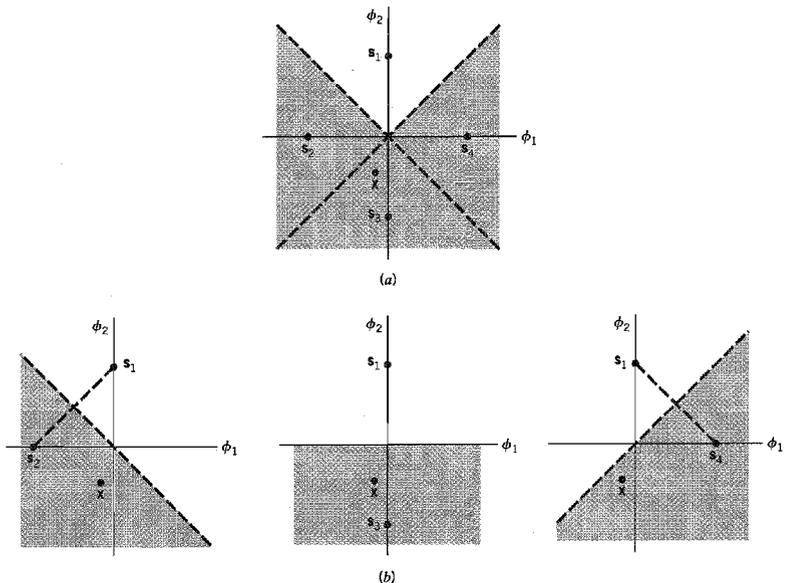


FIGURE 5.13 Illustrating the union bound. (a) Constellation of four message points. (b) Three constellations with a common message point and one other message point retained from the original constellation.

closer to the signal vector \mathbf{s}_k than every other, when \mathbf{s}_i (or m_i) is sent. On the other hand, the probability $P(A_{ik})$ depends on only two signal vectors, \mathbf{s}_i and \mathbf{s}_k . To emphasize this difference, we rewrite Equation (5.85) by adopting $P_2(\mathbf{s}_i, \mathbf{s}_k)$ in place of $P(A_{ik})$. We thus write

$$P_e(m_i) \leq \sum_{\substack{k=1 \\ k \neq i}}^M P_2(\mathbf{s}_i, \mathbf{s}_k), \quad i = 1, 2, \dots, M \quad (5.86)$$

The probability $P_2(\mathbf{s}_i, \mathbf{s}_k)$ is called the *pairwise error probability* in that if a data transmission system uses only a pair of signals, \mathbf{s}_i and \mathbf{s}_k , then $P_2(\mathbf{s}_i, \mathbf{s}_k)$ is the probability of the receiver mistaking \mathbf{s}_k for \mathbf{s}_i .

Consider then a simplified digital communication system that involves the use of two equally likely messages represented by the vectors \mathbf{s}_i and \mathbf{s}_k . Since white Gaussian noise is identically distributed along any set of orthogonal axes, we may temporarily choose the first axis in such a set as one that passes through the points \mathbf{s}_i and \mathbf{s}_k ; for three examples, see Figure 5.13*b*. The corresponding decision boundary is represented by the bisector that is perpendicular to the line joining the points \mathbf{s}_i and \mathbf{s}_k . Accordingly, when the symbol m_i (vector \mathbf{s}_i) is sent, and if the observation vector \mathbf{x} lies on the side of the bisector where \mathbf{s}_k lies, an error is made. The probability of this event is given by

$$\begin{aligned} P_2(\mathbf{s}_i, \mathbf{s}_k) &= P(\mathbf{x} \text{ is closer to } \mathbf{s}_k \text{ than } \mathbf{s}_i, \text{ when } \mathbf{s}_i \text{ is sent}) \\ &= \int_{d_{ik}/2}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{v^2}{N_0}\right) dv \end{aligned} \quad (5.87)$$

where d_{ik} is the Euclidean distance between \mathbf{s}_i and \mathbf{s}_k ; that is,

$$d_{ik} = \|\mathbf{s}_i - \mathbf{s}_k\| \quad (5.88)$$

From the definition of the complementary error function, we have

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2) dz$$

Thus, in terms of this function, with z set equal to $v/\sqrt{N_0}$, we find that Equation (5.87) takes on the compact form

$$P_2(\mathbf{s}_i, \mathbf{s}_k) = \frac{1}{2} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \quad (5.89)$$

Substituting Equation (5.89) into Equation (5.86), we get

$$P_e(m_i) \leq \frac{1}{2} \sum_{\substack{k=1 \\ k \neq i}}^M \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right), \quad i = 1, 2, \dots, M \quad (5.90)$$

The probability of symbol error, averaged over all the M symbols, is therefore overbounded as follows:

$$\begin{aligned} P_e &= \sum_{i=1}^M p_i P_e(m_i) \\ &\leq \frac{1}{2} \sum_{i=1}^M \sum_{\substack{k=1 \\ k \neq i}}^M p_i \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \end{aligned} \quad (5.91)$$

where p_i is the probability of transmitting symbol m_i .

There are two special forms of Equation (5.91) that we will find useful in Chapter 6 on passband data transmission:

1. Suppose that the signal constellation is *circularly symmetric* about the origin. Then the conditional probability of error $P_e(m_i)$ is the same for all i , in which case Equation (5.91) reduces to

$$P_e \leq \frac{1}{2} \sum_{k \neq i}^M \operatorname{erfc} \left(\frac{d_{ik}}{2\sqrt{N_0}} \right) \quad \text{for all } i \quad (5.92)$$

2. Define the *minimum distance* of a signal constellation, d_{\min} , as the smallest Euclidean distance between any two transmitted signal points in the constellation, as shown by

$$d_{\min} = \min_{k \neq i} d_{ik} \quad \text{for all } i \text{ and } k \quad (5.93)$$

Then, recognizing that the complementary error function $\operatorname{erfc}(u)$ is a monotonically decreasing function of its argument u , we may write

$$\operatorname{erfc} \left(\frac{d_{ik}}{2\sqrt{N_0}} \right) \leq \operatorname{erfc} \left(\frac{d_{\min}}{2\sqrt{N_0}} \right) \quad \text{for all } i \text{ and } k \quad (5.94)$$

We may therefore, in general, simplify the bound on the average probability of symbol error in Equation (5.91) as

$$P_e \leq \frac{(M-1)}{2} \operatorname{erfc} \left(\frac{d_{\min}}{2\sqrt{N_0}} \right) \quad (5.95)$$

The complementary error function is itself bounded as³

$$\operatorname{erfc} \left(\frac{d_{\min}}{2\sqrt{N_0}} \right) \leq \frac{1}{\sqrt{\pi}} \exp \left(-\frac{d_{\min}^2}{4N_0} \right) \quad (5.96)$$

Accordingly, we may further simplify the union bound on P_e given in Equation (5.95) as

$$P_e \leq \frac{(M-1)}{2\sqrt{\pi}} \exp \left(-\frac{d_{\min}^2}{4N_0} \right) \quad (5.97)$$

Equation (5.97) shows that for a prescribed AWGN channel, the average probability of symbol error P_e decreases exponentially as the squared minimum distance, d_{\min}^2 .

■ BIT VERSUS SYMBOL ERROR PROBABILITIES

Thus far, the only figure of merit we have used to assess the noise performance of a digital passband transmission system has been the average probability of symbol error. This figure of merit is the natural choice when messages of length $m = \log_2 M$ are transmitted, such as alphanumeric symbols. However, when the requirement is to transmit binary data such as digital computer data, it is often more meaningful to use another figure of merit called the *bit error rate* (BER). Although, in general, there are no unique relationships between these two figures of merit, it is fortunate that such relationships can be derived for two cases of practical interest, as discussed next.

Case 1

In the first case, we assume that it is possible to perform the mapping from binary to M -ary symbols in such a way that the two binary M -tuples corresponding to any pair of adjacent symbols in the M -ary modulation scheme differ in only one bit position. This mapping constraint is satisfied by using a *Gray code*. When the probability of symbol error P_e is acceptably small, we find that the probability of mistaking one symbol for either one of the two “nearest” symbols is much greater than any other kind of symbol error. Moreover, given a symbol error, the most probable number of bit errors is one, subject to the aforementioned mapping constraint. Since there are $\log_2 M$ bits per symbol, it follows that the average probability of symbol error is related to the bit error rate as follows:

$$\begin{aligned} P_e &= P\left(\bigcup_{i=1}^{\log_2 M} \{\text{ith bit is in error}\}\right) \\ &\leq \sum_{i=1}^{\log_2 M} P(\text{ith bit is in error}) \\ &= \log_2 M \cdot (\text{BER}) \end{aligned} \quad (5.97)$$

We also note that

$$P_e \geq P(\text{ith bit is in error}) = \text{BER} \quad (5.98)$$

It follows therefore that the bit error rate is bounded as follows:

$$\frac{P_e}{\log_2 M} \leq \text{BER} \leq P_e \quad (5.99)$$

Case 2

Let $M = 2^K$, where K is an integer. We assume that all symbol errors are equally likely and occur with probability

$$\frac{P_e}{M-1} = \frac{P_e}{2^K-1}$$

where P_e is the average probability of symbol error. What is the probability that the i th bit in a symbol is in error? Well, there are 2^{K-1} cases of symbol error in which this particular bit is changed, and there are 2^{K-1} cases in which it is not changed. Hence, the bit error rate is

$$\text{BER} = \left(\frac{2^{K-1}}{2^K-1}\right)P_e \quad (5.100)$$

or, equivalently,

$$\text{BER} = \left(\frac{M/2}{M-1}\right)P_e \quad (5.101)$$

Note that for large M , the bit error rate approaches the limiting value of $P_e/2$. The same idea described here also shows that bit errors are not independent, since we have

$$P(\text{ith and } j\text{th bits are in error}) = \frac{2^{K-2}}{2^K-1} P_e \neq (\text{BER})^2$$

5.8 Summary and Discussion

The primary goal of the material presented in this chapter is the formulation of a systematic procedure for the analysis and design of a digital communication receiver in the presence of *additive white Gaussian noise* (AWGN). The procedure, known as *maximum likelihood detection*, decides which particular transmitted symbol is the most likely cause of the noisy signal observed at the channel output. The approach that led to the formulation of the maximum likelihood detector (receiver) is called *signal-space analysis*. The basic idea of the approach is to represent each member of a set of transmitted signals by an N -dimensional vector, where N is the number of orthonormal basis functions needed for a unique geometric representation of the transmitted signals. The set of signal vectors so formed defines a *signal constellation* in an N -dimensional *signal space*.

For a given signal constellation, the (average) probability of symbol error P_e incurred in maximum likelihood signal detection over an AWGN channel is invariant to rotation of the signal constellation as well as its translation. However, except for a few simple (but important) cases, the numerical calculation of P_e is an impractical proposition. To overcome this difficulty, the customary practice is to resort to the use of bounds that lend themselves to computation in a straightforward manner. In this context, we described the *union bound* that follows directly from the signal-space diagram. The union bound is based on an intuitively satisfying idea: The probability of symbol error P_e is dominated by the nearest neighbors to the transmitted signal. The results obtained using the union bound are usually fairly accurate when the signal-to-noise ratio is high.

With the material on signal-space analysis and related issues on hand, we are well-equipped to study passband data transmission systems, which we do in Chapter 6.

NOTES AND REFERENCES

1. The geometric representation of signals was first developed by Kotel'nikov in 1947: V. A. Kotel'nikov, *The Theory of Optimum Noise Immunity* (Dover Publications, 1960), which is a translation of the original doctoral dissertation presented in January 1947 before the Academic Council of the Molotov Energy Institute in Moscow. In particular, see Part II of the book. This method was subsequently brought to fuller fruition in the classic book by Wozencraft and Jacobs (1965). Signal-space analysis is also discussed in Cioffi (1998), Anderson (1999), and Proakis (1995).
2. In Section 5.7, we derived the union bound on the average probability of symbol error; the classic reference for this bound is Wozencraft and Jacobs (1965). For the derivation of tighter bounds, see Viterbi and Omura (1979, pp. 58–59).
3. In Chapter 4, we used the following upper bound on the complementary error function

$$\operatorname{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi}u}$$

For large positive u , a second bound on the complementary error function is obtained by omitting the multiplying factor $1/u$ in the above upper bound, as shown by

$$\operatorname{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi}}$$

It is this second upper bound that is used in Equation (5.97).

PROBLEMS

Representation of Signals

5.1 In Section 3.7 we described line codes for pulse-code modulation. Referring to the material presented therein, formulate the signal constellations for the following line codes:

- (a) Unipolar nonreturn-to-zero code
- (b) Polar nonreturn-to-zero code
- (c) Unipolar return-to-zero code
- (d) Manchester code

5.2 An 8-level PAM signal is defined by

$$s_i(t) = A_i \operatorname{rect}\left(\frac{t}{T} - \frac{1}{2}\right)$$

where $A_i = \pm 1, \pm 3, \pm 5, \pm 7$. Formulate the signal constellation of $\{s_i(t)\}_{i=1}^8$.

5.3 Figure P5.3 displays the waveforms of four signals $s_1(t)$, $s_2(t)$, $s_3(t)$, and $s_4(t)$.

- (a) Using the Gram-Schmidt orthogonalization procedure, find an orthonormal basis for this set of signals.
- (b) Construct the corresponding signal-space diagram.

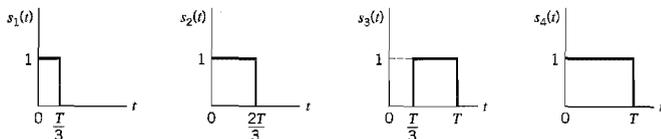


FIGURE P5.3

5.4 (a) Using the Gram-Schmidt orthogonalization procedure, find a set of orthonormal basis functions to represent the three signals $s_1(t)$, $s_2(t)$, and $s_3(t)$ shown in Figure P5.4.

- (b) Express each of these signals in terms of the set of basis functions found in part (a).

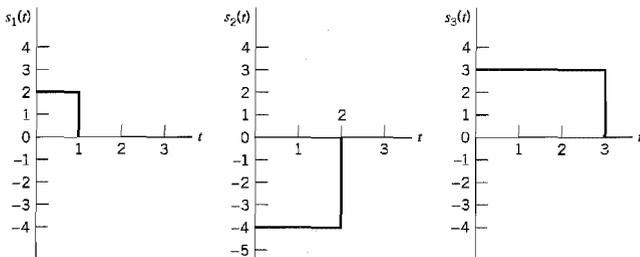


FIGURE P5.4

5.5 An *orthogonal set of signals* is characterized by the property that the inner product of any pair of signals in the set is zero. Figure P5.5 shows a pair of signals $s_1(t)$ and $s_2(t)$ that satisfy this condition. Construct the signal constellation for $s_1(t)$ and $s_2(t)$.

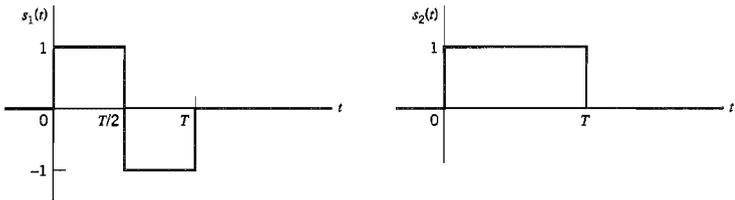


FIGURE P5.5

- 5.6 A source of information emits a set of symbols denoted by $\{m_i\}_{i=1}^M$. Two candidate modulation schemes, namely, pulse-duration modulation (PDM) and pulse-position modulation (PPM), are considered for the electrical representation of this set of symbols. In PDM, the i th symbol is represented by a pulse of unit amplitude and duration $(i/M)T$. On the other hand, in PPM, the i th symbol is represented by a short pulse of unit amplitude and fixed duration, which is transmitted at time $t = (i/M)T$. Show that PPM is the only one of the two that can produce an orthogonal set of signals over the interval $0 \leq t \leq T$.
- 5.7 A set of $2M$ biorthogonal signals is obtained from a set of M orthogonal signals by augmenting it with the negative of each signal in the set.
- The extension of orthogonal to biorthogonal signals leaves the dimensionality of the signal space unchanged. Why?
 - Construct the signal constellation for the biorthogonal signals corresponding to the pair of orthogonal signals shown in Figure P5.5.
- 5.8 (a) A pair of signals $s_i(t)$ and $s_k(t)$ have a common duration T . Show that the inner product of this pair of signals is given by

$$\int_0^T s_i(t)s_k(t)dt = \mathbf{s}_i^T \mathbf{s}_k$$

where \mathbf{s}_i and \mathbf{s}_k are the vector representations of $s_i(t)$ and $s_k(t)$, respectively.

- (b) As a followup to part (a), show that

$$\int_0^T (s_i(t) - s_k(t))^2 dt = \|\mathbf{s}_i - \mathbf{s}_k\|^2$$

- 5.9 Consider a pair of complex-valued signals $s_1(t)$ and $s_2(t)$ that are respectively represented by

$$\begin{aligned} s_1(t) &= a_{11}\phi_1(t) + a_{12}\phi_2(t), & -\infty < t < \infty \\ s_2(t) &= a_{21}\phi_1(t) + a_{22}\phi_2(t), & -\infty < t < \infty \end{aligned}$$

where the basis functions $\phi_1(t)$ and $\phi_2(t)$ are both real valued, but the coefficients a_{11} , a_{12} , a_{21} , and a_{22} are complex valued. Prove the complex form of the Schwarz inequality:

$$\left| \int_{-\infty}^{\infty} s_1(t)s_2^*(t)dt \right|^2 \leq \int_{-\infty}^{\infty} |s_1(t)|^2 dt \int_{-\infty}^{\infty} |s_2(t)|^2 dt$$

where the asterisk denotes complex conjugation. When is this relation satisfied with the equality sign?

Random Processes

- 5.10 Consider a random process $X(t)$ expanded in the form

$$X(t) = \sum_{j=1}^N X_j \phi_j(t) + W'(t), \quad 0 \leq t \leq T$$

where $W'(t)$ is a remainder noise term. The $\{\phi_j(t)\}_{j=1}^N$ form an orthonormal set over the interval $0 \leq t \leq T$, and the X_j are defined by

$$X_j = \int_0^T X(t)\phi_j(t)dt$$

Let $W'(t_k)$ denote a random variable obtained by observing $W'(t)$ at time $t = t_k$. Show that

$$E[X_j W'(t_k)] = 0, \quad \begin{cases} j = 1, 2, \dots, N \\ 0 \leq t_k \leq T \end{cases}$$

5.11 Consider the optimum detection of the sinusoidal signal

$$s(t) = \sin\left(\frac{8\pi t}{T}\right), \quad 0 \leq t \leq T$$

in additive white Gaussian noise.

- Determine the correlator output assuming a noiseless input.
- Determine the corresponding matched filter output, assuming that the filter includes a delay T to make it causal.
- Hence show that these two outputs are the same only at time instant $t = T$.

Probability of Error

5.12 Figure P5.12 shows a pair of signals $s_1(t)$ and $s_2(t)$ that are orthogonal to each other over the observation interval $0 \leq t \leq 3T$. The received signal is defined by

$$x(t) = s_k(t) + w(t), \quad \begin{matrix} 0 \leq t \leq 3T \\ k = 1, 2 \end{matrix}$$

where $w(t)$ is white Gaussian noise of zero mean and power spectral density $N_0/2$.

- Design a receiver that decides in favor of signals $s_1(t)$ or $s_2(t)$, assuming that these two signals are equiprobable.
- Calculate the average probability of symbol error incurred by this receiver for $E/N_0 = 4$, where E is the signal energy.

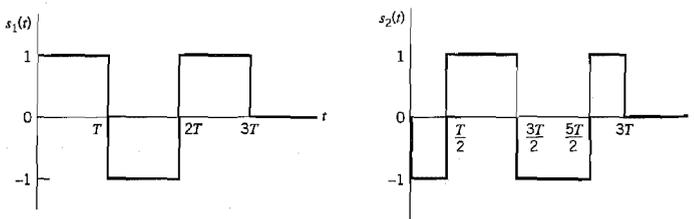


FIGURE P5.12

5.13 In the Manchester code, binary symbol 1 is represented by the doublet pulse $s(t)$ shown in Figure P5.13, and binary symbol 0 is represented by the negative of this pulse. Derive the formula for the probability of error incurred by the maximum likelihood detection procedure applied to this form of signaling over an AWGN channel.

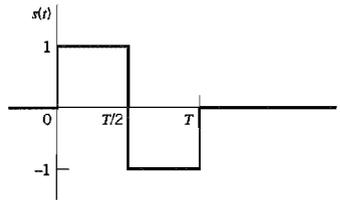


FIGURE P5.13

- 5.14 In the *Bayes test*, applied to a binary hypothesis testing problem where we have to choose one of two possible hypotheses H_0 and H_1 , we minimize the *risk* R defined by

$$\begin{aligned}
 R = & C_{00}p_0P(\text{say } H_0 | H_0 \text{ is true}) \\
 & + C_{10}p_0P(\text{say } H_1 | H_0 \text{ is true}) \\
 & + C_{11}p_1P(\text{say } H_1 | H_1 \text{ is true}) \\
 & + C_{01}p_1P(\text{say } H_0 | H_1 \text{ is true})
 \end{aligned}$$

The terms C_{00} , C_{10} , C_{11} , and C_{01} denote the costs assigned to the four possible outcomes of the experiment: The first subscript indicates the hypothesis chosen, and the second the hypothesis that is true. Assume that $C_{10} > C_{00}$ and $C_{01} > C_{11}$. The p_0 and p_1 denote the *a priori* probabilities of hypotheses H_0 and H_1 , respectively.

- (a) Given the observation vector \mathbf{x} , show that the partitioning of the observation space so as to minimize the risk R leads to the *likelihood ratio test*:

$$\begin{aligned}
 & \text{say } H_0 \text{ if } \Lambda(\mathbf{x}) < \lambda \\
 & \text{say } H_1 \text{ if } \Lambda(\mathbf{x}) > \lambda
 \end{aligned}$$

where $\Lambda(\mathbf{x})$ is the *likelihood ratio*

$$\Lambda(\mathbf{x}) = \frac{f_{\mathbf{x}}(\mathbf{x} | H_1)}{f_{\mathbf{x}}(\mathbf{x} | H_0)}$$

and λ is the *threshold* of the test defined by

$$\lambda = \frac{p_0(C_{10} - C_{00})}{p_1(C_{01} - C_{11})}$$

- (b) What are the cost values for which the Bayes' criterion reduces to the minimum probability of error criterion?

Principles of Rotational and Translational Invariance

- 5.15 Continuing with the four line codes considered in Problem 5.1, identify the line codes that have minimum average energy and those that do not. Compare your answers with the observations made on these line codes in Section 3.7.
- 5.16 Consider the two constellations shown in Figure 5.11. Determine the orthonormal matrix Q that transforms the constellation shown in Figure 5.11a into the one shown in Figure 5.11b.

- 5.17 (a) The two signal constellations shown in Figure P5.17 exhibit the same average probability of symbol error. Justify the validity of this statement.
- (b) Which of these two constellations has minimum average energy? Justify your answer. You may assume that the symbols pertaining to the message points displayed in Figure P5.17 are equally likely.

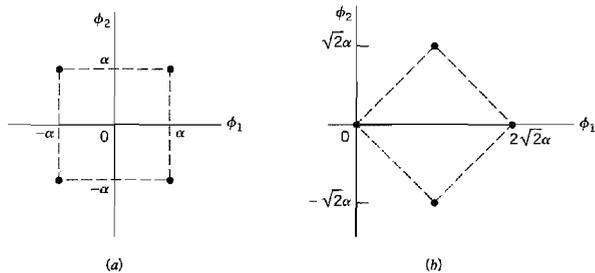


FIGURE P5.17

- 5.18 *Simplex (transorthogonal) signals* are equally likely highly-correlated signals with the most negative correlation that can be achieved with a set of M orthogonal signals. That is, the correlation coefficient between any pair of signals in the set is defined by

$$\rho_{ij} = \begin{cases} 1 & \text{for } i = j \\ -1 & \text{for } i \neq j \end{cases} \quad \text{for } i, j = 1, 2, \dots, M$$

One method of constructing simplex signals is to start with a set of M orthogonal signals, each with energy E_s , and then apply the minimum energy translate.

Consider a set of three equally likely symbols whose signal constellation consists of the vertices of an equilateral triangle. Show that these three symbols constitute a simplex code.

Bounds on Probability of Error

- 5.19 In this problem we explore the approximations to the probability of an error, P_e , for the pair of antipodal signals shown in Figure P5.19 in the presence of additive white Gaussian noise of power spectral density $N_0/2$. The exact formula for P_e is

$$P_e = \frac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{E_b}{N_0}} \right)$$

(This formula is derived in Section 6.3)

- (a) Using the two upper bounds for the complementary error function given in Note 3, derive the corresponding approximations to P_e .
- (b) Compare the approximations derived in part (a) for P_e to the exact formula for $E_b/N_0 = 9$. For the exact calculation of P_e , you may use Table A6.6 on the error function.

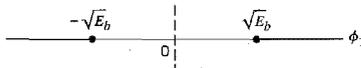


FIGURE P5.19

- 5.20 Consider the special case of a signal constellation that has a *symmetric geometry* with respect to the origin. Assume that the M message points of the constellation, pertaining to symbols m_1, m_2, \dots, m_M , are equally likely. Using the upper bound on the complementary error function given in Equation (5.94), show that the average probability of symbol error for the constellation is bounded as

$$P_e \leq \frac{M_{\min}}{2\sqrt{\pi}} \exp \left(-\min_{\substack{i, k \\ i \neq k}} \left(\frac{d_{ik}^2}{4N_0} \right) \right)$$

where d_{ik} is the Euclidean distance between message points i and k , and M_{\min} is the number of transmitted signals that attain the minimum Euclidean distance for each m_i .

PASSBAND DATA TRANSMISSION

This chapter builds on the material developed in Chapter 5 on signal-space analysis. It discusses the subject of digital data transmission over a band-pass channel that can be linear or nonlinear. As with analog communications, this mode of data transmission relies on the use of a sinusoidal carrier wave modulated by the data stream.

Specifically, the following topics are covered:

- ▶ *Different methods of digital modulation, namely, phase-shift keying, quadrature-amplitude modulation, and frequency-shift keying, and their individual variants.*
- ▶ *Coherent detection of modulated signals in additive white Gaussian noise, which requires the receiver to be synchronized to the transmitter with respect to both carrier phase and bit timing.*
- ▶ *Noncoherent detection of modulated signals in additive white Gaussian noise, disregarding phase information in the received signal.*
- ▶ *Modems for the transmission and reception of digital data over the public switched telephone network.*
- ▶ *Sophisticated modulation techniques, namely, carrierless amplitude/phase modulation and discrete multitone, for data transmission over a wideband channel with medium to severe intersymbol interference.*
- ▶ *Techniques for synchronizing the receiver to the transmitter.*

6.1 Introduction

In *baseband pulse transmission*, which we studied in Chapter 4, a data stream represented in the form of a discrete pulse-amplitude modulated (PAM) signal is transmitted directly over a low-pass channel. In *digital passband transmission*, on the other hand, the incoming data stream is modulated onto a carrier (usually sinusoidal) with fixed frequency limits imposed by a band-pass channel of interest; passband data transmission is studied in this chapter.

The communication channel used for passband data transmission may be a microwave radio link, a satellite channel, or the like. Yet other applications of passband data transmission are in the design of passband line codes for use on digital subscriber loops and orthogonal frequency-division multiplexing techniques for broadcasting. In any event, the modulation process making the transmission possible involves switching (keying) the amplitude, frequency, or phase of a sinusoidal carrier in some fashion in accordance with the incoming data. Thus there are three basic signaling schemes, and they are known as

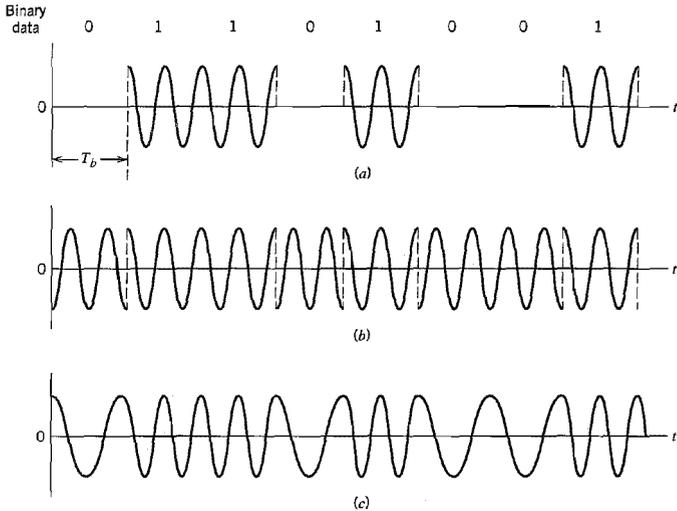


FIGURE 6.1 Illustrative waveforms for the three basic forms of signaling binary information. (a) Amplitude-shift keying. (b) Phase-shift keying. (c) Frequency-shift keying with continuous phase.

amplitude-shift keying (ASK), *frequency-shift keying (FSK)*, and *phase-shift keying (PSK)*. They may be viewed as special cases of amplitude modulation, frequency modulation, and phase modulation, respectively.

Figure 6.1 illustrates these three methods of modulation for the case of a source supplying binary data. The following points are noteworthy from Figure 6.1:

- ▶ Although in continuous-wave modulation it is usually difficult to distinguish between phase-modulated and frequency-modulated signals by merely looking at their waveforms, this is not true for PSK and FSK signals.
- ▶ Unlike ASK signals, both PSK and FSK signals have a constant envelope.

This latter property makes PSK and FSK signals impervious to amplitude nonlinearities, commonly encountered in microwave radio and satellite channels. It is for this reason, in practice, we find that PSK and FSK signals are preferred to ASK signals for passband data transmission over nonlinear channels.

■ HIERARCHY OF DIGITAL MODULATION TECHNIQUES¹

Digital modulation techniques may be classified into *coherent* and *noncoherent* techniques, depending on whether the receiver is equipped with a phase-recovery circuit or not. The phase-recovery circuit ensures that the oscillator supplying the locally generated carrier wave in the receiver is synchronized (in both frequency and phase) to the oscillator supplying the carrier wave used to originally modulate the incoming data stream in the transmitter.

As discussed in Chapter 4, in an M -ary signaling scheme, we may send any one of M possible signals $s_1(t)$, $s_2(t)$, \dots , $s_M(t)$, during each signaling interval of duration T . For

almost all applications, the number of possible signals $M = 2^n$, where n is an integer. The symbol duration $T = nT_b$, where T_b is the bit duration. In passband data transmission, these signals are generated by changing the amplitude, phase, or frequency of a sinusoidal carrier in M discrete steps. Thus we have M -ary ASK, M -ary PSK, and M -ary FSK digital modulation schemes. Another way of generating M -ary signals is to combine different methods of modulation into a hybrid form. For example, we may combine discrete changes in both the amplitude and phase of a carrier to produce M -ary amplitude-phase modulation (APK). A special form of this hybrid modulation is M -ary quadrature-amplitude modulation (QAM), which has some attractive properties. M -ary ASK is a special case of M -ary QAM.

M -ary signaling schemes are preferred over binary signaling schemes for transmitting digital information over band-pass channels when the requirement is to conserve bandwidth at the expense of increased power. In practice, we rarely find a communication channel that has the exact bandwidth required for transmitting the output of an information source by means of binary signaling schemes. Thus when the bandwidth of the channel is less than the required value, we may use M -ary signaling schemes for maximum efficiency. To illustrate the bandwidth-conservation capability of M -ary signaling schemes, consider the transmission of information consisting of a binary sequence with bit duration T_b . If we were to transmit this information by means of binary PSK, for example, we would require a bandwidth that is inversely proportional to T_b . However, if we take blocks of n bits and use an M -ary PSK scheme with $M = 2^n$ and symbol duration $T = nT_b$, the bandwidth required is proportional to $1/nT_b$. This shows that the use of M -ary PSK enables a reduction in transmission bandwidth by the factor $n = \log_2 M$ over binary PSK.

M -ary PSK and M -ary QAM are examples of *linear modulation*. However, they differ from each other in one important respect: An M -ary PSK signal has a constant envelope, whereas an M -ary QAM signal involves changes in the carrier amplitude. Accordingly, M -ary PSK can be used to transmit digital data over a nonlinear band-pass channel, whereas M -ary QAM requires the use of a linear channel.

M -ary PSK, M -ary QAM, and M -ary FSK are commonly used in coherent systems. Amplitude-shift keying and frequency-shift keying lend themselves naturally to use in noncoherent systems whenever it is impractical to maintain carrier phase synchronization. But in the case of phase-shift keying, we cannot have “noncoherent PSK” because the term noncoherent means doing without carrier phase information. Instead, we employ a “pseudo PSK” technique known as *differential phase-shift keying* (DPSK), which (in a loose sense) may be viewed as the noncoherent form of PSK. In practice, M -ary FSK and M -ary DPSK are the commonly used forms of digital modulation in noncoherent systems.

■ PROBABILITY OF ERROR

A major goal of passband data transmission systems is the optimum design of the receiver so as to minimize the average probability of symbol error in the presence of *additive white Gaussian noise* (AWGN). With this goal in mind, much of the material presented in this chapter builds on the signal-space analysis tools presented in Chapter 5. Specifically, in the study of each system we begin with the formulation of a signal constellation and the construction of decision regions in accordance with maximum likelihood signal detection over an AWGN channel. These formulations set the stage for evaluating the probability of symbol error P_e . Depending on the method of digital modulation under study, the evaluation of P_e proceeds in one of two ways:

- ▶ In the case of certain simple methods such as coherent binary PSK and coherent binary FSK, exact formulas are derived for P_e .

- In the case of more elaborate methods such as coherent M -ary PSK and coherent M -ary FSK, we resort to the use of the union bound for deriving an approximate formula for P_e .

▣ POWER SPECTRA

To fully appreciate the practical virtues of different methods of digital modulation, we also need to study the power spectra of the resulting modulated signals. This latter issue is particularly important in two contexts: occupancy of the channel bandwidth and co-channel interference in multiplexed systems.

Given a modulated signal $s(t)$, we may describe it in terms of its in-phase and quadrature components as

$$\begin{aligned} s(t) &= s_I(t) \cos(2\pi f_c t) - s_Q(t) \sin(2\pi f_c t) \\ &= \operatorname{Re}[\tilde{s}(t) \exp(j2\pi f_c t)] \end{aligned} \quad (6.1)$$

where $\operatorname{Re}[\cdot]$ is the real part of the expression contained inside the square brackets. We also have

$$\tilde{s}(t) = s_I(t) + js_Q(t) \quad (6.2)$$

and

$$\exp(j2\pi f_c t) = \cos(2\pi f_c t) + j \sin(2\pi f_c t) \quad (6.3)$$

The signal $\tilde{s}(t)$ is the *complex envelope* (i.e., baseband version) of the modulated (band-pass) signal $s(t)$. The components $s_I(t)$ and $s_Q(t)$ and therefore $\tilde{s}(t)$ are all low-pass signals. They are uniquely defined in terms of the band-pass signal $s(t)$ and the carrier frequency f_c , provided that the half-bandwidth of $s(t)$ is less than the carrier frequency f_c .

Let $S_B(f)$ denote the power spectral density of the complex envelope $\tilde{s}(t)$. We refer to $S_B(f)$ as the *baseband power spectral density*. The power spectral density, $S_S(f)$, of the original band-pass signal $s(t)$ is a frequency-shifted version of $S_B(f)$, except for a scaling factor, as shown by

$$S_S(f) = \frac{1}{4} [S_B(f - f_c) + S_B(f + f_c)] \quad (6.4)$$

It is therefore sufficient to evaluate the baseband power spectral density $S_B(f)$. Since $\tilde{s}(t)$ is a low-pass signal, the calculation of $S_B(f)$ should be simpler than the calculation of $S_S(f)$. (See Example 1.7.)

▣ BANDWIDTH EFFICIENCY

Throughout this book we have emphasized that *channel bandwidth* and *transmitted power* constitute two primary “communication resources,” the efficient utilization of which provides the motivation for the search for *spectrally efficient* schemes. The primary objective of spectrally efficient modulation is to maximize the bandwidth efficiency defined as the *ratio of the data rate in bits per second to the effectively utilized channel bandwidth*. A secondary objective is to achieve this bandwidth efficiency at a minimum practical expenditure of average signal power or, equivalently, in a channel perturbed by additive white Gaussian noise, a minimum practical expenditure of average signal-to-noise ratio.

With the data rate denoted by R_b and the effectively used channel bandwidth by B , we may express the bandwidth efficiency, ρ , as

$$\rho = \frac{R_b}{B} \text{ bits/s/Hz} \quad (6.5)$$

Recall from Chapter 4 that bandwidth efficiency is the product of two independent factors: one due to the possible use of *multilevel encoding* and the other due to *spectral shaping*. In multilevel encoding, information transmission through the channel is carried out on the basis of blocks of bits rather than single bits. With efficient spectral shaping, bandwidth requirement on the channel is reduced by the use of pulse-shaping filters that smooth out the sharp transitions in the transmitted waveform. These two factors are therefore important in their own individual ways in determining the bandwidth efficiency of a passband data transmission system of interest.

6.2 Passband Transmission Model

In a functional sense, we may model a passband data transmission system as shown in Figure 6.2. First, there is assumed to exist a *message source* that emits one *symbol* every T seconds, with the symbols belonging to an alphabet of M symbols, which we denote by m_1, m_2, \dots, m_M . The *a priori probabilities* $P(m_1), P(m_2), \dots, P(m_M)$ specify the message source output. When the M symbols of the alphabet are *equally likely*, we write

$$\begin{aligned} p_i &= P(m_i) \\ &= \frac{1}{M} \quad \text{for all } i \end{aligned} \quad (6.6)$$

The M -ary output of the message source is presented to a *signal transmission encoder*, producing a corresponding vector s_i made up of N real elements, one such set for each of the M symbols of the source alphabet; the dimension N is less than or equal to M . With the vector s_i as input, the *modulator* then constructs a *distinct* signal $s_i(t)$ of duration T seconds as the representation of the symbol m_i , generated by the message source. The signal $s_i(t)$ is necessarily an energy signal, as shown by

$$E_i = \int_0^T s_i^2(t) dt, \quad i = 1, 2, \dots, M \quad (6.7)$$

Note that $s_i(t)$ is real valued. One such signal is transmitted every T seconds. The particular signal chosen for transmission depends in some fashion on the incoming message and possibly on the signals transmitted in preceding time slots. With a sinusoidal carrier, the feature that is used by the modulator to distinguish one signal from another is a *step* change in the amplitude, frequency, or phase of the carrier. (Sometimes, a hybrid form of modulation that combines changes in both amplitude and phase or amplitude and frequency is used.)

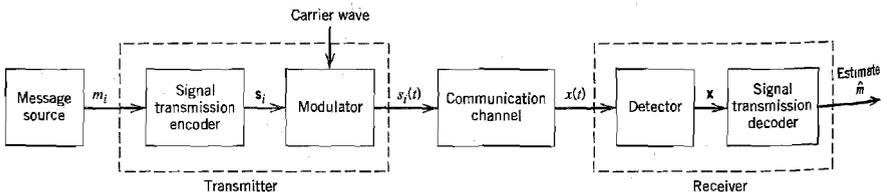


FIGURE 6.2 Functional model of passband data transmission system.

Returning to the functional model of Figure 6.2, the bandpass communication channel, coupling the transmitter to the receiver, is assumed to have two characteristics:

1. The channel is linear, with a bandwidth that is wide enough to accommodate the transmission of the modulated signal $s_i(t)$ with negligible or no distortion.
2. The channel noise $w(t)$ is the sample function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$.

The assumptions made herein are basically the same as those invoked in Chapter 5 dealing with signal-space analysis.

The receiver, which consists of a *detector* followed by a *signal transmission decoder*, performs two functions:

1. It reverses the operations performed in the transmitter.
2. It minimizes the effect of channel noise on the estimate \hat{m} computed for the transmitted symbol m_i .

6.3 Coherent Phase-Shift Keying

With the background material on the coherent detection of signals in additive white Gaussian noise that was presented in Chapter 5 at our disposal, we are now ready to study specific passband data transmission systems. In this section we focus on coherent phase-shift keying (PSK) by considering binary PSK, QPSK and its variants, and finish up with M -ary PSK.

■ BINARY PHASE-SHIFT KEYING

In a coherent binary PSK system, the pair of signals $s_1(t)$ and $s_2(t)$ used to represent binary symbols 1 and 0, respectively, is defined by

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \quad (6.8)$$

$$s_2(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t) \quad (6.9)$$

where $0 \leq t \leq T_b$, and E_b is the *transmitted signal energy per bit*. To ensure that each transmitted bit contains an integral number of cycles of the carrier wave, the carrier frequency f_c is chosen equal to n_c/T_b for some fixed integer n_c . A pair of sinusoidal waves that differ only in a relative phase-shift of 180 degrees, as defined in Equations (6.8) and (6.9), are referred to as *antipodal signals*.

From this pair of equations it is clear that, in the case of binary PSK, there is only one basis function of unit energy, namely,

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t), \quad 0 \leq t < T_b \quad (6.10)$$

Then we may express the transmitted signals $s_1(t)$ and $s_2(t)$ in terms of $\phi_1(t)$ as follows:

$$s_1(t) = \sqrt{E_b} \phi_1(t), \quad 0 \leq t < T_b \quad (6.11)$$

and

$$s_2(t) = -\sqrt{E_b} \phi_1(t), \quad 0 \leq t < T_b \quad (6.12)$$

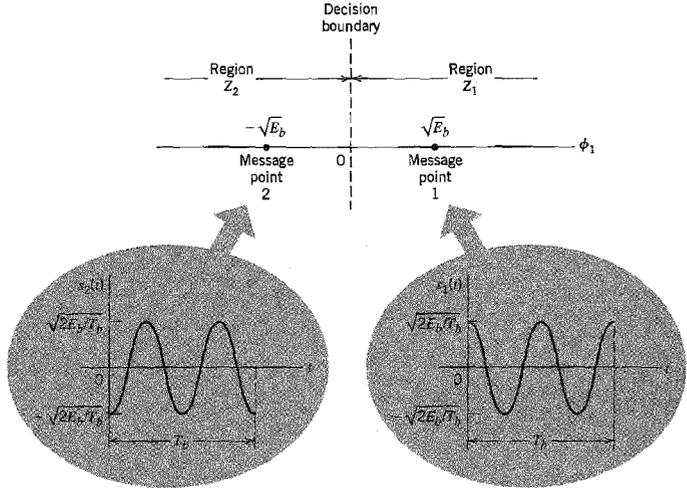


FIGURE 6.3 Signal-space diagram for coherent binary PSK system. The waveforms depicting the transmitted signals $s_1(t)$ and $s_2(t)$, displayed in the inserts, assume $n_c = 2$.

A coherent binary PSK system is therefore characterized by having a signal space that is one-dimensional (i.e., $N = 1$), with a signal constellation consisting of two message points (i.e., $M = 2$). The coordinates of the message points are

$$\begin{aligned} s_{11} &= \int_0^{T_b} s_1(t)\phi_1(t) dt \\ &= +\sqrt{E_b} \end{aligned} \quad (6.13)$$

and

$$\begin{aligned} s_{21} &= \int_0^{T_b} s_2(t)\phi_1(t) dt \\ &= -\sqrt{E_b} \end{aligned} \quad (6.14)$$

The message point corresponding to $s_1(t)$ is located at $s_{11} = +\sqrt{E_b}$, and the message point corresponding to $s_2(t)$ is located at $s_{21} = -\sqrt{E_b}$. Figure 6.3 displays the signal-space diagram for binary PSK. This figure also includes two inserts, showing example waveforms of antipodal signals representing $s_1(t)$ and $s_2(t)$. Note that the constellation of Figure 6.3 has minimum average energy.

Error Probability of Binary PSK

To realize a *rule for making a decision* in favor of symbol 1 or symbol 0, we apply Equation (5.59) of Chapter 5. Specifically, we partition the signal space of Figure 6.3 into two regions:

- ▶ The set of points closest to message point 1 at $+\sqrt{E_b}$.
- ▶ The set of points closest to message point 2 at $-\sqrt{E_b}$.

This is accomplished by constructing the midpoint of the line joining these two message points, and then marking off the appropriate decision regions. In Figure 6.3 these decision regions are marked Z_1 and Z_2 , according to the message point around which they are constructed.

The decision rule is now simply to decide that signal $s_1(t)$ (i.e., binary symbol 1) was transmitted if the received signal point falls in region Z_1 , and decide that signal $s_2(t)$ (i.e., binary symbol 0) was transmitted if the received signal point falls in region Z_2 . Two kinds of erroneous decisions may, however, be made. Signal $s_2(t)$ is transmitted, but the noise is such that the received signal point falls inside region Z_1 and so the receiver decides in favor of signal $s_1(t)$. Alternatively, signal $s_1(t)$ is transmitted, but the noise is such that the received signal point falls inside region Z_2 and so the receiver decides in favor of signal $s_2(t)$.

To calculate the probability of making an error of the first kind, we note from Figure 6.3 that the decision region associated with symbol 1 or signal $s_1(t)$ is described by

$$Z_1: 0 < x_1 < \infty$$

where the observable element x_1 is related to the received signal $x(t)$ by

$$x_1 = \int_0^{T_b} x(t)\phi_1(t) dt \quad (6.15)$$

The conditional probability density function of random variable X_1 , given that symbol 0 [i.e., signal $s_2(t)$] was transmitted, is defined by

$$\begin{aligned} f_{x_1}(x_1|0) &= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 - s_{21})^2\right] \\ &= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] \end{aligned} \quad (6.16)$$

The conditional probability of the receiver deciding in favor of symbol 1, given that symbol 0 was transmitted, is therefore

$$\begin{aligned} p_{10} &= \int_0^{\infty} f_{x_1}(x_1|0) dx_1 \\ &= \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] dx_1 \end{aligned} \quad (6.17)$$

Putting

$$z = \frac{1}{\sqrt{N_0}}(x_1 + \sqrt{E_b}) \quad (6.18)$$

and changing the variable of integration from x_1 to z , we may rewrite Equation (6.17) in the compact form

$$\begin{aligned} p_{10} &= \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp(-z^2) dz \\ &= \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \end{aligned} \quad (6.19)$$

where $\operatorname{erfc}(\cdot)$ is the complementary error function.

Consider next an error of the second kind. We note that the signal space of Figure 6.3 is symmetric with respect to the origin. It follows therefore that p_{01} , the conditional probability of the receiver deciding in favor of symbol 0, given that symbol 1 was transmitted, also has the same value as in Equation (6.19).

Thus, averaging the conditional error probabilities p_{10} and p_{01} , we find that the *average probability of symbol error* or, equivalently, the *bit error rate for coherent binary PSK* is (assuming equiprobable symbols)

$$P_e = \frac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{E_b}{N_0}} \right) \tag{6.20}$$

As we increase the transmitted signal energy per bit, E_b , for a specified noise spectral density N_0 , the message points corresponding to symbols 1 and 0 move further apart, and the average probability of error P_e is correspondingly reduced in accordance with Equation (6.20), which is intuitively satisfying.

Generation and Detection of Coherent Binary PSK Signals

To generate a binary PSK signal, we see from Equations (6.8)–(6.10) that we have to represent the input binary sequence in polar form with symbols 1 and 0 represented by constant amplitude levels of $+\sqrt{E_b}$ and $-\sqrt{E_b}$, respectively. This signal transmission encoding is performed by a polar nonreturn-to-zero (NRZ) level encoder. The resulting binary wave and a sinusoidal carrier $\phi_1(t)$, whose frequency $f_c = (n_c/T_b)$ for some fixed integer n_c , are applied to a product modulator, as in Figure 6.4a. The carrier and the timing pulses used to generate the binary wave are usually extracted from a common master clock. The desired PSK wave is obtained at the modulator output.

To detect the original binary sequence of 1s and 0s, we apply the noisy PSK signal $x(t)$ (at the channel output) to a correlator, which is also supplied with a locally generated coherent reference signal $\phi_1(t)$, as in Figure 6.4b. The correlator output, x_1 , is compared with a threshold of zero volts. If $x_1 > 0$, the receiver decides in favor of symbol 1. On the

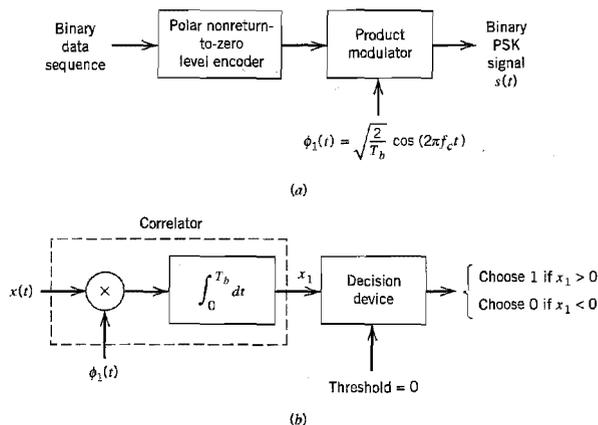


FIGURE 6.4 Block diagrams for (a) binary PSK transmitter and (b) coherent binary PSK receiver.

other hand, if $x_1 < 0$, it decides in favor of symbol 0. If x_1 is exactly zero, the receiver makes a random guess in favor of 0 or 1.

Power Spectra of Binary PSK Signals

From the modulator of Figure 6.4a, we see that the complex envelope of a binary PSK wave consists of an in-phase component only. Furthermore, depending on whether we have symbol 1 or symbol 0 at the modulator input during the signaling interval $0 \leq t \leq T_b$, we find that this in-phase component equals $+g(t)$ or $-g(t)$, respectively, where $g(t)$ is the *symbol shaping function* defined by

$$g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}}, & 0 \leq t \leq T_b \\ 0, & \text{otherwise} \end{cases} \quad (6.21)$$

We assume that the input binary wave is random, with symbols 1 and 0 equally likely and the symbols transmitted during the different time slots being statistically independent. In Example 1.6 of Chapter 1 it is shown that the power spectral density of a random binary wave so described is equal to the energy spectral density of the symbol shaping function divided by the symbol duration. The energy spectral density of a Fourier transformable signal $g(t)$ is defined as the squared magnitude of the signal's Fourier transform. Hence, the baseband power spectral density of a binary PSK signal equals

$$\begin{aligned} S_B(f) &= \frac{2E_b \sin^2(\pi T_b f)}{(\pi T_b f)^2} \\ &= 2E_b \operatorname{sinc}^2(T_b f) \end{aligned} \quad (6.22)$$

This power spectrum falls off as the inverse square of frequency, as shown in Figure 6.5.

Figure 6.5 also includes a plot of the baseband power spectral density of a binary FSK signal, details of which are presented in Section 6.5. Comparison of these two spectra is deferred to that section.

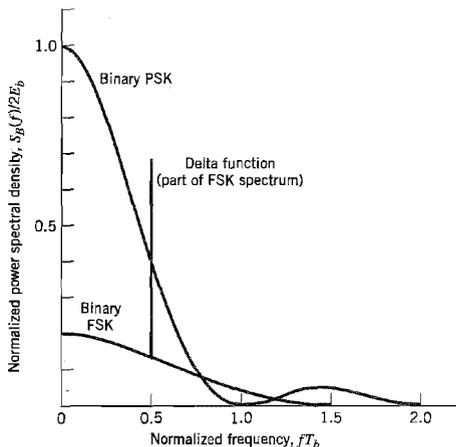


FIGURE 6.5 Power spectra of binary PSK and FSK signals.

■ **QUADRIPHASE-SHIFT KEYING**

The provision of reliable performance, exemplified by a very low probability of error, is one important goal in the design of a digital communication system. Another important goal is the efficient utilization of channel bandwidth. In this subsection, we study a bandwidth-conserving modulation scheme known as coherent quadriphase-shift keying, which is an example of *quadrature-carrier multiplexing*.

In *quadrature-shift keying* (QPSK), as with binary PSK, information carried by the transmitted signal is contained in the phase. In particular, the phase of the carrier takes on one of four equally spaced values, such as $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$. For this set of values we may define the transmitted signal as

$$s_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos \left[2\pi f_c t + (2i - 1) \frac{\pi}{4} \right], & 0 \leq t \leq T \\ 0, & \text{elsewhere} \end{cases} \quad (6.23)$$

where $i = 1, 2, 3, 4$; E is the transmitted signal energy per symbol, and T is the symbol duration. The carrier frequency f_c equals n_c/T for some fixed integer n_c . Each possible value of the phase corresponds to a unique dibit. Thus, for example, we may choose the foregoing set of phase values to represent the *Gray-encoded* set of dibits: 10, 00, 01, and 11, where only a single bit is changed from one dibit to the next.

Signal-Space Diagram of QPSK

Using a well-known trigonometric identity, we may use Equation (6.23) to redefine the transmitted signal $s_i(t)$ for the interval $0 \leq t \leq T$ in the equivalent form:

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos \left[(2i - 1) \frac{\pi}{4} \right] \cos(2\pi f_c t) - \sqrt{\frac{2E}{T}} \sin \left[(2i - 1) \frac{\pi}{4} \right] \sin(2\pi f_c t) \quad (6.24)$$

where $i = 1, 2, 3, 4$. Based on this representation, we can make the following observations:

- ▶ There are two orthonormal basis functions, $\phi_1(t)$ and $\phi_2(t)$, contained in the expansion of $s_i(t)$. Specifically, $\phi_1(t)$ and $\phi_2(t)$ are defined by a pair of *quadrature carriers*:

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \leq t \leq T \quad (6.25)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad 0 \leq t \leq T \quad (6.26)$$

TABLE 6.1 *Signal-space characterization of QPSK*

Gray-encoded Input Dibit	Phase of QPSK Signal (radians)	Coordinates of Message Points	
		s_{i1}	s_{i2}
10	$\pi/4$	$+\sqrt{E/2}$	$-\sqrt{E/2}$
00	$3\pi/4$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
01	$5\pi/4$	$-\sqrt{E/2}$	$+\sqrt{E/2}$
11	$7\pi/4$	$+\sqrt{E/2}$	$+\sqrt{E/2}$

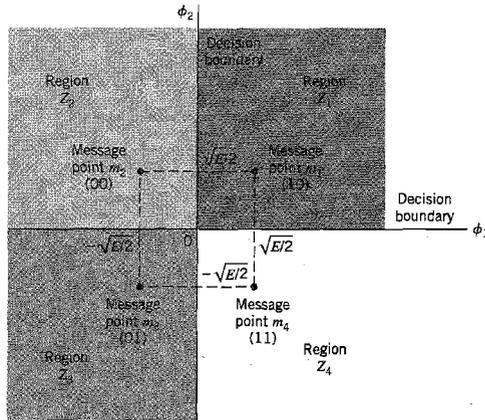


FIGURE 6.6 Signal-space diagram of coherent QPSK system.

- There are four message points, and the associated signal vectors are defined by

$$\mathbf{s}_i = \begin{bmatrix} \sqrt{E} \cos\left((2i-1)\frac{\pi}{4}\right) \\ -\sqrt{E} \sin\left((2i-1)\frac{\pi}{4}\right) \end{bmatrix}, \quad i = 1, 2, 3, 4 \quad (6.27)$$

The elements of the signal vectors, namely, s_{i1} and s_{i2} , have their values summarized in Table 6.1. The first two columns of this table give the associated dibit and phase of the QPSK signal.

Accordingly, a QPSK signal has a two-dimensional signal constellation (i.e., $N = 2$) and four message points (i.e., $M = 4$) whose phase angles increase in a counterclockwise direction, as illustrated in Figure 6.6. As with binary PSK, the QPSK signal has minimum average energy.

► **EXAMPLE 6.1**

Figure 6.7 illustrates the sequences and waveforms involved in the generation of a QPSK signal. The input binary sequence 01101000 is shown in Figure 6.7a. This sequence is divided into two other sequences, consisting of odd- and even-numbered bits of the input sequence. These two sequences are shown in the top lines of Figures 6.7b and 6.7c. The waveforms representing the two components of the QPSK signal, namely, $s_{i1}\phi_1(t)$ and $s_{i2}\phi_2(t)$, are also shown in Figures 6.7b and 6.7c, respectively. These two waveforms may individually be viewed as examples of a binary PSK signal. Adding them, we get the QPSK waveform shown in Figure 6.7d.

To define the decision rule for the detection of the transmitted data sequence, we partition the signal space into four regions, in accordance with Equation (5.59) of Chapter 5. The individual regions are defined by the set of points closest to the message point represented by signal vectors \mathbf{s}_1 , \mathbf{s}_2 , \mathbf{s}_3 , and \mathbf{s}_4 . This is readily accomplished by constructing the perpendicular bisectors of the square formed by joining the four message points and then marking

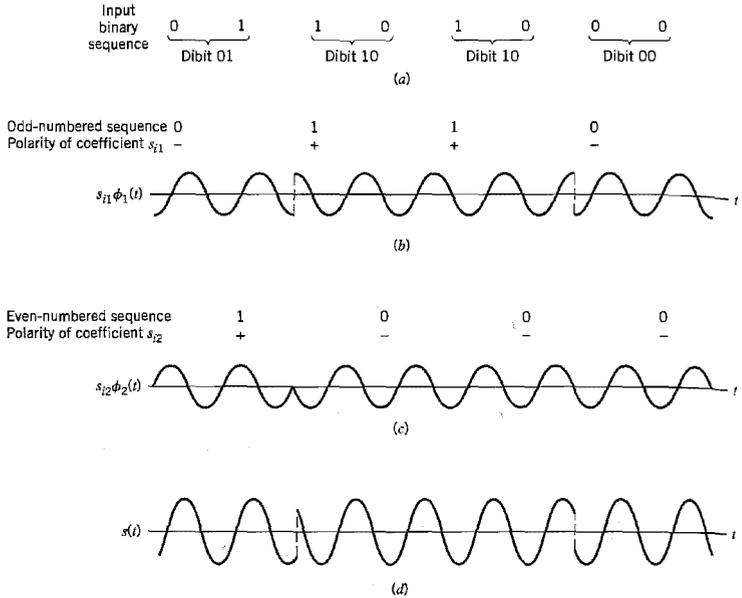


FIGURE 6.7 (a) Input binary sequence. (b) Odd-numbered bits of input sequence and associated binary PSK wave. (c) Even-numbered bits of input sequence and associated binary PSK wave. (d) QPSK waveform defined as $s(t) = s_{i1} \phi_1(t) + s_{i2} \phi_2(t)$.

off the appropriate regions. We thus find that the decision regions are quadrants whose vertices coincide with the origin. These regions are marked $Z_1, Z_2, Z_3,$ and Z_4 , in Figure 6.6, according to the message point around which they are constructed. ◀

Error Probability of QPSK

In a coherent QPSK system, the received signal $x(t)$ is defined by

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, 3, 4 \end{cases} \quad (6.28)$$

where $w(t)$ is the sample function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$. Correspondingly, the observation vector \mathbf{x} has two elements, x_1 and x_2 , defined by

$$\begin{aligned} x_1 &= \int_0^T x(t) \phi_1(t) dt \\ &= \sqrt{E} \cos \left[(2i - 1) \frac{\pi}{4} \right] + w_1 \\ &= \pm \sqrt{\frac{E}{2}} + w_1 \end{aligned} \quad (6.29)$$

and

$$\begin{aligned}
 x_2 &= \int_0^T x(t)\phi_2(t) dt \\
 &= -\sqrt{E} \sin\left[(2i-1)\frac{\pi}{4}\right] + w_2 \\
 &= \mp\sqrt{\frac{E}{2}} + w_2
 \end{aligned} \tag{6.30}$$

Thus the observable elements x_1 and x_2 are sample values of independent Gaussian random variables with mean values equal to $\pm\sqrt{E/2}$ and $\mp\sqrt{E/2}$, respectively, and with a common variance equal to $N_0/2$.

The decision rule is now simply to decide that $s_1(t)$ was transmitted if the received signal point associated with the observation vector \mathbf{x} falls inside region Z_1 , decide that $s_2(t)$ was transmitted if the received signal point falls inside region Z_2 , and so on. An erroneous decision will be made if, for example, signal $s_4(t)$ is transmitted but the noise $w(t)$ is such that the received signal point falls outside region Z_4 .

To calculate the average probability of symbol error, we note from Equation (6.24) that a coherent QPSK system is in fact equivalent to two coherent binary PSK systems working in parallel and using two carriers that are in phase quadrature; this is merely a statement of the quadrature-carrier multiplexing property of coherent QPSK. The in-phase channel output x_1 and the quadrature channel output x_2 (i.e., the two elements of the observation vector \mathbf{x}) may be viewed as the individual outputs of the two coherent binary PSK systems. Thus, according to Equations (6.29) and (6.30), these two binary PSK systems may be characterized as follows:

- ▶ The signal energy per bit is $E/2$.
- ▶ The noise spectral density is $N_0/2$.

Hence, using Equation (6.20) for the average probability of bit error of a coherent binary PSK system, we may now state that the average probability of bit error in *each* channel of the coherent QPSK system is

$$\begin{aligned}
 P' &= \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E/2}{N_0}}\right) \\
 &= \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)
 \end{aligned} \tag{6.31}$$

Another important point to note is that the bit errors in the in-phase and quadrature channels of the coherent QPSK system are statistically independent. The in-phase channel makes a decision on one of the two bits constituting a symbol (dibit) of the QPSK signal, and the quadrature channel takes care of the other bit. Accordingly, the *average probability of a correct decision* resulting from the combined action of the two channels working together is

$$\begin{aligned}
 P_c &= (1 - P')^2 \\
 &= \left[1 - \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)\right]^2 \\
 &= 1 - \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) + \frac{1}{4} \operatorname{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right)
 \end{aligned} \tag{6.32}$$

The average probability of symbol error for coherent QPSK is therefore

$$P_e = 1 - P_c = \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) - \frac{1}{4} \operatorname{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) \quad (6.33)$$

In the region where $(E/2N_0) \gg 1$, we may ignore the quadratic term on the right-hand side of Equation (6.33), so we approximate the formula for the average probability of symbol error for coherent QPSK as

$$P_e \approx \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \quad (6.34)$$

The formula of Equation (6.34) may also be derived in another insightful way, using the signal-space diagram of Figure 6.6. Since the four message points of this diagram are circularly symmetric with respect to the origin, we may apply Equation (5.92), reproduced here in the form

$$P_e \leq \frac{1}{2} \sum_{\substack{k=1 \\ k \neq i}}^4 \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \quad \text{for all } i \quad (6.35)$$

Consider, for example, message point m_1 (corresponding to dibit 10) chosen as the transmitted message point. The message points m_2 and m_4 (corresponding to dibits 00 and 11) are the *closest* to m_1 . From Figure 6.6 we readily find that m_1 is equidistant from m_2 and m_4 in a Euclidean sense, as shown by

$$d_{12} = d_{14} = \sqrt{2E}$$

Assuming that E/N_0 is large enough to ignore the contribution of the most distant message point m_3 (corresponding to dibit 01) relative to m_1 , we find that the use of Equation (6.35) yields an approximate expression for P_e that is the same as Equation (6.34). Note that in mistaking either m_2 or m_4 for m_1 , a single bit error is made; on the other hand, in mistaking m_3 for m_1 , two bit errors are made. For a high enough E/N_0 , the likelihood of both bits of a symbol being in error is much less than a single bit, which is a further justification for ignoring m_3 in calculating P_e when m_1 is sent.

In a QPSK system, we note that since there are two bits per symbol, the transmitted signal energy per symbol is twice the signal energy per bit, as shown by

$$E = 2E_b \quad (6.36)$$

Thus expressing the average probability of symbol error in terms of the ratio E_b/N_0 , we may write

$$P_e \approx \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \quad (6.37)$$

With Gray encoding used for the incoming symbols, we find from Equations (6.31) and (6.36) that the *bit error rate* of QPSK is exactly

$$\text{BER} = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \quad (6.38)$$

We may therefore state that a coherent QPSK system achieves the same average probability of bit error as a coherent binary PSK system for the same bit rate and the same E_b/N_0 , but uses only half the channel bandwidth. Stated in a different way, for the same E_b/N_0 and therefore the same average probability of bit error, a coherent QPSK system transmits information at twice the bit rate of a coherent binary PSK system for the same channel

bandwidth. For a prescribed performance, QPSK uses channel bandwidth better than binary PSK, which explains the preferred use of QPSK over binary PSK in practice.

Generation and Detection of Coherent QPSK Signals

Consider next the generation and detection of QPSK signals. Figure 6.8a shows a block diagram of a typical QPSK transmitter. The incoming binary data sequence is first transformed into polar form by a *nonreturn-to-zero level encoder*. Thus, symbols 1 and 0 are represented by $+\sqrt{E_b}$ and $-\sqrt{E_b}$, respectively. This binary wave is next divided by means of a *demultiplexer* into two separate binary waves consisting of the odd- and even-numbered input bits. These two binary waves are denoted by $a_1(t)$ and $a_2(t)$. We note that in any signaling interval, the amplitudes of $a_1(t)$ and $a_2(t)$ equal s_{i1} and s_{i2} , respectively, depending on the particular dibit that is being transmitted. The two binary waves $a_1(t)$ and $a_2(t)$ are used to modulate a pair of quadrature carriers or orthonormal basis functions: $\phi_1(t)$ equal to $\sqrt{2/T} \cos(2\pi f_c t)$ and $\phi_2(t)$ equal to $\sqrt{2/T} \sin(2\pi f_c t)$. The result is a pair of

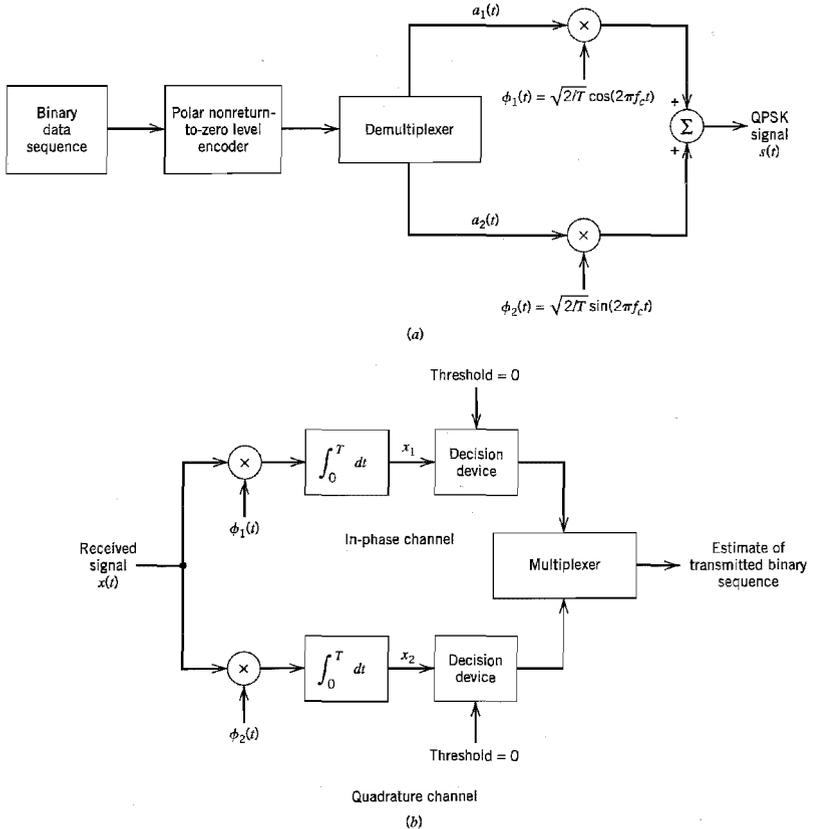


FIGURE 6.8 Block diagrams of (a) QPSK transmitter and (b) coherent QPSK receiver.

binary PSK signals, which may be detected independently due to the orthogonality of $\phi_1(t)$ and $\phi_2(t)$. Finally, the two binary PSK signals are added to produce the desired QPSK signal.

The QPSK receiver consists of a pair of correlators with a common input and supplied with a locally generated pair of coherent reference signals $\phi_1(t)$ and $\phi_2(t)$, as in Figure 6.8b. The correlator outputs x_1 and x_2 , produced in response to the received signal $x(t)$, are each compared with a threshold of zero. If $x_1 > 0$, a decision is made in favor of symbol 1 for the in-phase channel output, but if $x_1 < 0$, a decision is made in favor of symbol 0. Similarly, if $x_2 > 0$, a decision is made in favor of symbol 1 for the quadrature channel output, but if $x_2 < 0$, a decision is made in favor of symbol 0. Finally, these two binary sequences at the in-phase and quadrature channel outputs are combined in a *multiplexer* to reproduce the original binary sequence at the transmitter input with the minimum probability of symbol error in an AWGN channel.

Power Spectra of QPSK Signals

Assume that the binary wave at the modulator input is random, with symbols 1 and 0 being equally likely, and with the symbols transmitted during adjacent time slots being statistically independent. We make the following observations pertaining to the in-phase and quadrature components of a QPSK signal:

1. Depending on the dibit sent during the signaling interval $-T_b \leq t \leq T_b$, the in-phase component equals $+g(t)$ or $-g(t)$, and similarly for the quadrature component. The $g(t)$ denotes the symbol shaping function, defined by

$$g(t) = \begin{cases} \sqrt{\frac{E}{T}}, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \quad (6.39)$$

Hence, the in-phase and quadrature components have a common power spectral density, namely, $E \operatorname{sinc}^2(Tf)$.

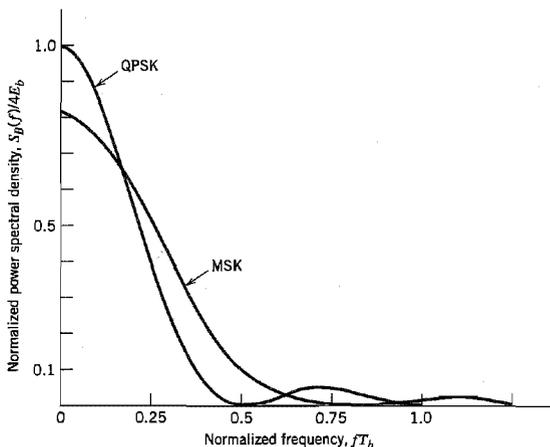


FIGURE 6.9 Power spectra of QPSK and MSK signals.

2. The in-phase and quadrature components are statistically independent. Accordingly, the baseband power spectral density of the QPSK signal equals the sum of the individual power spectral densities of the in-phase and quadrature components, so we may write

$$\begin{aligned} S_B(f) &= 2E \operatorname{sinc}^2(Tf) \\ &= 4E_b \operatorname{sinc}^2(2T_b f) \end{aligned} \quad (6.40)$$

Figure 6.9 plots $S_B(f)$, normalized with respect to $4E_b$, versus the normalized frequency fT_b . This figure also includes a plot of the baseband power spectral density of a certain form of binary FSK called minimum shift keying, the evaluation of which is presented in Section 6.5. Comparison of these two spectra is deferred to that section.

■ OFFSET QPSK

The signal space diagram of Figure 6.10a embodies all the possible phase transitions that can arise in the generation of a QPSK signal. More specifically, examining the QPSK waveform illustrated in Figure 6.7 for Example 6.1, we may make the following observations:

1. The carrier phase changes by ± 180 degrees whenever both the in-phase and quadrature components of the QPSK signal changes sign. An example of this situation is illustrated in Figure 6.7 when the input binary sequence switches from dibit 01 to dibit 10.
2. The carrier phase changes by ± 90 degrees whenever the in-phase or quadrature component changes sign. An example of this second situation is illustrated in Figure 6.7 when the input binary sequence switches from dibit 10 to dibit 00, during which the in-phase component changes sign, whereas the quadrature component is unchanged.
3. The carrier phase is unchanged when neither the in-phase component nor the quadrature component changes sign. This last situation is illustrated in Figure 6.7 when dibit 10 is transmitted in two successive symbol intervals.

Situation 1 and, to a much lesser extent, situation 2 can be of a particular concern when the QPSK signal is filtered during the course of transmission, prior to detection. Specifically, the 180- and 90-degree shifts in carrier phase can result in changes in the carrier amplitude (i.e., envelope of the QPSK signal), thereby causing additional symbol errors on detection.

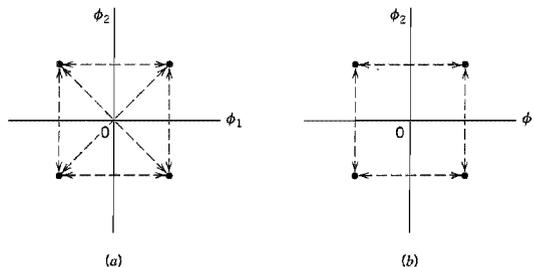


FIGURE 6.10 Possible paths for switching between the message points in (a) QPSK and (b) offset QPSK.

The extent of amplitude fluctuations exhibited by QPSK signals may be reduced by using *offset QPSK*.² In this variant of QPSK, the bit stream responsible for generating the quadrature component is delayed (i.e., offset) by half a symbol interval with respect to the bit stream responsible for generating the in-phase component. Specifically, the two basis functions of offset QPSK are defined by

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \leq t \leq T \quad (6.41)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad \frac{T}{2} \leq t \leq \frac{3T}{2} \quad (6.42)$$

Accordingly, unlike QPSK, the phase transitions likely to occur in offset QPSK are confined to ± 90 degrees, as indicated in the signal space diagram of Figure 6.10*b*. However, ± 90 degree phase transitions in offset QPSK occur twice as frequently but with half the intensity encountered in QPSK. Since, in addition to ± 90 -degree phase transitions, ± 180 -degree phase transitions also occur in QPSK, we find that amplitude fluctuations in offset QPSK due to filtering have a smaller amplitude than in the case of QPSK.

Despite the delay $T/2$ applied to the basis function $\phi_2(t)$ in Equation (6.42) compared to that in Equation (6.26), the offset QPSK has exactly the same probability of symbol error in an AWGN channel as QPSK. The equivalence in noise performance between these phase-shift keying schemes assumes the use of coherent detection. The reason for the equivalence is that the statistical independence of the in-phase and quadrature components applies to both QPSK and offset QPSK. We may therefore say that the error probability in the in-phase or quadrature channel of a coherent offset QPSK receiver is still equal to $(1/2) \operatorname{erfc}(\sqrt{E}/2N_0)$. Hence the formula of Equation (6.34) applies equally well to the offset QPSK.

■ $\pi/4$ -SHIFTED QPSK

An ordinary QPSK signal may reside in either one of the two commonly used constellations shown in Figures 6.11*a* and 6.11*b*, which are shifted by $\pi/4$ radians with respect to each other. In another variant of QPSK known as *$\pi/4$ -shifted QPSK*,³ the carrier phase used for the transmission of successive symbols (i.e., dibits) is alternately picked from one of the two QPSK constellations in Figure 6.11 and then the other. It follows therefore that a $\pi/4$ -shifted QPSK signal may reside in any one of eight possible phase states, as indicated

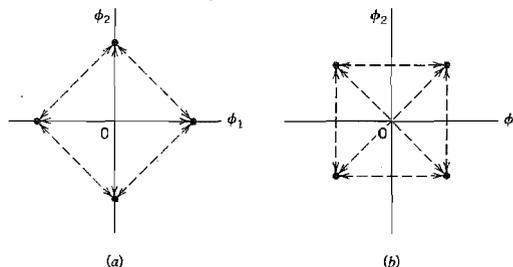


FIGURE 6.11 Two commonly used signal constellations for QPSK; the arrows indicate the paths along which the QPSK modulator can change its state.

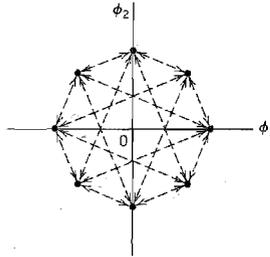


FIGURE 6.12 Eight possible phase states for the $\pi/4$ -shifted QPSK modulator.

in Figure 6.12. The four dashed lines emanating from each possible message point in Figure 6.12 define the phase transitions that are feasible in $\pi/4$ -shifted QPSK.

Table 6.2 summarizes a possible set of relationships between the phase transitions in this new digital modulation scheme and the incoming Gray-encoded dibits. For example, if the modulator is in one of the phase states portrayed in Figure 6.11*b*, then on receiving the dibit 00 it shifts into a phase state portrayed in Figure 6.11*a* by rotating through $\pi/4$ radians in a counterclockwise direction.

Attractive features of the $\pi/4$ -shifted QPSK scheme include the following:

- ▶ The phase transitions from one symbol to the next are restricted to $\pm\pi/4$ and $\pm 3\pi/4$ radians, which is to be contrasted with the $\pm\pi/2$ and $\pm\pi$ phase transitions in QPSK. Consequently, envelope variations of $\pi/4$ -shifted QPSK signals due to filtering are significantly reduced, compared to those in QPSK.
- ▶ Unlike offset QPSK signals, $\pi/4$ -shifted QPSK signals can be noncoherently detected, thereby considerably simplifying the receiver design. Moreover, like QPSK signals, $\pi/4$ -shifted QPSK can be differently encoded, in which case we should really speak of $\pi/4$ -shifted DQPSK.

The generation of $\pi/4$ -shifted DQPSK symbols, represented by the symbol pair (I, Q) , is described by the following pair of relationships (see Problem 6.13):

$$\begin{aligned} I_k &= \cos(\theta_{k-1} + \Delta\theta_k) \\ &= \cos \theta_k \end{aligned} \quad (6.43)$$

$$\begin{aligned} Q_k &= \sin(\theta_{k-1} + \Delta\theta_k) \\ &= \sin \theta_k \end{aligned} \quad (6.44)$$

TABLE 6.2 Correspondence between input dibit and phase change for $\pi/4$ -shifted DQPSK

Gray-Encoded Input Dibit	Phase Change, $\Delta\theta$ (radians)
00	$\pi/4$
01	$3\pi/4$
11	$-3\pi/4$
10	$-\pi/4$

TABLE 6.3 $\pi/4$ -shifted DQPSK results for Example 6.2

Step k	Phase θ_{k-1} (radians)	Input Dibit	Phase Change $\Delta\theta_k$ (radians)	Transmitted Phase θ_k (radians)
1	$\pi/4$	00	$\pi/4$	$\pi/2$
2	$\pi/2$	10	$-\pi/4$	$\pi/4$
3	$\pi/4$	10	$-\pi/4$	0
4	0	01	$3\pi/4$	$3\pi/4$

where θ_{k-1} is the absolute phase angle of symbol $k - 1$, and $\Delta\theta_k$ is the differentially encoded phase change defined in accordance with Table 6.2.

► **EXAMPLE 6.2**

Continuing with the input binary sequence of Example 6.1, namely, 01101000, suppose that the phase angle $\theta_0 = \pi/4$ in the constellation of Figure 6.11b is assigned as the *initial* phase state of the $\pi/4$ -shifted DQPSK modulator. Then, arranging the input binary sequence as a sequence of dibits and following the convention of Table 6.2, we get the results presented in Table 6.3 for the example at hand. ◀

Detection of $\pi/4$ -Shifted DQPSK Signals

Having familiarized ourselves with the generation of $\pi/4$ -shifted DQPSK signals, we go on to consider their differential detection. Given the noisy channel output $x(t)$, the receiver first computes the projections of $x(t)$ onto the basis functions $\phi_1(t)$ and $\phi_2(t)$. The resulting outputs, denoted by I and Q , respectively, are applied to a *differential detector* that consists of the following components, as indicated in Figure 6.13:

- ▶ *Arctangent computer* for extracting the phase angle θ of the channel output (received signal).
- ▶ *Phase-difference computer* for determining the change in the phase θ occurring over one symbol interval.
- ▶ *Modulo- 2π correction logic* for correcting errors due to the possibility of phase angles wrapping around the real axis.

Elaborating further on the latter point, let $\Delta\theta_k$ denote the computed phase difference between θ_k and θ_{k-1} representing the phase angles of the channel output for symbols k and $k - 1$, respectively. Then the modulo- 2π correction logic operates as follows:

$$\begin{aligned} \text{IF } \Delta\theta_k < -180 \text{ degrees THEN } \Delta\theta_k &= \Delta\theta_k + 360 \text{ degrees} \\ \text{IF } \Delta\theta_k > 180 \text{ degrees THEN } \Delta\theta_k &= \Delta\theta_k - 360 \text{ degrees} \end{aligned} \tag{6.45}$$

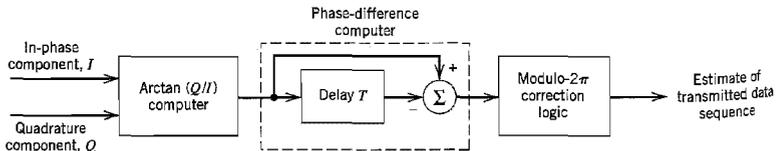


FIGURE 6.13 Block diagram of the $\pi/4$ -shifted DQPSK detector.

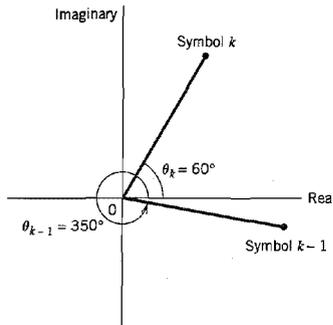


FIGURE 6.14 Illustrating the possibility of phase angles wrapping around the positive real axis.

To illustrate the need for this phase correction, consider the situation depicted in Figure 6.14, where $\theta_{k-1} = 350$ degrees and $\theta_k = 60$ degrees, both phase angles being measured in a counterclockwise direction. From this figure we readily see that the phase change $\Delta\theta_k$, measured in a counterclockwise direction, is 70 degrees. However, without correction the phase change $\Delta\theta_k$ is computed as 60 degrees $-$ 350 degrees $= -290$ degrees. Applying the first line of Equation (6.45), the modulo- 2π correction logic compensates for the wrap-around the positive real axis, yielding the corrected result

$$\Delta\theta_k = -290 \text{ degrees} + 360 \text{ degrees} = 70 \text{ degrees}$$

The tangent type differential detector of Figure 6.13 for the demodulation of $\pi/4$ -shifted DQPSK signals is relatively simple to implement. It offers a satisfactory performance in a Rayleigh fading channel as in a static multipath environment. However, when the multipath environment is time varying as experienced in a commercial digital wireless communication system, computer simulation results appear to show that the receiver performance degrades very rapidly.⁴

■ M-ARY PSK

QPSK is a special case of M -ary PSK, where the phase of the carrier takes on one of M possible values, namely, $\theta_i = 2(i-1)\pi/M$, where $i = 1, 2, \dots, M$. Accordingly, during each signaling interval of duration T , one of the M possible signals

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(2\pi f_c t + \frac{2\pi}{M}(i-1)\right), \quad i = 1, 2, \dots, M \quad (6.46)$$

is sent, where E is the signal energy per symbol. The carrier frequency $f_c = n_c/T$ for some fixed integer n_c .

Each $s_i(t)$ may be expanded in terms of the same two basis functions $\phi_1(t)$ and $\phi_2(t)$ defined in Equations (6.25) and (6.26), respectively. The signal constellation of M -ary PSK is therefore two-dimensional. The M message points are equally spaced on a circle of radius \sqrt{E} and center at the origin, as illustrated in Figure 6.15a, for the case of *octaphase-shift-keying* (i.e., $M = 8$).

From Figure 6.15a we note that the signal-space diagram is circularly symmetric. We may therefore apply Equation (5.92), based on the union bound, to develop an approximate formula for the average probability of symbol error for M -ary PSK. Suppose that the

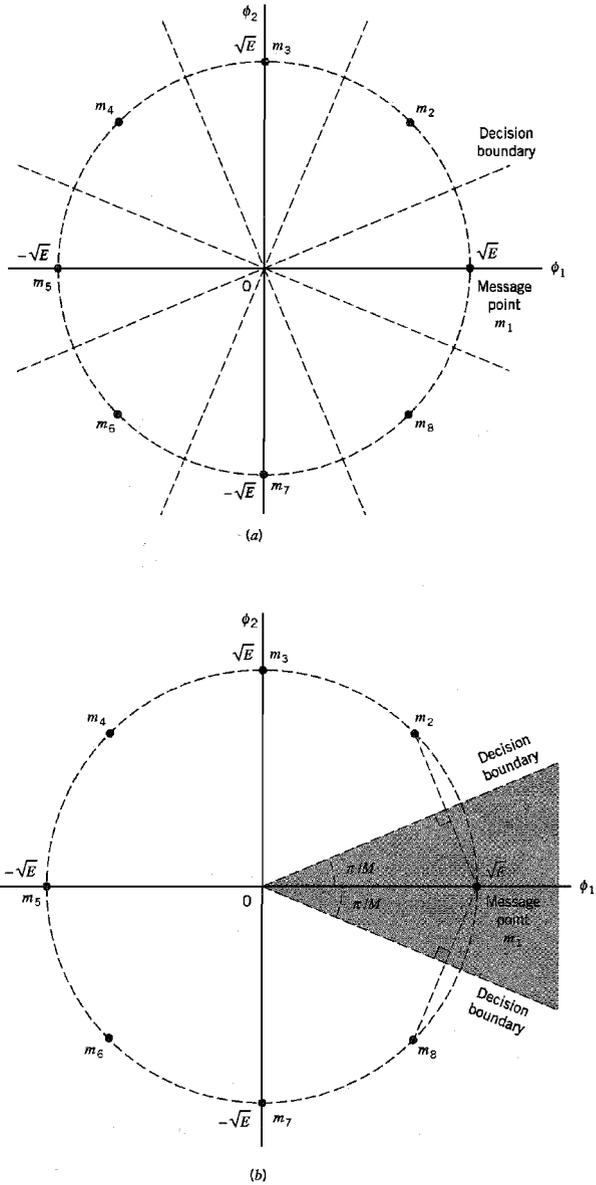


FIGURE 6.15 (a) Signal-space diagram for octaphase-shift keying (i.e., $M = 8$). The decision boundaries are shown as dashed lines. (b) Signal-space diagram illustrating the application of the union bound for octaphase-shift keying.

transmitted signal corresponds to the message point m_1 , whose coordinates along the ϕ_1 - and ϕ_2 -axes are $+\sqrt{E}$ and 0, respectively. Suppose that the ratio E/N_0 is large enough to consider the nearest two message points, one on either side of m_1 , as potential candidates for being mistaken for m_1 due to channel noise. This is illustrated in Figure 6.15b for the case of $M = 8$. The Euclidean distance of each of these two points from m_1 is (for $M = 8$)

$$d_{12} = d_{18} = 2\sqrt{E} \sin\left(\frac{\pi}{M}\right)$$

Hence, the use of Equation (5.92) of Chapter 5 yields the average probability of symbol error for coherent M -ary PSK as

$$P_e \approx \operatorname{erfc}\left(\sqrt{\frac{E}{N_0}} \sin\left(\frac{\pi}{M}\right)\right) \quad (6.47)$$

where it is assumed that $M \geq 4$. The approximation becomes extremely tight, for fixed M , as E/N_0 is increased. For $M = 4$, Equation (6.47) reduces to the same form given in Equation (6.34) for QPSK.

Power Spectra of M -ary PSK Signals

The symbol duration of M -ary PSK is defined by

$$T = T_b \log_2 M \quad (6.48)$$

where T_b is the bit duration. Proceeding in a manner similar to that described for a QPSK signal, we may show that the baseband power spectral density of an M -ary PSK signal is given by

$$\begin{aligned} S_B(f) &= 2E \operatorname{sinc}^2(Tf) \\ &= 2E_b \log_2 M \operatorname{sinc}^2(T_b f \log_2 M) \end{aligned} \quad (6.49)$$

In Figure 6.16, we show the normalized power spectral density $S_B(f)/2E_b$ plotted versus the normalized frequency fT_b for three different values of M , namely, $M = 2, 4, 8$.

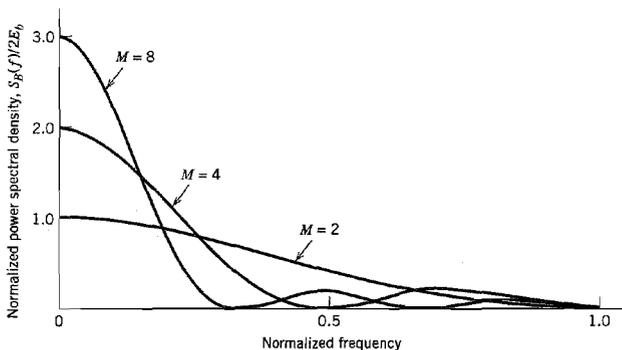


FIGURE 6.16 Power spectra of M -ary PSK signals for $M = 2, 4, 8$.

TABLE 6.4 Bandwidth efficiency of *M*-ary PSK signals

<i>M</i>	2	4	8	16	32	64
ρ (bits/s/Hz)	0.5	1	1.5	2	2.5	3

■ BANDWIDTH EFFICIENCY OF *M*-ARY PSK SIGNALS

The power spectra of *M*-ary PSK signals possess a *main lobe* bounded by well-defined *spectral nulls* (i.e., frequencies at which the power spectral density is zero). Accordingly, the spectral width of the main lobe provides a simple and popular measure for the bandwidth of *M*-ary PSK signals. This definition is referred to as the *null-to-null bandwidth*. With the null-to-null bandwidth encompassing the main lobe of the power spectrum of an *M*-ary signal, we find that it contains most of the signal power. This is readily seen by looking at the power spectral plots of Figure 6.16.

For the passband basis functions defined in Equations (6.25) and (6.26), the channel bandwidth required to pass *M*-ary PSK signals (more precisely, the main spectral lobe of *M*-ary signals) is given by

$$B = \frac{2}{T} \quad (6.50)$$

where *T* is the symbol duration. But the symbol duration *T* is related to the bit duration *T_b* by Equation (6.48). Moreover, the bit rate $R_b = 1/T_b$. Hence, we may redefine the channel bandwidth of Equation (6.50) in terms of the bit rate *R_b* as

$$B = \frac{2R_b}{\log_2 M} \quad (6.51)$$

Based on this formula, the bandwidth efficiency of *M*-ary PSK signals is given by

$$\begin{aligned} \rho &= \frac{R_b}{B} \\ &= \frac{\log_2 M}{2} \end{aligned} \quad (6.52)$$

Table 6.4 gives the values of ρ calculated from Equation (6.52) for varying *M*.

In light of Equation (6.47) and Table 6.4, we make the following observation in the context of *M*-ary PSK: As the number of states, *M*, is increased, the bandwidth efficiency is improved at the expense of error performance. To ensure that there is no degradation in error performance, we have to increase E_b/N_0 to compensate for the increase in *M*.

6.4 Hybrid Amplitude/Phase Modulation Schemes

In an *M*-ary PSK system, the in-phase and quadrature components of the modulated signal are interrelated in such a way that the envelope is constrained to remain constant. This constraint manifests itself in a circular constellation for the message points. However, if

this constraint is removed, and the in-phase and quadrature components are thereby permitted to be independent, we get a new modulation scheme called *M*-ary quadrature amplitude modulation (QAM). This latter modulation scheme is *hybrid* in nature in that the carrier experiences amplitude as well as phase modulation.

The passband basis functions in *M*-ary QAM may not be periodic for an arbitrary choice of the carrier frequency f_c with respect to the symbol rate $1/T$. Ordinarily, this aperiodicity is of no real concern. By reformulating the expression for the transmitted signal in a certain way, it is possible to eliminate the time variation of the basis functions on successive symbol transmissions, and thereby simplify implementation of the transmitter. In particular, the transmitter is made to appear “carrierless,” while fully retaining the essence of the hybridized amplitude and phase modulation process. This is indeed the idea behind the carrierless amplitude/phase modulation (CAP).

Despite the differences between QAM and CAP in their implementation details, they have exactly the same signal constellations. Accordingly, they are fundamentally equivalent in performance for a prescribed receiver complexity. In what follows we first discuss QAM and then CAP.

■ M-ARY QUADRATURE AMPLITUDE MODULATION

In Chapters 4 and 5, we studied *M*-ary pulse amplitude modulation (PAM), which is one-dimensional. *M*-ary QAM is a two-dimensional generalization of *M*-ary PAM in that its formulation involves two orthogonal passband basis functions, as shown by

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t), \quad 0 \leq t \leq T \quad (6.53)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t), \quad 0 \leq t \leq T \quad (6.54)$$

Let the i th message point s_i in the (ϕ_1, ϕ_2) plane be denoted by $(a_i d_{\min}/2, b_i d_{\min}/2)$, where d_{\min} is the minimum distance between any two message points in the constellation, a_i and b_i are integers, and $i = 1, 2, \dots, M$. Let $(d_{\min}/2) = \sqrt{E_0}$, where E_0 is the energy of the signal with the lowest amplitude. The transmitted *M*-ary QAM signal for symbol k , say, is then defined by

$$s_k(t) = \sqrt{\frac{2E_0}{T}} a_k \cos(2\pi f_c t) - \sqrt{\frac{2E_0}{T}} b_k \sin(2\pi f_c t), \quad 0 \leq t \leq T \quad (6.55)$$

$$k = 0, \pm 1, \pm 2, \dots$$

The signal $s_k(t)$ consists of two phase-quadrature carriers with each one being modulated by a set of discrete amplitudes, hence the name *quadrature amplitude modulation*.

Depending on the number of possible symbols M , we may distinguish two distinct QAM constellations: square constellations for which the number of bits per symbol is even, and cross constellations for which the number of bits per symbol is odd. These two cases are considered in the sequel in that order.

QAM Square Constellations

With an *even* number of bits per symbol, we may write

$$L = \sqrt{M} \quad (6.56)$$

where L is a positive integer. Under this condition, an *M*-ary QAM square constellation can always be viewed as the *Cartesian product* of a one-dimensional L -ary PAM constel-

lation with itself. By definition, the Cartesian product of two sets of coordinates (representing a pair of one-dimensional constellations) is made up of the set of all possible ordered pairs of coordinates with the first coordinate in each such pair taken from the first set involved in the product and the second coordinate taken from the second set in the product.

In the case of a QAM square constellation, the ordered pairs of coordinates naturally form a square matrix, as shown by

$$\{a_i, b_i\} = \begin{bmatrix} (-L + 1, L - 1) & (-L + 3, L - 1) & \dots & (L - 1, L - 1) \\ (-L + 1, L - 3) & (-L + 3, L - 3) & \dots & (L - 1, L - 3) \\ \vdots & \vdots & & \vdots \\ (-L + 1, -L + 1) & (-L + 3, -L + 1) & \dots & (L - 1, -L + 1) \end{bmatrix} \quad (6.57)$$

► **EXAMPLE 6.3**

Consider a 16-QAM whose signal constellation is depicted in Figure 6.17a. The encoding of the message points shown in this figure is as follows:

- Two of the four bits, namely, the left-most two bits, specify the quadrant in the (ϕ_1, ϕ_2) -plane in which a message point lies. Thus, starting from the first quadrant and proceeding counterclockwise, the four quadrants are represented by the dibits 11, 10, 00, and 01.
- The remaining two bits are used to represent one of the four possible symbols lying within each quadrant of the (ϕ_1, ϕ_2) -plane.

Note that the encoding of the four quadrants and also the encoding of the symbols in each quadrant follow the Gray coding rule.

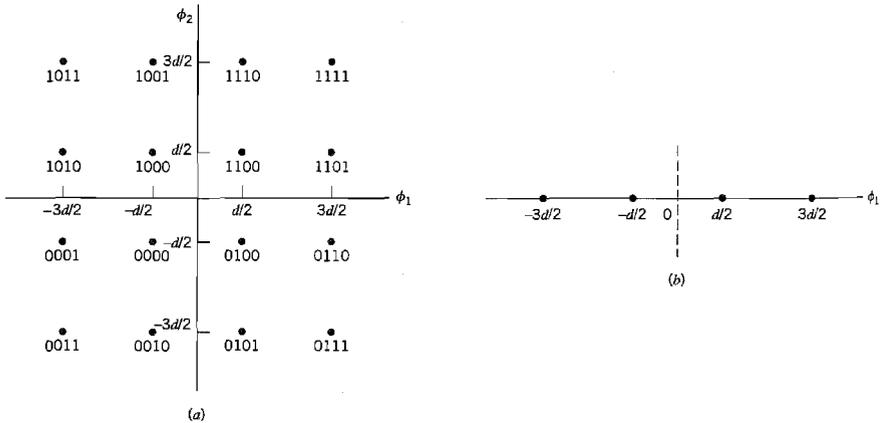


FIGURE 6.17 (a) Signal-space diagram of M -ary QAM for $M = 16$; the message points in each quadrant are identified with Gray-encoded quadsbits. (b) Signal-space diagram of the corresponding 4-PAM signal.

For the example at hand, we have $L = 4$. Thus the square constellation of Figure 6.17a is the Cartesian product of the 4-PAM constellation shown in Figure 6.17b with itself. Moreover, the matrix of Equation (6.57) has the value

$$\{a_i, b_i\} = \begin{bmatrix} (-3, 3) & (-1, 3) & (1, 3) & (3, 3) \\ (-3, 1) & (-1, 1) & (1, 1) & (3, 1) \\ (-3, -1) & (-1, -1) & (1, -1) & (3, -1) \\ (-3, -3) & (-1, -3) & (1, -3) & (3, -3) \end{bmatrix}$$

To calculate the probability of symbol error for M -ary QAM, we exploit the property that a QAM square constellation can be factored into the product of the corresponding PAM constellation with itself. We may thus proceed as follows:

1. The probability of correct detection for M -ary QAM may be written as

$$P_c = (1 - P'_e)^2 \quad (6.58)$$

where P'_e is the probability of symbol error for the corresponding L -ary PAM with $L = \sqrt{M}$.

2. The probability of symbol error P'_e is defined by

$$P'_e = \left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right) \quad (6.59)$$

(Note that $L = \sqrt{M}$ in the M -ary QAM corresponds to M in the M -ary PAM considered in Problem 4.27.)

3. The probability of symbol error for M -ary QAM is given by

$$\begin{aligned} P_e &= 1 - P_c \\ &= 1 - (1 - P'_e)^2 \\ &\approx 2P'_e \end{aligned} \quad (6.60)$$

where it is assumed that P'_e is small enough compared to unity to justify ignoring the quadratic term.

Hence, using Equations (6.58) and (6.59) in Equation (6.60), we find that the probability of symbol error for M -ary QAM is approximately given by

$$P_e \approx 2 \left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{E_0}{N_0}}\right) \quad (6.61)$$

The transmitted energy in M -ary QAM is variable in that its instantaneous value depends on the particular symbol transmitted. It is therefore more logical to express P_e in terms of the *average* value of the transmitted energy rather than E_0 . Assuming that the L amplitude levels of the in-phase or quadrature component are equally likely, we have

$$E_{\text{av}} = 2 \left[\frac{2E_0}{L} \sum_{i=1}^{L/2} (2i - 1)^2 \right] \quad (6.62)$$

where the multiplying factor of 2 outside the square brackets accounts for the equal contributions made by the in-phase and quadrature components. The limits of the summation and the multiplying factor of 2 inside the square brackets take account of the symmetric nature of the pertinent amplitude levels around zero. Summing the series in Equation (6.62), we get

$$\begin{aligned} E_{av} &= \frac{2(L^2 - 1)E_0}{3} \\ &= \frac{2(M - 1)E_0}{3} \end{aligned} \quad (6.63)$$

Accordingly, we may rewrite Equation (6.61) in terms of E_{av} as

$$P_e \approx 2 \left(1 - \frac{1}{\sqrt{M}} \right) \operatorname{erfc} \left(\sqrt{\frac{3E_{av}}{2(M - 1)N_0}} \right) \quad (6.64)$$

which is the desired result.

The case of $M = 4$ is of special interest. The signal constellation for this value of M is the same as that for QPSK. Indeed, putting $M = 4$ in Equation (6.64) and noting that for this special case E_{av} equals E , where E is the energy per symbol, we find that the resulting formula for the probability of symbol error becomes identical to that in Equation (6.34), and so it should.

QAM Cross Constellation

To generate an M -ary QAM signal with an odd number of bits per symbol, we require the use of a *cross constellation*. As illustrated in Figure 6.18, we may construct such a signal constellation with n bits per symbol by proceeding as follows:

- ▶ Start with a QAM square constellation with $n-1$ bits per symbol.
- ▶ Extend each side of the QAM square constellation by adding 2^{n-3} symbols.
- ▶ Ignore the corners in the extension.

The inner square represents 2^{n-1} symbols. The four side extensions add $4 \times 2^{n-3} = 2^{n-1}$ symbols. The total number of symbols in the cross constellation is therefore $2^{n-1} + 2^{n-1}$, which equals 2^n and therefore represents n bits per symbol as desired.

Unlike QAM square constellation, it is *not* possible to express a QAM cross constellation as the product of a PAM constellation with itself. The absence of such a factor-

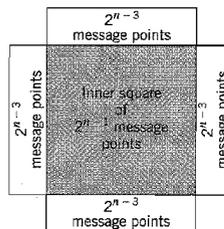


FIGURE 6.18 Illustrating how a square QAM constellation can be expanded to form a QAM cross-constellation.

ization complicates the determination of the probability of symbol error P_e incurred in the use of M -ary QAM characterized by a cross constellation. We therefore simply state the formula for P_e without proof, as shown here

$$P_e \approx 2 \left(1 - \frac{1}{\sqrt{2M}} \right) \operatorname{erfc} \left(\sqrt{\frac{E_0}{N_0}} \right) \text{ for high } E_0/N_0 \quad (6.65)$$

which agrees with the formula of Equation (6.61) for a square constellation, except for the inclusion of an extra 0.5 bit per dimension in the constellation.⁵ Note also that it is not possible to perfectly Gray code a QAM cross constellation.

■ CARRIERLESS AMPLITUDE/PHASE MODULATION

The passband basis functions of Equations (6.53) and (6.54) assume the use of a rectangular pulse for the pulse-shaping function. For reasons that will become apparent, we redefine the transmitted M -ary QAM signal of Equation (6.55) in terms of a general pulse-shaping function $g(t)$ as

$$s_k(t) = a_k g(t - kT) \cos(2\pi f_c t) - b_k g(t - kT) \sin(2\pi f_c t), \quad \begin{array}{l} 0 \leq t \leq T \\ k = 0, \pm 1, \pm 2, \dots \end{array} \quad (6.66)$$

It is assumed that carrier frequency f_c has an arbitrary value with respect to the symbol rate $1/T$. On the basis of Equation (6.66), we may express the transmitted M -ary QAM signal $s(t)$ for an infinite succession of symbols as

$$\begin{aligned} s(t) &= \sum_{k=-\infty}^{\infty} s_k(t) \\ &= \sum_{k=-\infty}^{\infty} [a_k g(t - kT) \cos(2\pi f_c t) - b_k g(t - kT) \sin(2\pi f_c t)] \end{aligned} \quad (6.67)$$

This equation shows that for an arbitrary f_c , the passband functions $g(t - kT) \cos(2\pi f_c t)$ and $g(t - kT) \sin(2\pi f_c t)$ are *aperiodic* in that they vary from one symbol to another.

How can we eliminate the time variations of these passband basis functions from symbol to symbol? To answer this question, we find it convenient to change our formalism from real to complex notation. Specifically, we rewrite Equation (6.67) in the equivalent form

$$\begin{aligned} s(t) &= \operatorname{Re} \left\{ \sum_{k=-\infty}^{\infty} (a_k + jb_k) g(t - kT) \exp(j2\pi f_c t) \right\} \\ &= \operatorname{Re} \left\{ \sum_{k=-\infty}^{\infty} A_k g(t - kT) \exp(j2\pi f_c t) \right\} \end{aligned} \quad (6.68)$$

where A_k is a complex number defined by

$$A_k = a_k + jb_k \quad (6.69)$$

and $\operatorname{Re}\{\cdot\}$ denotes the real part of the complex quantity enclosed inside the braces. Clearly, Equation (6.68) is unchanged by multiplying the summand in this equation by unity ex-

pressed as the product of the complex exponential $\exp(-j2\pi f_c kT)$ and its complex conjugate $\exp(j2\pi f_c kT)$. We may thus rewrite Equation (6.68) in the new form

$$\begin{aligned} s(t) &= \operatorname{Re} \left\{ \sum_{k=-\infty}^{\infty} A_k g(t - kT) \exp(j2\pi f_c t) \exp(-j2\pi f_c kT) \exp(j2\pi f_c kT) \right\} \\ &= \operatorname{Re} \left\{ \sum_{k=-\infty}^{\infty} A_k \exp(j2\pi f_c kT) g(t - kT) \exp(j2\pi f_c (t - kT)) \right\} \end{aligned} \quad (6.70)$$

Define

$$\bar{A}_k = A_k \exp(j2\pi f_c kT) \quad (6.71)$$

$$g_+(t) = g(t) \exp(j2\pi f_c t) \quad (6.72)$$

The scalar \bar{A}_k is simply a *rotated* version of the complex representation of the coordinates of the k th transmitted symbol in the (ϕ_1, ϕ_2) -plane. Before presenting an interpretation of the complex-valued signal $g_+(t)$, we assume that the pulse-shaping function $g(t)$ is a low-pass signal whose highest frequency component is smaller than the carrier frequency f_c . Then following the material presented in Appendix 2, we recognize $g_+(t)$ as the *analytic signal*, or *pre-envelope*, representation of the band-pass signal $g(t) \cos(2\pi f_c t)$. To be more specific, we expand $g_+(t)$ as

$$\begin{aligned} g_+(t) &= g(t) \cos(2\pi f_c t) + jg(t) \sin(2\pi f_c t) \\ &= p(t) + j\hat{p}(t) \end{aligned} \quad (6.73)$$

where $p(t)$ and $\hat{p}(t)$ are defined by

$$p(t) = g(t) \cos(2\pi f_c t) \quad (6.74)$$

and

$$\hat{p}(t) = g(t) \sin(2\pi f_c t) \quad (6.75)$$

We may then say that the quadrature (imaginary) component $\hat{p}(t)$ of the analytic signal $g_+(t)$ is the *Hilbert transform* of the in-phase (real) component $p(t)$. Note that whereas the pulse-shaping function $g(t)$ is a baseband function, the in-phase and quadrature components of the corresponding analytic signal $g_+(t)$ are both passband functions. Henceforth, $g(t)$ is referred to as the *baseband pulse*, and $p(t)$ and $\hat{p}(t)$ are referred to as *passband in-phase* and *passband quadrature pulses*, respectively.

With the definitions of \bar{A}_k and $g_+(t)$ at hand, we are ready to finally redefine the transmitted signal of Equation (6.70) simply as

$$s(t) = \operatorname{Re} \left\{ \sum_{k=-\infty}^{\infty} \bar{A}_k g_+(t - kT) \right\} \quad (6.76)$$

Three important observations are noteworthy from this new formulation of the transmitted signal:

- ▶ The transmitted signal $s(t)$ appears to be carrierless.
- ▶ Since the formulation of $s(t)$ in Equation (6.76) is indeed equivalent to that of the M -ary QAM signal presented in Equation (6.67), the new formulation of $s(t)$ fully retains the hybridized amplitude and phase modulation characterizing the original M -ary QAM signal.
- ▶ The transmitted signal $s(t)$ represents a symbol-time-invariant realization of this hybrid modulation process.

For a prescribed carrier frequency f_c , the sequence of rotations described in Equation (6.71) is known. Hence, the receiver need only detect \tilde{A}_k , in which case we may compute the corresponding value of A_k by applying the reverse rotations as described here:

$$A_k = \tilde{A}_k \exp(-j2\pi f_c kT), \quad k = 0, \pm 1, \pm 2, \dots$$

In practice, however, the rotations are ignored because they do *not* have any bearing on operation or performance of the hybrid modulation system; application of the rotations is in fact necessary only when its equivalence to QAM is an issue of interest. Accordingly, we may ignore Equation (6.71) and redefine the transmitted signal of Equation (6.76) simply as

$$\begin{aligned} s(t) &= \operatorname{Re} \left\{ \sum_{k=-\infty}^{\infty} A_k g_+(t - kT) \right\} \\ &= \operatorname{Re} \left\{ \sum_{k=-\infty}^{\infty} (a_k + jb_k) (p(t - kT) + j\hat{p}(t - kT)) \right\} \\ &= \sum_{k=-\infty}^{\infty} [a_k p(t - kT) - b_k \hat{p}(t - kT)] \end{aligned} \quad (6.77)$$

where, as mentioned previously, $\hat{p}(t)$ is the Hilbert transform of $p(t)$. For obvious reasons, the transmitted signal of Equation (6.77) is referred to as *carrierless amplitude/phase modulation* (CAP).⁶

Properties of the Passband In-phase and Quadrature Pulses

From the definitions of the passband in-phase and quadrature pulses given in Equations (6.74) and (6.75), we deduce the following properties:

Property 1

The passband in-phase pulse $p(t)$ and quadrature pulse $\hat{p}(t)$ are even and odd functions of time t , respectively, given that the baseband pulse $g(t)$ is an even function of time t .

This property follows directly from Equations (6.74) and (6.75).

Property 2

The passband pulses $p(t)$ and $\hat{p}(t)$ form an orthogonal set over the entire interval $(-\infty, \infty)$ as shown by

$$\int_{-\infty}^{\infty} p(t)\hat{p}(t) dt = 0 \quad (6.78)$$

The reason for not restricting the integration interval in Equation (6.78) to a symbol period T is that the pulses $p(t)$ and $\hat{p}(t)$ are bandwidth-efficient for high-performing CAP systems. To prove Property 2, we transform Equation (6.78) into the frequency domain by using the Fourier transform to write

$$\int_{-\infty}^{\infty} p(t)\hat{p}(t) dt = \int_{-\infty}^{\infty} P(f)\hat{P}^*(f) df \quad (6.79)$$

where $p(t) \Leftrightarrow P(f)$ and $\hat{p}(t) \Leftrightarrow \hat{P}(f)$. The asterisk in $\hat{P}^*(f)$ denotes complex conjugation. (The frequency function $\hat{P}(f)$ should *not* be viewed as the Hilbert transform of $P(f)$;

rather, following the usual terminology in Fourier analysis, $\hat{P}(f)$ is simply the Fourier transform of $\hat{p}(t)$.) The Fourier transform $\hat{P}(f)$ is related to the Fourier transform $P(f)$ by (see Appendix 2)

$$\hat{P}(f) = -j \operatorname{sgn}(f)P(f) \quad (6.80)$$

where $\operatorname{sgn}(f)$ is the signum function. The Fourier transforms $P(f)$ and $\hat{P}(f)$ have the same magnitude spectrum, but their phase spectra differ by +90 degrees for negative frequencies and -90 degrees for positive frequencies. We may therefore rewrite Equation (6.79) as

$$\begin{aligned} \int_{-\infty}^{\infty} p(t)\hat{p}(t) dt &= j \int_{-\infty}^{\infty} P(f)P^*(f) \operatorname{sgn}(f) df \\ &= j \int_{-\infty}^{\infty} |P(f)|^2 \operatorname{sgn}(f) df \end{aligned} \quad (6.81)$$

Recognizing that the magnitude response $|P(f)|$ is an even function of frequency f and the signum function $\operatorname{sgn}(f)$ is an odd function of frequency f , the integral of Equation (6.81) is zero, proving Property 2.

Next, let the passband pulses $p(t)$ and $\hat{p}(t)$ be passed through a linear time-invariant channel of impulse response $h(t)$, yielding the following passband outputs:

$$u(t) = p(t) \star h(t) \quad (6.82)$$

and

$$\hat{u}(t) = \hat{p}(t) \star h(t) \quad (6.83)$$

where the symbol \star denotes convolution. We may then formulate the third property.

Property 3

The passband pulses $u(t)$ and $\hat{u}(t)$, defined in Equations (6.82) and (6.83), form a Hilbert-transform pair and are therefore orthogonal over the entire interval $(-\infty, \infty)$ for any $h(t)$.

It is this important property that makes it possible for the CAP receiver to separate the transmitted real and imaginary symbols, a_k and b_k , given the channel output. To prove Property 3, we again use the Fourier transform (in a manner similar to Equations (6.79) and (6.81)) to write

$$\begin{aligned} \int_{-\infty}^{\infty} u(t)\hat{u}(t) dt &= \int_{-\infty}^{\infty} U(f)\hat{U}^*(f) df \\ &= \int_{-\infty}^{\infty} (P(f)H(f))(-j \operatorname{sgn}(f)P(f)H(f))^* df \\ &= \int_{-\infty}^{\infty} j |P(f)|^2 |H(f)|^2 \operatorname{sgn}(f) df \\ &= 0 \end{aligned}$$

► EXAMPLE 6.4 Bandwidth-Efficient Spectral Shaping

The CAP signal of Equation (6.77) uses multilevel encoding via the complex scalar A_k for spectral efficiency. We may further improve the bandwidth efficiency of CAP by using a spectrally efficient formulation of the baseband pulse $g(t)$. For the selection of $g(t)$, we may draw upon the *raised-cosine family* of pulse-shaping functions discussed in Chapter 4.

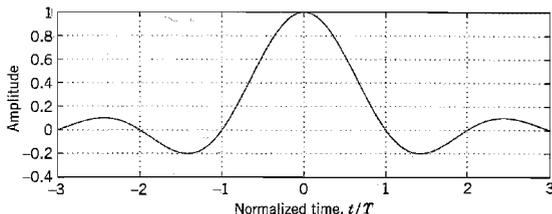


FIGURE 6.19 The baseband pulse $g(t)$ for rolloff factor $\alpha = 0.2$.

Consider, for example, a baseband raised-cosine shaping filter with a rolloff factor $\alpha = 0.2$ (i.e., excess bandwidth of 20 percent). The formula for the baseband pulse $g(t)$ is*

$$g(t) = (\text{sinc}(t)) \frac{\cos(\pi\alpha t)}{1 - 4\alpha^2 t^2} \quad (6.84)$$

where time t is normalized with respect to the symbol duration T . Hence putting $\alpha = 0.2$ in Equation (6.84), we get the plot of Figure 6.19. Using Equations (6.74) and (6.75), we may compute the passband in-phase pulse $p(t)$ and quadrature pulse $\hat{p}(t)$ plotted in Figures 6.20a and 6.20b for a normalized carrier frequency $f_c T = 0.5(1 + \alpha) = 0.6$. The waveforms of Figure 6.20 show that $p(t)$ and $\hat{p}(t)$ are even and odd function of time t , respectively, in accordance with Property 1 and they are orthogonal over the interval $(-\infty, \infty)$ in accordance with Property 2. \blacktriangleleft

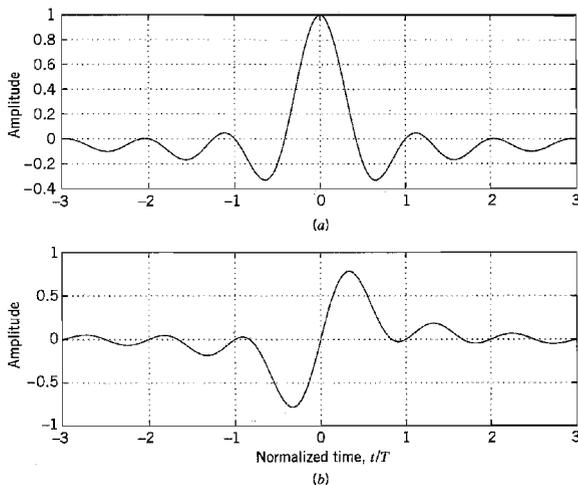


FIGURE 6.20 (a) In-phase pulse $p(t)$, and (b) quadrature pulse $\hat{p}(t)$ for rolloff factor $\alpha = 0.2$.

*Equation (6.84) is obtained by using the normalized time t in place of $2Wt$ in Eq. (4.62) of Chapter 4. Recall that the Nyquist bandwidth W is equal to 0.5 times the symbol (bit) rate. Note also that the $p(t)$ in Chapter 4 is not to be confused with the $p(t)$ in this chapter.

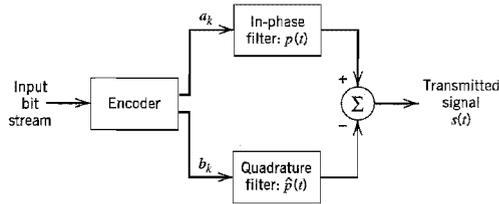


FIGURE 6.21 Block diagram of CAP transmitter.

Basic Structure of the CAP System

Figure 6.21 shows a functional block diagram of the CAP transmitter, which builds on the material embodied in Equations (6.74), (6.75), and (6.77). The transmitter consists of a multilevel encoder and a pair of passband filters. The *multilevel encoder* partitions the incoming serial data stream into successive blocks of n bits each; these blocks are, in turn, mapped into multilevel symbols a_k and b_k , where k refers to the k th symbol period. The *passband in-phase and quadrature filters* process the symbol streams $\{a_k\}$ and $\{b_k\}$ in parallel, respectively. The impulse responses of these two filters, namely, $p(t)$ and $\hat{p}(t)$, are designed in accordance with Equations (6.74) and (6.75), for a chosen baseband pulse $g(t)$. The resulting outputs of these two filters are subtracted to produce the transmitted CAP signal $s(t)$ in accordance with Equation (6.77).

The transmitted signal $s(t)$ propagates through a channel characterized by impulse response $h(t)$ and additive noise $w(t)$. The resulting channel output is defined by

$$\begin{aligned}
 x(t) &= s(t) \star h(t) + w(t) \\
 &= \sum_{k=-\infty}^{\infty} [a_k p(t - kT) \star h(t) - b_k \hat{p}(t - kT) \star h(t)] + w(t) \quad (6.85) \\
 &= \sum_{k=-\infty}^{\infty} [a_k u(t - kT) - b_k \hat{u}(t - kT)] + w(t)
 \end{aligned}$$

where $u(t)$ and $\hat{u}(t)$ are defined by Equations (6.82) and (6.83), respectively. Given $x(t)$ as the input signal, the function of the receiver is to recover the transmitted symbols a_k and b_k in an optimum fashion and on a symbol-by-symbol basis.

The receiver consists of a two-dimensional optimum receiver, the details of which depend on the channel impairments. Specifically, we may mention two different situations:

- ▶ *Additive white Gaussian noise is the only impairment.* In this idealized situation, the optimum CAP receiver consists of a *two-dimensional matched filter*, as indicated in

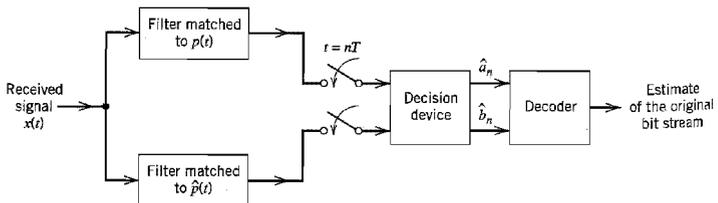


FIGURE 6.22 Block diagram of CAP receiver using a two-dimensional matched filter for the case of an ideal white Gaussian noise channel.

Figure 6.22. The optimum in-phase filter consists of a filter matched to the passband in-phase pulse $p(t)$. The optimum quadrature filter consists of a filter matched to the passband quadrature pulse $\hat{p}(t)$.

- ▶ *Intersymbol interference and additive white Gaussian noise are the channel impairments.* In this more realistic situation in practical terms, the structure of the optimum CAP receiver follows from the optimum linear receiver theory presented in Section 4.9. In particular, the CAP receiver consists of a two-dimensional matched filter, followed by a pair of identical equalizers and a pair of synchronous samplers, as shown in Figure 6.23. The equalizers (implemented in tapped-delay-line form) compensate for dispersion in the channel.

From Property 3, we deduce that the output of the in-phase matched filter due to the passband quadrature pulse is zero, and vice versa. Accordingly, in both of these situations, the transmitted symbols a_k and b_k can be detected separately by the two-dimensional optimum CAP receiver.

Digital Implementation of the CAP Receiver

The receiver structure of Figure 6.23 can be further simplified by first reformulating it in the form shown in Figure 6.24a, where we have introduced an analog-to-digital (A/D) converter at the receiver input to facilitate the use of digital signal processing. Next, we recognize that the matched filters and equalizers in Figure 6.24a are all linear systems. Accordingly, the samples at points A and B in Figure 6.24a are *linear combinations* of the input samples $x[n] = x(nT_s)$, where T_s is the sampling period. It follows therefore that we may replace the combination of matched filter and equalizer in each receiver path in Figure 6.24a by a single *finite-duration impulse response (FIR) filter* operating at the sampling rate $1/T_s$, as shown in Figure 6.24b. With a common input, it is natural for the two FIR filters in Figure 6.24b to share a common set of unit-delay elements but different sets of coefficients of their own. Note also that the receiver structure of Figure 6.24b is the mirror image of the transmitter structure in Figure 6.21.

One last comment is in order. In practice, the two FIR filters in the receiver of Figure 6.24b are made *adaptive* so as to accommodate operation of the CAP receiver in an unknown environment. With adaptive equalization (filtering) as the method of choice, we have two options in light of the material presented in Chapter 4: linear equalization and decision feedback equalization (DFE). When the frequency response of the channel is approximately flat, the use of linear equalization is adequate for the task at hand. However, when the frequency response of the channel and/or the noise power spectrum are not approximately flat, performance of the CAP receiver can be improved significantly by the use of DFE.

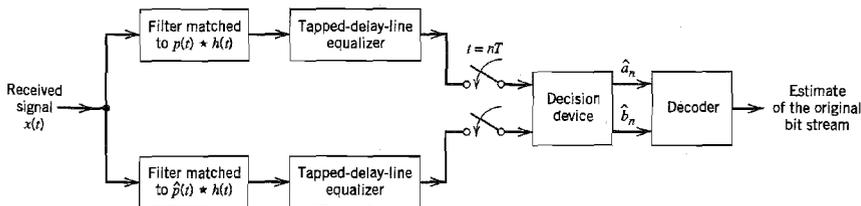


FIGURE 6.23 Block diagram of CAP receiver using a pair of optimum linear receivers for the case of a noisy, dispersive channel.

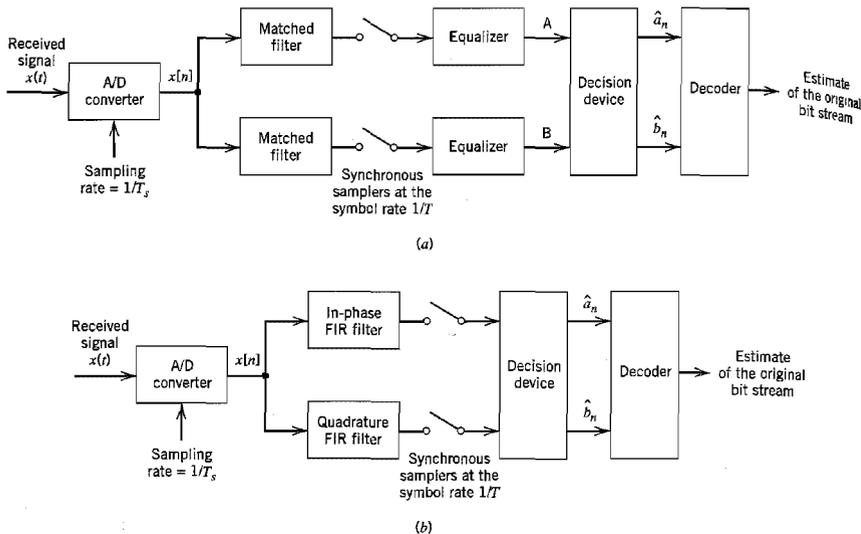


FIGURE 6.24 Digital implementation of the CAP receiver. (a) Reformulation of the receiver in Figure 6.23 using an A/D converter and associated circuitry. (b) Replacement of the matched filter/equalizer pairs with equivalent (digitally implemented) FIR filters.

An Application of CAP

An important application of CAP (using 32 or 64 constellation points) is in the passband transmission of digital data over twisted-pair wiring of lengths less than 100 m, as in local area networks (LANs) suitable for premises' distribution systems. (All modern LAN standards limit the length of the wiring to a maximum of 100 m; the CAP systems can actually operate over longer loops, if so required.) The data rates may range from 51 up to 155 Mb/s, with the usable channel bandwidth being strictly limited to 30 MHz.

The two major impairments for transceivers providing duplex operation over twisted-pairs cables are propagation loss and near-end crosstalk (NEXT); these impairments were discussed in Section 4.8, which dealt with digital subscriber lines.

In the environment described herein, a practical issue is how to adapt the receiver to wide variations in twisted-pair cables. To cater to this requirement, adaptive filters are used to implement the optimum in-phase and quadrature filters in the receiver, as remarked earlier. The wider the range of twisted-pair cables that have to be accommodated, the more complex these adaptive filters must be.

6.5 Coherent Frequency-Shift Keying

M -ary PSK and M -ary QAM share a common property: Both are examples of linear modulation. In this section we study a nonlinear method of passband data transmission, namely, *coherent frequency-shift keying* (FSK). We begin the study by considering the simple case of binary FSK.

■ BINARY FSK

In a *binary FSK system*, symbols 1 and 0 are distinguished from each other by transmitting one of two sinusoidal waves that differ in frequency by a fixed amount. A typical pair of sinusoidal waves is described by

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t), & 0 \leq t \leq T_b \\ 0, & \text{elsewhere} \end{cases} \quad (6.86)$$

where $i = 1, 2$, and E_b is the transmitted signal energy per bit; the transmitted frequency is

$$f_i = \frac{n_c + i}{T_b} \quad \text{for some fixed integer } n_c \text{ and } i = 1, 2 \quad (6.87)$$

Thus symbol 1 is represented by $s_1(t)$, and symbol 0 by $s_2(t)$. The FSK signal described here is known as *Sunde's FSK*. It is a *continuous-phase signal* in the sense that phase continuity is always maintained, including the inter-bit switching times. This form of digital modulation is an example of *continuous-phase frequency-shift keying* (CPFSK), on which we have more to say later on in the section.

From Equations (6.86) and (6.87), we observe directly that the signals $s_1(t)$ and $s_2(t)$ are orthogonal, but not normalized to have unit energy. We therefore deduce that the most useful form for the set of orthonormal basis functions is

$$\phi_i(t) = \begin{cases} \sqrt{\frac{2}{T_b}} \cos(2\pi f_i t), & 0 \leq t \leq T_b \\ 0, & \text{elsewhere} \end{cases} \quad (6.88)$$

where $i = 1, 2$. Correspondingly, the coefficient s_{ij} for $i = 1, 2$, and $j = 1, 2$ is defined by

$$\begin{aligned} s_{ij} &= \int_0^{T_b} s_i(t) \phi_j(t) dt \\ &= \int_0^{T_b} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t) \sqrt{\frac{2}{T_b}} \cos(2\pi f_j t) dt \\ &= \begin{cases} \sqrt{E_b}, & i = j \\ 0, & i \neq j \end{cases} \end{aligned} \quad (6.89)$$

Thus, unlike coherent binary PSK, a coherent binary FSK system is characterized by having a signal space that is two-dimensional (i.e., $N = 2$) with two message points (i.e., $M = 2$), as shown in Figure 6.25. The two message points are defined by the

$$s_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix} \quad (6.90)$$

and

$$s_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix} \quad (6.91)$$

with the Euclidean distance between them equal to $\sqrt{2E_b}$. Figure 6.25 also includes a couple of inserts, which show waveforms representative of signals $s_1(t)$ and $s_2(t)$.

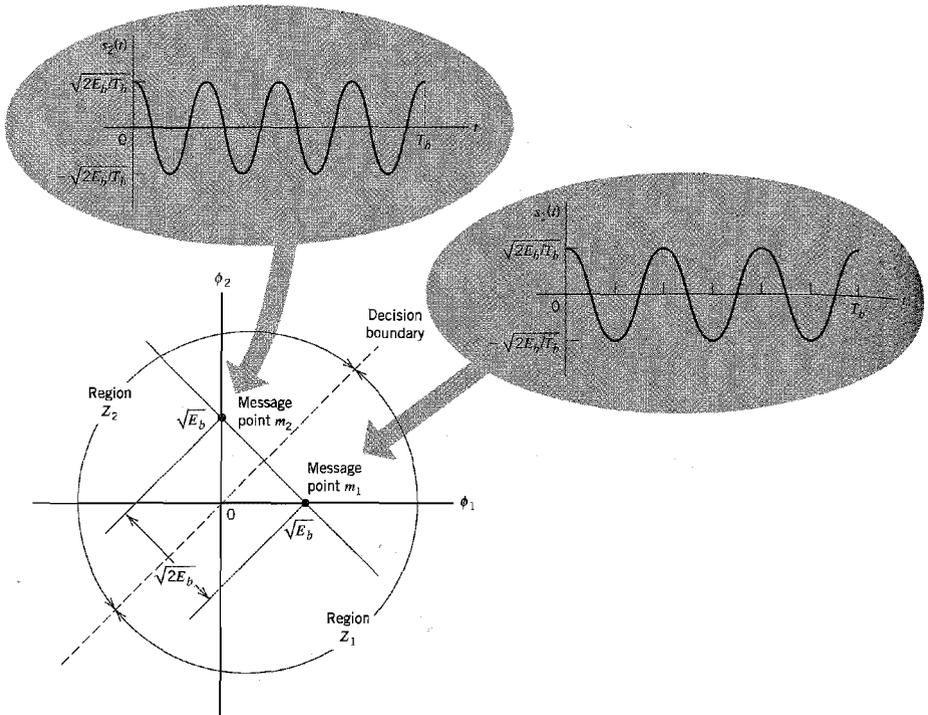


FIGURE 6.25 Signal-space diagram for binary FSK system. The diagram also includes two inserts showing example waveforms of the two modulated signals $s_1(t)$ and $s_2(t)$.

Error Probability of Binary FSK

The observation vector \mathbf{x} has two elements x_1 and x_2 that are defined by, respectively,

$$x_1 = \int_0^{T_b} x(t)\phi_1(t) dt \tag{6.92}$$

and

$$x_2 = \int_0^{T_b} x(t)\phi_2(t) dt \tag{6.93}$$

where $x(t)$ is the received signal, the form of which depends on which symbol was transmitted. Given that symbol 1 was transmitted, $x(t)$ equals $s_1(t) + w(t)$, where $w(t)$ is the same function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$. If, on the other hand, symbol 0 was transmitted, $x(t)$ equals $s_2(t) + w(t)$.

Now, applying the decision rule of Equation (5.59), we find that the observation space is partitioned into two decision regions, labeled Z_1 and Z_2 in Figure 6.25. The decision boundary, separating region Z_1 from region Z_2 is the perpendicular bisector of

the line joining the two message points. The receiver decides in favor of symbol 1 if the received signal point represented by the observation vector \mathbf{x} falls inside region Z_1 . This occurs when $x_1 > x_2$. If, on the other hand, we have $x_1 < x_2$, the received signal point falls inside region Z_2 , and the receiver decides in favor of symbol 0. On the decision boundary, we have $x_1 = x_2$, in which case the receiver makes a random guess in favor of symbol 1 or 0.

Define a new Gaussian random variable Y whose sample value y is equal to the difference between x_1 and x_2 ; that is,

$$y = x_1 - x_2 \quad (6.94)$$

The mean value of the random variable Y depends on which binary symbol was transmitted. Given that symbol 1 was transmitted, the Gaussian random variables X_1 and X_2 , whose sample values are denoted by x_1 and x_2 , have mean values equal to $\sqrt{E_b}$ and zero, respectively. Correspondingly, the conditional mean of the random variable Y , given that symbol 1 was transmitted, is

$$\begin{aligned} E[Y|1] &= E[X_1|1] - E[X_2|1] \\ &= +\sqrt{E_b} \end{aligned} \quad (6.95)$$

On the other hand, given that symbol 0 was transmitted, the random variables X_1 and X_2 have mean values equal to zero and $\sqrt{E_b}$, respectively. Correspondingly, the conditional mean of the random variable Y , given that symbol 0 was transmitted, is

$$\begin{aligned} E[Y|0] &= E[X_1|0] - E[X_2|0] \\ &= -\sqrt{E_b} \end{aligned} \quad (6.96)$$

The variance of the random variable Y is independent of which binary symbol was transmitted. Since the random variables X_1 and X_2 are statistically independent, each with a variance equal to $N_0/2$, it follows that

$$\begin{aligned} \text{var}[Y] &= \text{var}[X_1] + \text{var}[X_2] \\ &= N_0 \end{aligned} \quad (6.97)$$

Suppose we know that symbol 0 was transmitted. The conditional probability density function of the random variable Y is then given by

$$f_Y(y|0) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{(y + \sqrt{E_b})^2}{2N_0}\right] \quad (6.98)$$

Since the condition $x_1 > x_2$, or equivalently, $y > 0$, corresponds to the receiver making a decision in favor of symbol 1, we deduce that the conditional probability of error, given that symbol 0 was transmitted, is

$$\begin{aligned} p_{10} &= P(y > 0 | \text{symbol 0 was sent}) \\ &= \int_0^{\infty} f_Y(y|0) dy \\ &= \frac{1}{\sqrt{2\pi N_0}} \int_0^{\infty} \exp\left[-\frac{(y + \sqrt{E_b})^2}{2N_0}\right] dy \end{aligned} \quad (6.99)$$

Put

$$\frac{y + \sqrt{E_b}}{\sqrt{2N_0}} = z \quad (6.100)$$

Then, changing the variable of integration from y to z , we may rewrite Equation (6.99) as follows:

$$\begin{aligned} p_{10} &= \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/2N_0}}^{\infty} \exp(-z^2) dz \\ &= \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \end{aligned} \quad (6.101)$$

Similarly, we may show the p_{01} , the conditional probability of error given that symbol 1 was transmitted, has the same value as in Equation (6.101). Accordingly, averaging p_{10} and p_{01} , we find that the *average probability of bit error* or, equivalently, the *bit error rate for coherent binary FSK* is (assuming equiprobable symbols)

$$P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \quad (6.102)$$

Comparing Equations (6.20) and (6.102), we see that, in a coherent binary FSK system, we have to double the *bit energy-to-noise density ratio*, E_b/N_0 , to maintain the same bit error rate as in a coherent binary PSK system. This result is in perfect accord with the signal-space diagrams of Figures 6.3 and 6.25, where we see that in a binary PSK system the Euclidean distance between the two message points is equal to $2\sqrt{E_b}$, whereas in a binary FSK system the corresponding distance is $\sqrt{2E_b}$. For a prescribed E_b , the minimum distance d_{\min} in binary PSK is therefore $\sqrt{2}$ times that in binary FSK. Recall from Chapter 5 that the probability of error decreases exponentially as d_{\min}^2 , hence the difference between the formulas of Equations (6.20) and (6.102).

Generation and Detection of Coherent Binary FSK Signals

To generate a binary FSK signal, we may use the scheme shown in Figure 6.26a. The incoming binary data sequence is first applied to an *on-off level encoder*, at the output of which symbol 1 is represented by a constant amplitude of $\sqrt{E_b}$ volts and symbol 0 is represented by zero volts. By using an *inverter* in the lower channel in Figure 6.26a, we in effect make sure that when we have symbol 1 at the input, the oscillator with frequency f_1 in the upper channel is switched on while the oscillator with frequency f_2 in the lower channel is switched off, with the result that frequency f_1 is transmitted. Conversely, when we have symbol 0 at the input, the oscillator in the upper channel is switched off and the oscillator in the lower channel is switched on, with the result that frequency f_2 is transmitted. The two frequencies f_1 and f_2 are chosen to equal different integer multiples of the bit rate $1/T_b$, as in Equation (6.87).

In the transmitter of Figure 6.26a, we assume that the two oscillators are synchronized, so that their outputs satisfy the requirements of the two orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$, as in Equation (6.88). Alternatively, we may use a single keyed (voltage-controlled) oscillator. In either case, the frequency of the modulated wave is shifted with a continuous phase, in accordance with the input binary wave.

To detect the original binary sequence given the noisy received signal $x(t)$, we may use the receiver shown in Figure 6.26b. It consists of two correlators with a common input, which are supplied with locally generated coherent reference signals $\phi_1(t)$ and $\phi_2(t)$. The correlator outputs are then subtracted, one from the other, and the resulting difference, y , is compared with a threshold of zero volts. If $y > 0$, the receiver decides in favor of 1. On the other hand, if $y < 0$, it decides in favor of 0. If y is exactly zero, the receiver makes a random guess in favor of 1 or 0.

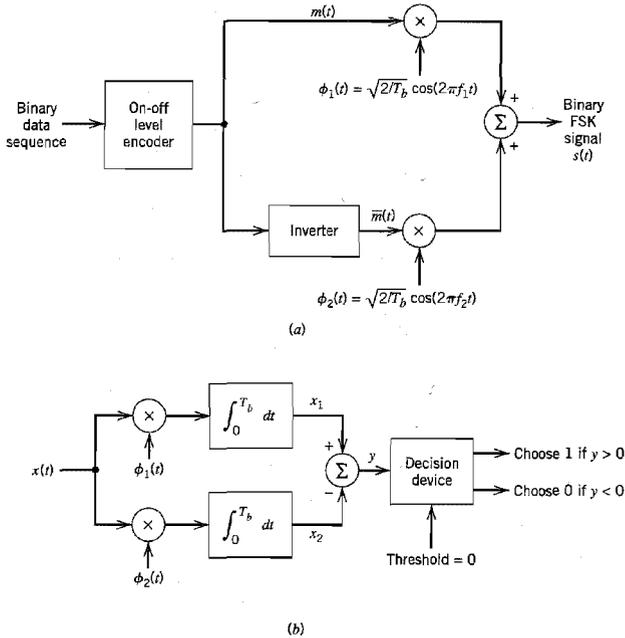


FIGURE 6.26 Block diagrams for (a) binary FSK transmitter and (b) coherent binary FSK receiver.

Power Spectra of Binary FSK Signals

Consider the case of Sunde's FSK, for which the two transmitted frequencies f_1 and f_2 differ by an amount equal to the bit rate $1/T_b$, and their arithmetic mean equals the nominal carrier frequency f_c ; phase continuity is always maintained, including inter-bit switching times. We may express this special binary FSK signal as follows:

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos\left(2\pi f_c t \pm \frac{\pi t}{T_b}\right), \quad 0 \leq t \leq T_b \quad (6.103)$$

and using a well-known trigonometric identity, we get

$$\begin{aligned} s(t) &= \sqrt{\frac{2E_b}{T_b}} \cos\left(\pm \frac{\pi t}{T_b}\right) \cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}} \sin\left(\pm \frac{\pi t}{T_b}\right) \sin(2\pi f_c t) \\ &= \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi t}{T_b}\right) \cos(2\pi f_c t) \mp \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi t}{T_b}\right) \sin(2\pi f_c t) \end{aligned} \quad (6.104)$$

In the last line of Equation (6.104), the plus sign corresponds to transmitting symbol 0, and the minus sign corresponds to transmitting symbol 1. As before, we assume that the symbols 1 and 0 in the random binary wave at the modulator input are equally likely, and that the symbols transmitted in adjacent time slots are statistically independent. Then,

based on the representation of Equation (6.104), we may make the following observations pertaining to the in-phase and quadrature components of a binary FSK signal with continuous phase:

1. The in-phase component is completely independent of the input binary wave. It equals $\sqrt{2E_b/T_b} \cos(\pi t/T_b)$ for all values of time t . The power spectral density of this component therefore consists of two delta functions, weighted by the factor $E_b/2T_b$, and occurring at $f = \pm 1/2T_b$.
2. The quadrature component is directly related to the input binary wave. During the signaling interval $0 \leq t \leq T_b$, it equals $-g(t)$ when we have symbol 1, and $+g(t)$ when we have symbol 0. The symbol shaping function $g(t)$ is defined by

$$g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi t}{T_b}\right), & 0 \leq t \leq T_b \\ 0, & \text{elsewhere} \end{cases} \quad (6.105)$$

The energy spectral density of this symbol shaping function equals

$$\Psi_g(f) = \frac{8E_b T_b \cos^2(\pi T_b f)}{\pi^2 (4T_b^2 f^2 - 1)^2} \quad (6.106)$$

The power spectral density of the quadrature component equals $\Psi_g(f)/T_b$. It is also apparent that the in-phase and quadrature components of the binary FSK signal are independent of each other. Accordingly, the baseband power spectral density of Sunde's FSK signal equals the sum of the power spectral densities of these two components, as shown by

$$S_B(f) = \frac{E_b}{2T_b} \left[\delta\left(f - \frac{1}{2T_b}\right) + \delta\left(f + \frac{1}{2T_b}\right) \right] + \frac{8E_b \cos^2(\pi T_b f)}{\pi^2 (4T_b^2 f^2 - 1)^2} \quad (6.107)$$

Substituting Equation (6.107) in Equation (6.4), we find that the power spectrum of the binary FSK signal contains two discrete frequency components located at $(f_c + 1/2T_b) = f_1$ and $(f_c - 1/2T_b) = f_2$, with their average powers adding up to one-half the total power of the binary FSK signal. The presence of these two discrete frequency components provides a means of synchronizing the receiver with the transmitter.

Note also that the baseband power spectral density of a binary FSK signal with continuous phase ultimately falls off as the inverse fourth power of frequency. This is readily established by taking the limit in Equation (6.107) as f approaches infinity. If, however, the FSK signal exhibits phase discontinuity at the inter-bit switching instants (this arises when the two oscillators applying frequencies f_1 and f_2 operate independently of each other), the power spectral density ultimately falls off as the inverse square of frequency; see Problem 6.23. Accordingly, an FSK signal with continuous phase does not produce as much interference outside the signal band of interest as an FSK signal with discontinuous phase.

In Figure 6.5, we have plotted the baseband power spectra of Equations (6.22) and (6.107). (To simplify matters, we have only plotted the results for positive frequency.) In both cases, $S_B(f)$ is shown normalized with respect to $2E_b$, and the frequency is normalized with respect to the bit rate $R_b = 1/T_b$. The difference in the falloff rates of these spectra can be explained on the basis of the pulse shape $g(t)$. The smoother the pulse, the faster the drop of spectral tails to zero. Thus, since binary FSK (with continuous phase) has a smoother pulse shape, it has lower sidelobes than binary PSK.

■ MINIMUM SHIFT KEYING

In the coherent detection of binary FSK signal, the phase information contained in the received signal is not fully exploited, other than to provide for synchronization of the receiver to the transmitter. We now show that by proper use of the phase when performing detection, it is possible to improve the noise performance of the receiver significantly. This improvement is, however, achieved at the expense of increased receiver complexity.

Consider a continuous-phase frequency-shift keying (CPFSK) signal, which is defined for the interval $0 \leq t \leq T_b$ as follows:

$$s(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_1 t + \theta(0)] & \text{for symbol 1} \\ \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_2 t + \theta(0)] & \text{for symbol 0} \end{cases} \quad (6.108)$$

where E_b is the transmitted signal energy per bit, and T_b is the bit duration. The phase $\theta(0)$, denoting the value of the phase at time $t = 0$, sums up the past history of the modulation process up to time $t = 0$. The frequencies f_1 and f_2 are sent in response to binary symbols 1 and 0 appearing at the modulator input, respectively.

Another useful way of representing the CPFSK signal $s(t)$ is to express it in the conventional form of an *angle-modulated signal* as follows:

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos[2\pi f_c t + \theta(t)] \quad (6.109)$$

where $\theta(t)$ is the phase of $s(t)$. When the phase $\theta(t)$ is a continuous function of time, we find that the modulated signal $s(t)$ itself is also continuous at all times, including the inter-bit switching times. The phase $\theta(t)$ of a CPFSK signal increases or decreases linearly with time during each bit duration of T_b seconds, as shown by

$$\theta(t) = \theta(0) \pm \frac{\pi h}{T_b} t, \quad 0 \leq t \leq T_b \quad (6.110)$$

where the plus sign corresponds to sending symbol 1, and the minus sign corresponds to sending symbol 0; the parameter h is to be defined. Substituting Equation (6.110) into (6.109), and then comparing the angle of the cosine function with that of Equation (6.108), we deduce the following pair of relations:

$$f_c + \frac{h}{2T_b} = f_1 \quad (6.111)$$

$$f_c - \frac{h}{2T_b} = f_2 \quad (6.112)$$

Solving Equations (6.111) and (6.112) for f_c and h , we thus get

$$f_c = \frac{1}{2} (f_1 + f_2) \quad (6.113)$$

and

$$h = T_b(f_1 - f_2) \quad (6.114)$$

The nominal carrier frequency f_c is therefore the arithmetic mean of the frequencies f_1 and f_2 . The difference between the frequencies f_1 and f_2 , normalized with respect to the bit rate $1/T_b$, defines the dimensionless parameter h , which is referred to as the *deviation ratio*.

Phase Trellis

From Equation (6.110) we find that at time $t = T_b$,

$$\theta(T_b) - \theta(0) = \begin{cases} \pi b & \text{for symbol 1} \\ -\pi b & \text{for symbol 0} \end{cases} \quad (6.115)$$

That is to say, the sending of symbol 1 increases the phase of a CPFSK signal $s(t)$ by πb radians, whereas the sending of symbol 0 reduces it by an equal amount.

The variation of phase $\theta(t)$ with time t follows a path consisting of a sequence of straight lines, the slopes of which represent frequency changes. Figure 6.27 depicts possible paths starting from time $t = 0$. A plot like that shown in Figure 6.27 is called a *phase tree*. The tree makes clear the transitions of phase across interval boundaries of the incoming sequence of data bits. Moreover, it is evident from Figure 6.27 that the phase of a CPFSK signal is an odd or even multiple of πb radians at odd or even multiples of the bit duration T_b , respectively.

The phase tree described in Figure 6.27 is a manifestation of phase continuity, which is an inherent characteristic of a CPFSK signal. To appreciate the notion of phase continuity, let us go back for a moment to Sunde's FSK, which is a CPFSK scheme as previously described. In this case, the deviation ratio b is exactly unity. Hence, according to Figure 6.27 the phase change over one bit interval is $\pm\pi$ radians. But, a change of $+\pi$ radians is exactly the same as a change of $-\pi$ radians, modulo 2π . It follows therefore that in the case of Sunde's FSK there is *no memory*; that is, knowing which particular change occurred in the *previous* bit interval provides no help in the *current* bit interval.

In contrast, we have a completely different situation when the deviation ratio b is assigned the special value of $1/2$. We now find that the phase can take on only the two values $\pm\pi/2$ at odd multiples of T_b , and only the two values 0 and π at even multiples of T_b , as in Figure 6.28. This second graph is called a *phase trellis*, since a "trellis" is a treelike structure with remerging branches. Each path from left to right through the trellis of Figure 6.28 corresponds to a specific binary sequence input. For example, the path shown in boldface in Figure 6.28 corresponds to the binary sequence 1101000 with $\theta(0) = 0$. Henceforth, we assume that $b = 1/2$.

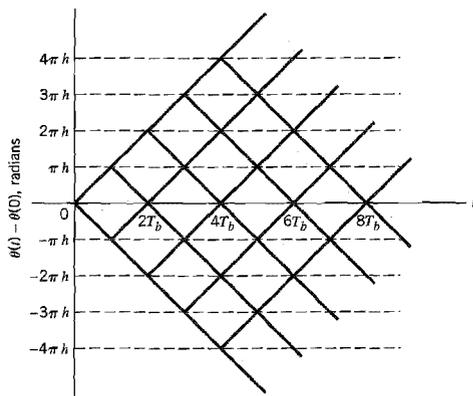


FIGURE 6.27 Phase tree.

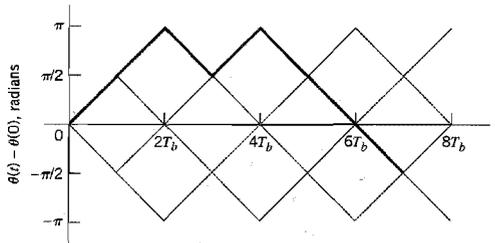


FIGURE 6.28 Phase trellis; boldfaced path represents the sequence 1101000.

With $h = 1/2$, we find from Equation (6.114) that the frequency deviation (i.e., the difference between the two signaling frequencies f_1 and f_2) equals half the bit rate. This is the minimum frequency spacing that allows the two FSK signals representing symbols 1 and 0, as in Equation (6.108), to be coherently orthogonal in the sense that they do not interfere with one another in the process of detection. It is for this reason that a CPFSK signal with a deviation ratio of one half is commonly referred to as *minimum shift keying* (MSK).⁷

Signal-Space Diagram of MSK

Using a well-known trigonometric identity in Equation (6.109), we may express the CPFSK signal $s(t)$ in terms of its in-phase and quadrature components as follows:

$$s(t) = \sqrt{\frac{2E_b}{T_b}} \cos[\theta(t)] \cos(2\pi f_c t) - \sqrt{\frac{2E_b}{T_b}} \sin[\theta(t)] \sin(2\pi f_c t) \quad (6.116)$$

Consider first the in-phase component $\sqrt{2E_b/T_b} \cos[\theta(t)]$. With the deviation ratio $h = 1/2$, we have from Equation (6.110) that

$$\theta(t) = \theta(0) \pm \frac{\pi}{2T_b} t, \quad 0 \leq t \leq T_b \quad (6.117)$$

where the plus sign corresponds to symbol 1 and the minus sign corresponds to symbol 0. A similar result holds for $\theta(t)$ in the interval $-T_b \leq t \leq 0$, except that the algebraic sign is not necessarily the same in both intervals. Since the phase $\theta(0)$ is 0 or π , depending on the past history of the modulation process, we find that, in the interval $-T_b \leq t \leq T_b$, the polarity of $\cos[\theta(t)]$ depends only on $\theta(0)$, regardless of the sequence of 1s and 0s transmitted before or after $t = 0$. Thus, for this time interval, the in-phase component $s_I(t)$ consists of a *half-cycle cosine pulse* defined as follows:

$$\begin{aligned} s_I(t) &= \sqrt{\frac{2E_b}{T_b}} \cos[\theta(t)] \\ &= \sqrt{\frac{2E_b}{T_b}} \cos[\theta(0)] \cos\left(\frac{\pi}{2T_b} t\right) \\ &= \pm \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi}{2T_b} t\right), \quad -T_b \leq t \leq T_b \end{aligned} \quad (6.118)$$

where the plus sign corresponds to $\theta(0) = 0$ and the minus sign corresponds to $\theta(0) = \pi$. In a similar way, we may show that, in the interval $0 \leq t \leq 2T_b$, the quadrature component

$s_Q(t)$ consists of a *half-cycle sine pulse*, whose polarity depends only on $\theta(T_b)$, as shown by

$$\begin{aligned} s_Q(t) &= \sqrt{\frac{2E_b}{T_b}} \sin[\theta(t)] \\ &= \sqrt{\frac{2E_b}{T_b}} \sin[\theta(T_b)] \sin\left(\frac{\pi}{2T_b} t\right) \\ &= \pm \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi}{2T_b} t\right), \quad 0 \leq t \leq 2T_b \end{aligned} \quad (6.119)$$

where the plus sign corresponds to $\theta(T_b) = \pi/2$ and the minus sign corresponds to $\theta(T_b) = -\pi/2$.

From the foregoing discussion we see that since the phase states $\theta(0)$ and $\theta(T_b)$ can each assume one of two possible values, any one of four possibilities can arise, as described here:

- ▶ The phase $\theta(0) = 0$ and $\theta(T_b) = \pi/2$, corresponding to the transmission of symbol 1.
- ▶ The phase $\theta(0) = \pi$ and $\theta(T_b) = \pi/2$, corresponding to the transmission of symbol 0.
- ▶ The phase $\theta(0) = \pi$ and $\theta(T_b) = -\pi/2$ (or, equivalently, $3\pi/2$ modulo 2π), corresponding to the transmission of symbol 1.
- ▶ The phase $\theta(0) = 0$ and $\theta(T_b) = -\pi/2$, corresponding to the transmission of symbol 0.

This, in turn, means that the MSK signal itself may assume any one of four possible forms, depending on the values of $\theta(0)$ and $\theta(T_b)$.

From the expansion of Equation (6.116), we deduce that the orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$ for MSK are defined by a pair of sinusoidally modulated quadrature carriers:

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos\left(\frac{\pi}{2T_b} t\right) \cos(2\pi f_c t), \quad 0 \leq t \leq T_b \quad (6.120)$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}} \sin\left(\frac{\pi}{2T_b} t\right) \sin(2\pi f_c t), \quad 0 \leq t \leq T_b \quad (6.121)$$

Correspondingly, we may express the MSK signal in the expanded form

$$s(t) = s_1 \phi_1(t) + s_2 \phi_2(t), \quad 0 \leq t \leq T_b \quad (6.122)$$

where the coefficients s_1 and s_2 are related to the phase states $\theta(0)$ and $\theta(T_b)$, respectively. To evaluate s_1 , we integrate the product $s(t)\phi_1(t)$ between the limits $-T_b$ and T_b , as shown by

$$\begin{aligned} s_1 &= \int_{-T_b}^{T_b} s(t) \phi_1(t) dt \\ &= \sqrt{E_b} \cos[\theta(0)], \quad -T_b \leq t \leq T_b \end{aligned} \quad (6.123)$$

Similarly, to evaluate s_2 we integrate the product $s(t)\phi_2(t)$ between the limits 0 and $2T_b$ as shown by

$$\begin{aligned} s_2 &= \int_0^{2T_b} s(t) \phi_2(t) dt \\ &= -\sqrt{E_b} \sin[\theta(T_b)], \quad 0 \leq t \leq 2T_b \end{aligned} \quad (6.124)$$

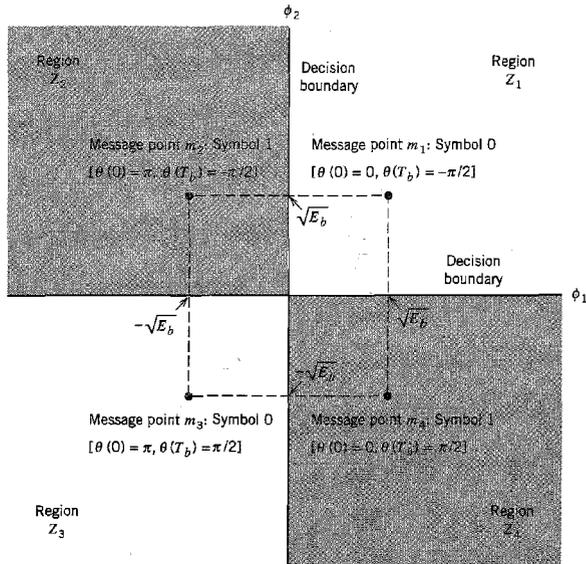


FIGURE 6.29 Signal-space diagram for MSK system.

Note that in Equations (6.123) and (6.124):

- ▶ Both integrals are evaluated for a time interval equal to twice the bit duration.
- ▶ Both the lower and upper limits of the product integration used to evaluate the coefficient s_1 are shifted by the bit duration T_b with respect to those used to evaluate the coefficient s_2 .
- ▶ The time interval $0 \leq t \leq T_b$, for which the phase states $\theta(0)$ and $\theta(T_b)$ are defined, is common to both integrals.

Accordingly, the signal constellation for an MSK signal is two-dimensional (i.e., $N = 2$), with four possible message points (i.e., $M = 4$), as illustrated in Figure 6.29. The coordinates of the message points are as follows in a counterclockwise direction: $(+\sqrt{E_b}, +\sqrt{E_b})$, $(-\sqrt{E_b}, +\sqrt{E_b})$, $(-\sqrt{E_b}, -\sqrt{E_b})$, and $(+\sqrt{E_b}, -\sqrt{E_b})$. The possible values of $\theta(0)$ and $\theta(T_b)$, corresponding to these four message points, are also included in Figure 6.29. The signal-space diagram of MSK is thus similar to that of QPSK in that both of them have four message points. However, they differ in a subtle way that should be carefully noted: In QPSK the transmitted symbol is represented by any one of the four message points, whereas in MSK one of two message points is used to represent the transmitted symbol at any one time, depending on the value of $\theta(0)$.

Table 6.5 presents a summary of the values of $\theta(0)$ and $\theta(T_b)$, as well as the corresponding values of s_1 and s_2 that are calculated for the time intervals $-T_b \leq t \leq T_b$ and $0 \leq t \leq 2T_b$, respectively. The first column of this table indicates whether symbol 1 or symbol 0 was sent in the interval $0 \leq t \leq T_b$. Note that the coordinates of the message points, s_1 and s_2 , have opposite signs when symbol 1 is sent in this interval, but the same sign when symbol 0 is sent. Accordingly, for a given input data sequence, we may use the

TABLE 6.5 *Signal-space characterization of MSK*

Transmitted Binary Symbol, $0 \leq t \leq T_b$	Phase States (radians)		Coordinates of Message Points	
	$\theta(0)$	$\theta(T_b)$	s_1	s_2
0	0	$-\pi/2$	$+\sqrt{E_b}$	$+\sqrt{E_b}$
1	π	$-\pi/2$	$-\sqrt{E_b}$	$+\sqrt{E_b}$
0	π	$+\pi/2$	$-\sqrt{E_b}$	$-\sqrt{E_b}$
1	0	$+\pi/2$	$+\sqrt{E_b}$	$-\sqrt{E_b}$

entries of Table 6.5 to derive, on a bit-by-bit basis, the two sequences of coefficients required to scale $\phi_1(t)$ and $\phi_2(t)$, and thereby determine the MSK signal $s(t)$.

▶ **EXAMPLE 6.5**

Figure 6.30 shows the sequences and waveforms involved in the generation of an MSK signal for the binary sequence 1101000. The input binary sequence is shown in Figure 6.30a. The two modulation frequencies are: $f_1 = 5/4T_b$ and $f_2 = 3/4T_b$. Assuming that, at time $t = 0$ the phase $\theta(0)$ is zero, the sequence of phase states is as shown in Figure 6.30, modulo 2π . The polarities of the two sequences of factors used to scale the time functions $\phi_1(t)$ and $\phi_2(t)$ are shown in the top lines of Figures 6.30b and 6.30c. Note that these two sequences are *offset* relative to each other by an interval equal to the bit duration T_b . The waveforms of the resulting two components of $s(t)$, namely, $s_1\phi_1(t)$ and $s_2\phi_2(t)$, are also shown in Figures 6.30b and 6.30c. Adding these two modulated waveforms, we get the desired MSK signal $s(t)$ shown in Figure 6.30d. ◀

Error Probability of MSK

In the case of an AWGN channel, the received signal is given by

$$x(t) = s(t) + w(t)$$

where $s(t)$ is the transmitted MSK signal, and $w(t)$ is the sample function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$. To decide whether symbol 1 or symbol 0 was transmitted in the interval $0 \leq t \leq T_b$, say, we have to establish a procedure for the use of $x(t)$ to detect the phase states $\theta(0)$ and $\theta(T_b)$. For the optimum detection of $\theta(0)$, we first determine the projection of the received signal $x(t)$ onto the reference signal $\phi_1(t)$ over the interval $-T_b \leq t \leq T_b$, obtaining

$$\begin{aligned}
 x_1 &= \int_{-T_b}^{T_b} x(t)\phi_1(t) dt \\
 &= s_1 + w_1, \quad -T_b \leq t \leq T_b
 \end{aligned}
 \tag{6.125}$$

where s_1 is as defined by Equation (6.123) and w_1 is the sample value of a Gaussian random variable of zero mean and variance $N_0/2$. From the signal-space diagram of Figure 6.29, we observe that if $x_1 > 0$, the receiver chooses the estimate $\hat{\theta}(0) = 0$. On the other hand, if $x_1 < 0$, it chooses the estimate $\hat{\theta}(0) = \pi$.

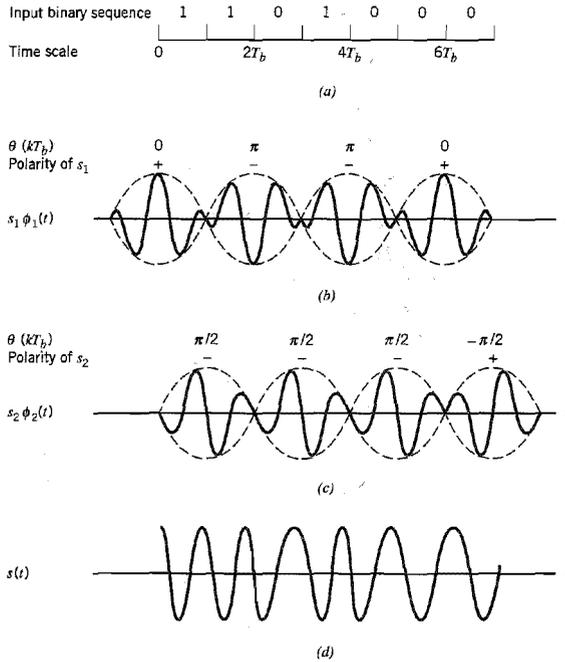


FIGURE 6.30 (a) Input binary sequence. (b) Waveform of scaled time function $s_1 \phi_1(t)$. (c) Waveform of scaled time function $s_2 \phi_2(t)$. (d) Waveform of the MSK signal $s(t)$ obtained by adding $s_1 \phi_1(t)$ and $s_2 \phi_2(t)$ on a bit-by-bit basis.

Similarly, for the optimum detection of $\theta(T_b)$, we determine the projection of the received signal $x(t)$ onto the second reference signal $\phi_2(t)$ over the interval $0 \leq t \leq 2T_b$, obtaining

$$\begin{aligned} x_2 &= \int_0^{2T_b} x(t) \phi_2(t) dt \\ &= s_2 + w_2, \quad 0 \leq t \leq 2T_b \end{aligned} \quad (6.126)$$

where s_2 is as defined by Equation (6.124) and w_2 is the sample value of another independent Gaussian random variable of zero mean and variance $N_0/2$. Referring again to the signal space diagram of Figure 6.29, we observe that if $x_2 > 0$, the receiver chooses the estimate $\hat{\theta}(T_b) = -\pi/2$. If, on the other hand, $x_2 < 0$, it chooses the estimate $\hat{\theta}(T_b) = \pi/2$.

To reconstruct the original binary sequence, we interleave the above two sets of phase decisions, as described next (see Table 6.5):

- If we have the estimates $\hat{\theta}(0) = 0$ and $\hat{\theta}(T_b) = -\pi/2$, or alternatively if we have the estimates $\hat{\theta}(0) = \pi$ and $\hat{\theta}(T_b) = \pi/2$, the receiver makes a decision in favor of symbol 0.

- ▷ If we have the estimates $\hat{\theta}(0) = \pi$ and $\hat{\theta}(T_b) = -\pi/2$, or alternatively if we have the estimates $\hat{\theta}(0) = 0$ and $\hat{\theta}(T_b) = \pi/2$, the receiver makes a decision in favor of symbol 1.

Thus referring to the signal-space diagram of Figure 6.29, we see that the decision made by the receiver is between the message points m_1 and m_3 for symbol 1, or between the message points m_2 and m_4 for symbol 0. The corresponding decisions whether $\theta(0)$ is 0 or π and whether $\theta(T_b)$ is $-\pi/2$ or $+\pi/2$ (i.e., the bit decisions) are made *alternately* in the *I*- and *Q*-channels of the receiver, with each channel looking at the input signal for $2T_b$ seconds. The signal from *other bits* does not interfere with the receiver's decision for a given bit in either channel. The receiver makes an error when the *I*-channel assigns the wrong value to $\theta(0)$ or the *Q*-channel assigns the wrong value to $\theta(T_b)$. Accordingly, using the statistical characterizations of the product-integrator outputs x_1 and x_2 of these two channels, defined by Equations (6.125) and (6.126), respectively, we readily find that the bit error rate for coherent MSK is given by

$$P_e = \frac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{E_b}{N_0}} \right) \tag{6.127}$$

which is exactly the same as that for binary PSK and QPSK. It is important to note, however, that this good performance is the result of the detection of the MSK signal being performed in the receiver on the basis of observations over $2T_b$ seconds.

Generation and Detection of MSK Signals

Consider next the generation and demodulation of MSK. Figure 6.31*a* shows the block diagram of a typical MSK transmitter. The advantage of this method of generating MSK signals is that the signal coherence and deviation ratio are largely unaffected by variations in the input data rate. Two input sinusoidal waves, one of frequency $f_c = n_c/4T_b$ for some fixed integer n_c , and the other of frequency $1/4T_b$, are first applied to a product modulator. This produces two phase-coherent sinusoidal waves at frequencies f_1 and f_2 , which are related to the carrier frequency f_c and the bit rate $1/T_b$ by Equations (6.111) and (6.112) for $h = 1/2$. These two sinusoidal waves are separated from each other by two narrowband filters, one centered at f_1 and the other at f_2 . The resulting filter outputs are next linearly combined to produce the pair of quadrature carriers or orthonormal basis functions $\phi_1(t)$ and $\phi_2(t)$. Finally, $\phi_1(t)$ and $\phi_2(t)$ are multiplied with two binary waves $a_1(t)$ and $a_2(t)$, both of which have a bit rate equal to $1/2T_b$. These two binary waves are extracted from the incoming binary sequence in the manner described in Example 6.5.

Figure 6.31*b* shows the block diagram of a typical MSK receiver. The received signal $x(t)$ is correlated with locally generated replicas of the coherent reference signals $\phi_1(t)$ and $\phi_2(t)$. Note that in both cases the integration interval is $2T_b$ seconds, and that the integration in the quadrature channel is delayed by T_b seconds with respect to that in the in-phase channel. The resulting in-phase and quadrature channel correlator outputs, x_1 and x_2 , are each compared with a threshold of zero, and estimates of the phase $\theta(0)$ and $\theta(T_b)$ are derived in the manner described previously. Finally, these phase decisions are interleaved so as to reconstruct the original input binary sequence with the minimum average probability of symbol error in an AWGN channel.

Power Spectra of MSK Signals

As with the binary FSK signal, we assume that the input binary wave is random with symbols 1 and 0 equally likely, and the symbols transmitted during different time slots being statistically independent. In this case, we make the following observations:

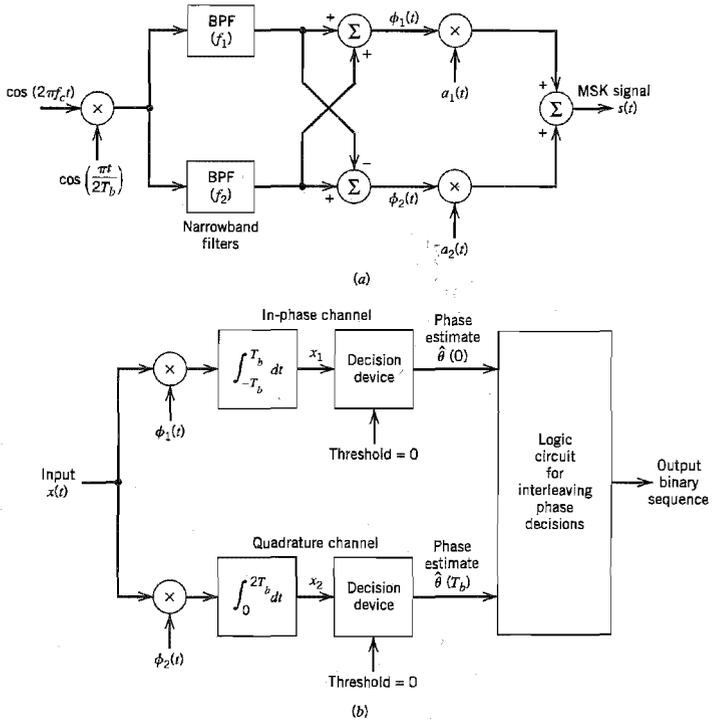


FIGURE 6.31 Block diagrams for (a) MSK transmitter and (b) coherent MSK receiver.

1. Depending on the value of phase state $\theta(0)$, the in-phase component equals $+g(t)$ or $-g(t)$, where

$$g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos\left(\frac{\pi t}{2T_b}\right), & -T_b \leq t \leq T_b \\ 0, & \text{otherwise} \end{cases} \quad (6.128)$$

The energy spectral density of this symbol-shaping function is

$$\psi_g(f) = \frac{32E_b T_b}{\pi^2} \left[\frac{\cos(2\pi T_b f)}{16T_b^2 f^2 - 1} \right]^2 \quad (6.129)$$

Hence, the power spectral density of the in-phase component equals $\psi_g(f)/2T_b$.

2. Depending on the value of the phase state $\theta(T_b)$, the quadrature component equals $+g(t)$ or $-g(t)$, where we now have

$$g(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \sin\left(\frac{\pi t}{2T_b}\right), & 0 \leq t \leq 2T_b \\ 0, & \text{otherwise} \end{cases} \quad (6.130)$$

The energy spectral density of this second symbol-shaping function is also given by Equation (6.129). Hence, the in-phase and quadrature components have the same power spectral density.

3. The in-phase and quadrature component of the MSK signal are also statistically independent. Hence, the baseband power spectral density of the MSK signal is given by

$$\begin{aligned} S_B(f) &= 2 \left[\frac{\psi_g(f)}{2T_b} \right]^2 \\ &= \frac{32E_b}{\pi^2} \left[\frac{\cos(2\pi T_b f)}{16T_b^2 f^2 - 1} \right]^2 \end{aligned} \quad (6.131)$$

The baseband power spectrum of Equation (6.131) is plotted in Figure 6.9, where the power spectrum is normalized with respect to $4E_b$ and the frequency f is normalized with respect to the bit rate $1/T_b$. This figure also includes the corresponding plot of Equation (6.40) for the QPSK signal, which was considered earlier. For $f \gg 1/T_b$, the baseband power spectral density of the MSK signal falls off as the inverse fourth power of frequency, whereas in the case of the QPSK signal it falls off as the inverse square of frequency. Accordingly, MSK does not produce as much interference outside the signal band of interest as QPSK. This is a desirable characteristic of MSK, especially when the digital communication system operates with a bandwidth limitation.

■ GAUSSIAN-FILTERED MSK

From the detailed study of minimum shift keying (MSK) just presented, we may summarize the desirable properties of the MSK signal as follows:

- ▶ Constant envelope
- ▶ Relatively narrow bandwidth
- ▶ Coherent detection performance equivalent to that of QPSK

However, the out-of-band spectral characteristics of MSK signals, as good as they are, still do not satisfy the stringent requirements of certain applications such as wireless communications. To illustrate this limitation, we find from Equation (6.131) that at $fT_b = 0.5$, the baseband power spectral density of the MSK signal drops by only $10 \log_{10} 9 = 9.54$ dB below its midband value. Hence, when the MSK signal is assigned a transmission bandwidth of $1/T_b$, the adjacent channel interference of a wireless communication system using MSK is not low enough to satisfy the practical requirements of such a multiuser communications environment.

Recognizing that the MSK signal can be generated by direct frequency modulation of a voltage-controlled oscillator, we may overcome this serious limitation of MSK by modifying its power spectrum into a *compact* form, while maintaining the constant-envelope property of the MSK signal. This modification can be achieved through the use of a premodulation low-pass filter, hereafter referred to as a baseband *pulse-shaping filter*. Desirably, the pulse-shaping filter should satisfy the following properties:

1. Frequency response with narrow bandwidth and sharp cutoff characteristics.
2. Impulse response with relatively low overshoot.

3. Evolution of a phase trellis where the carrier phase of the modulated signal assumes the two values $\pm\pi/2$ at odd multiples of T_b and the two values 0 and π at even multiples of T_b as in MSK.

Condition 1 is needed to suppress the high-frequency components of the transmitted signal. Condition 2 avoids excessive deviations in the instantaneous frequency of the FM signal. Finally, condition 3 ensures that the modified FM signal can be coherently detected in the same way as the MSK signal, or it can be noncoherently detected as a simple binary FSK signal.

These desirable properties can be achieved by passing a nonreturn-to-zero (NRZ) binary data stream through a baseband pulse-shaping filter whose impulse response (and likewise its frequency response) is defined by a *Gaussian* function. The resulting method of binary frequency modulation is naturally referred to as *Gaussian-filtered MSK* or just *GMSK*.⁸

Let W denote the 3 dB baseband bandwidth of the pulse-shaping filter. We may then define the transfer function $H(f)$ and impulse response $h(t)$ of the pulse-shaping filter as follows, respectively:

$$H(f) = \exp\left(-\frac{\log 2}{2} \left(\frac{f}{W}\right)^2\right) \quad (6.132)$$

and

$$h(t) = \sqrt{\frac{2\pi}{\log 2}} W \exp\left(-\frac{2\pi^2}{\log 2} W^2 t^2\right) \quad (6.133)$$

The response of this Gaussian filter to a rectangular pulse of unit amplitude and duration T_b (centered on the origin) is given by (see Problem 6.28)

$$\begin{aligned} g(t) &= \int_{-T_b/2}^{T_b/2} h(t - \tau) d\tau \\ &= \sqrt{\frac{2\pi}{\log 2}} W \int_{-T_b/2}^{T_b/2} \exp\left(-\frac{2\pi^2}{\log 2} W^2 (t - \tau)^2\right) d\tau \end{aligned} \quad (6.134)$$

which may be expressed as the difference between two complementary error functions, as shown by

$$g(t) = \frac{1}{2} \left[\operatorname{erfc}\left(\pi \sqrt{\frac{2}{\log 2}} W T_b \left(\frac{t}{T_b} - \frac{1}{2}\right)\right) - \operatorname{erfc}\left(\pi \sqrt{\frac{2}{\log 2}} W T_b \left(\frac{t}{T_b} + \frac{1}{2}\right)\right) \right] \quad (6.135)$$

The pulse response $g(t)$ constitutes the *frequency shaping pulse* of the GMSK modulator, with the dimensionless *time-bandwidth product* $W T_b$ playing the role of a design parameter.

The frequency-shaping pulse $g(t)$, as defined in Equation (6.135), is noncausal in that it is nonzero for $t < -T_b/2$, where $t = -T_b/2$ is the time at which the input rectangular pulse (symmetrically positioned around the origin) is applied to the Gaussian filter. For a causal response, $g(t)$ must be truncated and shifted in time. Figure 6.32 presents plots of $g(t)$, which has been truncated at $t = \pm 2.5T_b$ and then shifted in time by $2.5T_b$. The plots shown here are for $W T_b = 0.2, 0.25,$ and 0.3 . Note that as $W T_b$ is reduced, the time spread of the frequency-shaping pulse is correspondingly increased.

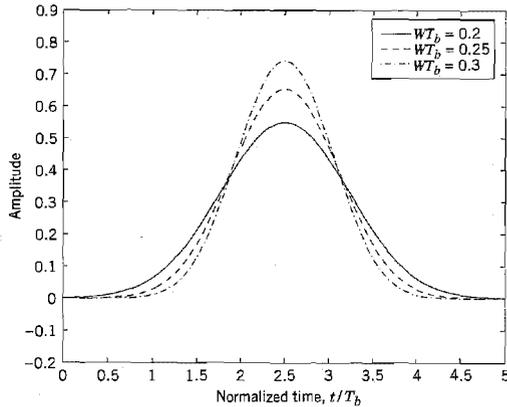


FIGURE 6.32 Frequency-shaping pulse $g(t)$ of Equation (6.135) shifted in time by $2.5T_b$ and truncated at $\pm 2.5T_b$, for varying time-bandwidth product WT_b .

Figure 6.33 shows the machine-computed power spectra of MSK signals (expressed in decibels) versus the normalized frequency difference $(f - f_c)T_b$, where f_c is the mid-band frequency and T_b is the bit duration.⁹ The results plotted in Figure 6.33 are for varying values of the time-bandwidth product WT_b . From this figure we may make the following observations:

- ▶ The curve for the limiting condition $WT_b = \infty$ corresponds to the case of ordinary MSK.
- ▶ When WT_b is less than unity, increasingly more of the transmit power is concentrated inside the passband of the GMSK signal.

An undesirable feature of GMSK is that the processing of NRZ binary data by a Gaussian filter generates a modulating signal that is no longer confined to a single bit interval as in ordinary MSK, which is readily apparent from Figure 6.32. Stated in another way, the tails of the Gaussian impulse response of the pulse-shaping filter cause the modulating signal to spread out to adjacent symbol intervals. The net result is the generation of *intersymbol interference*, the extent of which increases with decreasing WT_b . In light of this observation and the observation we made on the basis of Figure 6.33 on the power spectra of GMSK signals, we may say that the choice of the time-bandwidth product WT_b offers a trade-off between spectral compactness and performance loss.

To explore the issue of performance degradation, consider the probability of error P_e of GMSK using coherent detection in the presence of additive white Gaussian noise. Recognizing that GMSK is a special kind of binary frequency modulation, we may express P_e by the empirical formula

$$P_e = \frac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{\alpha E_b}{2N_0}} \right) \quad (6.136)$$

where, as before, E_b is the signal energy per bit and $N_0/2$ is the noise spectral density. The factor α is a constant whose value depends on the time-bandwidth product WT_b . Comparing the formula of Equation (6.136) for GMSK with that of Equation (6.127) for

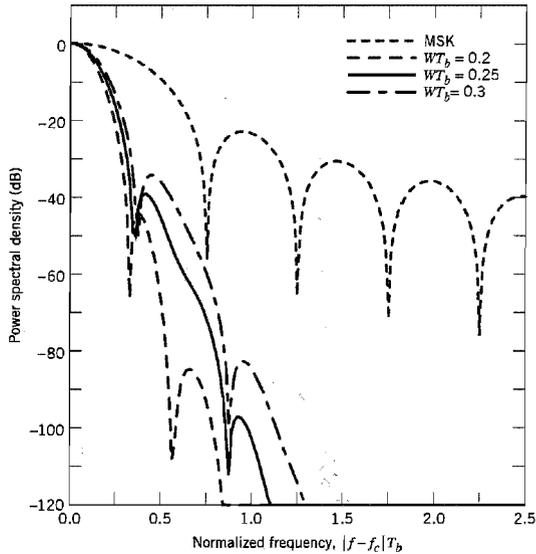


FIGURE 6.33 Power spectra of MSK and GMSK signals for varying time-bandwidth product. (Reproduced with permission from Dr. Gordon Stüber, Georgia Tech.)

ordinary MSK, we may view $10\log_{10}(\alpha/2)$, expressed in decibels, as a measure of performance degradation of GMSK (with prescribed WT_b) compared to ordinary MSK. Figure 6.34 shows the machine-computed value of $10\log_{10}(\alpha/2)$ versus WT_b . For ordinary MSK we have $WT_b = \infty$, in which case Equation (6.136) with $\alpha = 2$ assumes exactly the same form as Equation (6.127) and there is no degradation in performance, which is confirmed by Figure 6.34. For GMSK with $WT_b = 0.3$ we find from Figure 6.34 that there is a

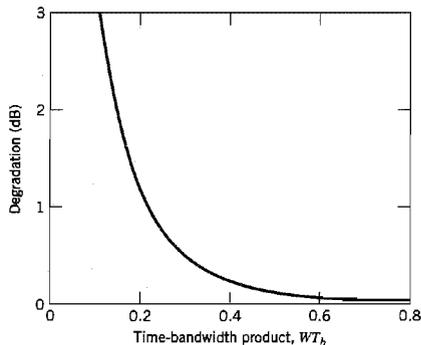


FIGURE 6.34 Theoretical E_b/N_0 degradation of GMSK for varying time-bandwidth product. (Taken from Murata and Hirade, 1981, with permission of the IEEE.)

degradation in performance of about 0.46 dB, which corresponds to $(\alpha/2) = 0.9$. This degradation in performance is a small price to pay for the highly desirable spectral compactness of the GMSK signal.

► **EXAMPLE 6.6 GMSK for GSM Wireless Communications**

An important application of GMSK is in a standardized wireless communication system widely known as GSM, which is a time-division multiple-access system that is discussed in Chapter 8. For this application, the time-bandwidth product WT_b of GMSK is standardized at 0.3, which provides the best compromise between increased bandwidth occupancy and resistance to co-channel interference. Ninety-nine percent of the radio frequency (RF) power of GMSK signals so specified is confined to a bandwidth of 250 kHz, which means that, for all practical purposes, the sidelobes are virtually zero outside this frequency band.

The available spectrum is divided into 200 kHz-wide subchannels. Each subchannel is assigned to a GSM system transmitting data at 271 kb/s. Figure 6.35 depicts the power spectrum of a subchannel in relation to its two adjacent subchannels; this plot is the passband version of the baseband power spectrum of Figure 6.33 corresponding to $WT_b = 0.3$. From Figure 6.35 we may make the following important observation: The RF power spectrum of the subchannel shown shaded is down by an amount larger than 40 dB at the carrier frequencies of both adjacent subchannels, which means that the effect of co-channel interference is practically negligible. ◀

■ **M-ARY FSK**

Consider next the M -ary version of FSK, for which the transmitted signals are defined by

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos \left[\frac{\pi}{T} (n_c + i)t \right], \quad 0 \leq t \leq T \quad (6.137)$$

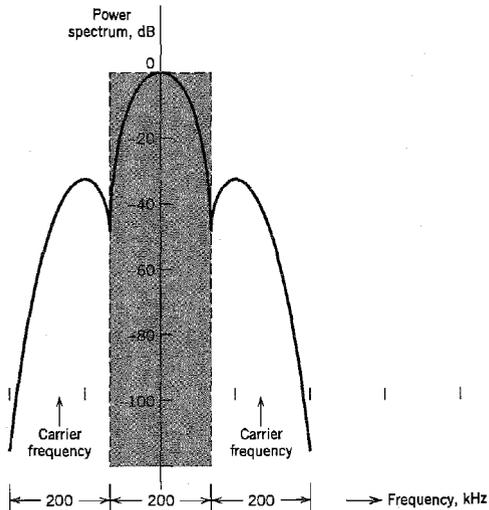


FIGURE 6.35 Power spectrum of GMSK signal for GSM wireless communications.

where $i = 1, 2, \dots, M$, and the carrier frequency $f_c = n_c/2T$ for some fixed integer n_c . The transmitted symbols are of equal duration T and have equal energy E . Since the individual signal frequencies are separated by $1/2T$ Hz, the signals in Equation (6.137) are orthogonal; that is

$$\int_0^T s_i(t)s_j(t) dt = 0, \quad i \neq j \quad (6.138)$$

This property of M -ary FSK suggests that we may use the transmitted signals $s_i(t)$ themselves, except for energy normalization, as a complete orthonormal set of basis functions, as shown by

$$\phi_i(t) = \frac{1}{\sqrt{E}} s_i(t), \quad \begin{matrix} 0 \leq t \leq T \\ i = 1, 2, \dots, M \end{matrix} \quad (6.139)$$

Accordingly, the M -ary FSK is described by an M -dimensional signal-space diagram.

For coherent M -ary FSK, the optimum receiver consists of a bank of M correlators or matched filters, with the $\phi_i(t)$ of Equation (6.139) providing the pertinent reference signals. At the sampling times $t = kT$, the receiver makes decisions based on the largest matched filter output in accordance with the maximum likelihood decoding rule. An exact formula for the probability of symbol error is however difficult to derive for a coherent M -ary FSK system. Nevertheless, we may use the union bound of Equation (5.96) of Chapter 5 to place an upper bound on the average probability of symbol error for M -ary FSK. Specifically, noting that the minimum distance d_{\min} in M -ary FSK is $\sqrt{2E}$, the use of Equation (5.96) yields (assuming equiprobable symbols)

$$P_e \leq \frac{1}{2} (M - 1) \operatorname{erfc} \left(\sqrt{\frac{E}{2N_0}} \right) \quad (6.140)$$

For fixed M , this bound becomes increasingly tight as E/N_0 is increased. Indeed, it becomes a good approximation to P_e for values of $P_e \leq 10^{-3}$. Moreover, for $M = 2$ (i.e., binary FSK), the bound of Equation (6.140) becomes an equality.

Power Spectra of M -ary FSK Signals

The spectral analysis of M -ary FSK signals¹⁰ is much more complicated than that of M -ary PSK signals. A case of particular interest occurs when the frequencies assigned to the multilevels make the frequency spacing uniform and the frequency deviation $k = 0.5$. That is, the M signal frequencies are separated by $1/2T$, where T is the symbol duration. For $k = 0.5$, the baseband power spectral density of M -ary FSK signals is plotted in Figure 6.36 for $M = 2, 4, 8$.

Bandwidth Efficiency of M -ary FSK Signals

When the orthogonal signals of an M -ary FSK signal are detected coherently, the adjacent signals need only be separated from each other by a frequency difference $1/2T$ so as to maintain orthogonality. Hence, we may define the channel bandwidth required to transmit M -ary FSK signals as

$$B = \frac{M}{2T} \quad (6.141)$$

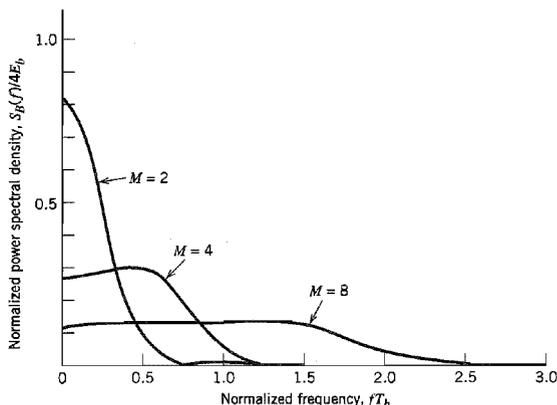


FIGURE 6.36 Power spectra of M -ary PSK signals for $M = 2, 4, 8$.

For multilevels with frequency assignments that make the frequency spacing uniform and equal to $1/2T$, the bandwidth B of Equation (6.141) contains a large fraction of the signal power. This is readily confirmed by looking at the baseband power spectral plots shown in Figure 6.36. From Equation (6.48) we recall that the symbol period T is equal to $T_b \log_2 M$. Hence, using $R_b = 1/T_b$, we may redefine the channel bandwidth B for M -ary FSK signals as

$$B = \frac{R_b M}{2 \log_2 M} \quad (6.142)$$

The bandwidth efficiency of M -ary signals is therefore

$$\begin{aligned} \rho &= \frac{R_b}{B} \\ &= \frac{2 \log_2 M}{M} \end{aligned} \quad (6.143)$$

Table 6.6 gives the values of ρ calculated from Equation (6.143) for varying M .

Comparing Tables 6.4 and 6.6, we see that increasing the number of levels M tends to increase the bandwidth efficiency of M -ary PSK signals, but it also tends to decrease the bandwidth efficiency of M -ary FSK signals. In other words, M -ary PSK signals are spectrally efficient, whereas M -ary FSK signals are spectrally inefficient.

TABLE 6.6 Bandwidth efficiency of M -ary FSK signals

M	2	4	8	16	32	64
ρ (bits/s/Hz)	1	1	0.75	0.5	0.3125	0.1875

6.6 Detection of Signals with Unknown Phase

Up to this point in our discussion, we have assumed that the receiver is perfectly synchronized to the transmitter, and the only channel impairment is noise. In practice, however, it is often found that in addition to the uncertainty due to channel noise, there is also uncertainty due to the randomness of certain signal parameters. The usual cause of this uncertainty is distortion in the transmission medium. Perhaps the most common random signal parameter is the carrier phase, which is especially true for narrowband signals. For example, transmission over a multiplicity of paths of different and variable lengths, or rapidly varying delays in the propagating medium from transmitter to receiver, may cause the phase of the received signal to change in a way that the receiver cannot follow. Synchronization with the phase of the transmitted carrier may then be too costly, and the designer may simply choose to disregard the phase information in the received signal at the expense of some degradation in noise performance. A digital communication receiver with no provision made for carrier phase recovery is said to be *noncoherent*.

✻ OPTIMUM QUADRATIC RECEIVER

Consider a binary digital communication system in which the transmitted signal is

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_i t), \quad 0 \leq t \leq T \quad (6.144)$$

$$i = 1, 2$$

where E is the signal energy, T is the duration of the signaling interval, and the carrier frequency f_i for symbol i is an integral multiple of $1/2T$. The system is assumed to be noncoherent, in which case the received signal for an AWGN channel may be written in the form

$$x(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_i t + \theta) + w(t), \quad 0 \leq t \leq T \quad (6.145)$$

$$i = 1, 2,$$

where θ is the unknown carrier phase, and $w(t)$ is the sample function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$. In a real-life situation it is realistic to assume complete lack of prior information about θ and to treat it as a random variable with *uniform distribution*:

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi < \theta \leq \pi \\ 0, & \text{otherwise} \end{cases} \quad (6.146)$$

The binary detection problem to be solved may now be stated as follows:

Given the received signal $x(t)$ and confronted with the unknown carrier phase θ , design an optimum receiver for detecting symbol s_i represented by the signal component $\sqrt{E/2T} \cos(2\pi f_i t + \theta)$ that is contained in $x(t)$.

Proceeding in a manner similar to that described in Sections 5.3–5.6, we may formulate the *conditional likelihood function* of symbol s_i , given the carrier phase θ , as

$$L(s_i(\theta)) = \exp\left(\sqrt{\frac{E}{N_0 T}} \int_0^T x(t) \cos(2\pi f_i t + \theta) dt\right) \quad (6.147)$$

To proceed further, we have to remove dependence of $L(s_i(\theta))$ on phase θ , which is achieved by integrating it over all possible values of θ . We may thus write

$$\begin{aligned} L(s_i) &= \int_{-\pi}^{\pi} L(s_i(\theta)) f_{\theta}(\theta) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\sqrt{\frac{E}{N_0 T}} \int_0^T x(t) \cos(2\pi f_i t + \theta) dt\right) d\theta \end{aligned} \quad (6.148)$$

Note that the dependence on θ was removed by integrating the likelihood function and *not* the log-likelihood function.

Using a well-known trigonometric formula, we may expand $\cos(2\pi f_i t + \theta)$ as

$$\cos(2\pi f_i t + \theta) = \cos(2\pi f_i t) \cos \theta - \sin(2\pi f_i t) \sin \theta$$

Correspondingly, we may rewrite the integral in the exponent of Equation (6.148) as

$$\int_0^T x(t) \cos(2\pi f_i t + \theta) dt = \cos \theta \int_0^T x(t) \cos(2\pi f_i t) dt - \sin \theta \int_0^T x(t) \sin(2\pi f_i t) dt \quad (6.149)$$

Define

$$I_i = \left[\left(\int_0^T x(t) \cos(2\pi f_i t) dt \right)^2 + \left(\int_0^T x(t) \sin(2\pi f_i t) dt \right)^2 \right]^{1/2} \quad (6.150)$$

$$\beta_i = \tan^{-1} \left(\frac{\int_0^T x(t) \sin(2\pi f_i t) dt}{\int_0^T x(t) \cos(2\pi f_i t) dt} \right) \quad (6.151)$$

Hence, we may go one step further and simplify Equation (6.149) to

$$\begin{aligned} \int_0^T x(t) \cos(2\pi f_i t + \theta) dt &= I_i (\cos \theta \cos \beta_i - \sin \theta \sin \beta_i) \\ &= I_i \cos(\theta + \beta_i) \end{aligned} \quad (6.152)$$

Accordingly, using Equation (6.152) in Equation (6.148), we obtain

$$\begin{aligned} L(s_i) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\sqrt{\frac{E}{N_0 T}} I_i \cos(\theta + \beta_i)\right) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi+\beta_i}^{\pi+\beta_i} \exp\left(\sqrt{\frac{E}{N_0 T}} I_i \cos \theta\right) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\sqrt{\frac{E}{N_0 T}} I_i \cos \theta\right) d\theta \end{aligned} \quad (6.153)$$

From Appendix 3 on Bessel functions, we immediately recognize the integral of Equation (6.153) as the *modified Bessel function of zero order*:

$$I_0\left(\sqrt{\frac{E}{N_0 T}} I_i\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(\sqrt{\frac{E}{N_0 T}} I_i \cos \theta\right) d\theta \quad (6.154)$$

Hence, we may express the likelihood function for the signal detection problem described herein in the compact form

$$L(s_i) = I_0 \left(\sqrt{\frac{E}{N_0 T}} l_i \right) \quad (6.155)$$

The *binary hypothesis test* (i.e., the hypothesis that signal $s_1(t)$ or signal $s_2(t)$ was transmitted) can now be written as

$$I_0 \left(\sqrt{\frac{E}{N_0 T}} l_1 \right) \underset{H_2}{\overset{H_1}{\gtrless}} I_0 \left(\sqrt{\frac{E}{N_0 T}} l_2 \right) \quad (6.156)$$

where hypothesis H_1 and H_2 correspond to signals $s_1(t)$ and $s_2(t)$, respectively. From Appendix 3 we note that the modified Bessel function $I(\cdot)$ is a monotonically increasing function of its argument. Hence the hypothesis test can be carried out in terms of either $I_0(\sqrt{E/N_0 T} l_i)$ or simply l_i . For convenience of implementation, however, the hypothesis test is carried out in terms of l_i^2 instead of l_i , as shown by

$$l_1^2 \underset{H_2}{\overset{H_1}{\gtrless}} l_2^2 \quad (6.157)$$

A receiver based on Equation (6.157) is known as the *quadratic receiver*. In light of the definition of l_i given in Equation (6.150), the receiver structure for computing l_i is as shown in Figure 6.37a. Note that the test described in Equation (6.157) is independent of the symbol energy E . It is for this reason that this hypothesis test is said to be *uniformly most powerful* with respect to the symbol energy E .

■ TWO EQUIVALENT FORMS OF THE QUADRATIC RECEIVER

We next derive two equivalent forms of the quadrature receiver shown in Figure 6.37a. The first form is obtained easily by replacing each correlator in Figure 6.37a with a corresponding equivalent matched filter. We thus obtain the alternative form of quadrature receiver shown in Figure 6.37b. In one branch of this receiver, we have a filter matched to the signal $\cos(2\pi f_i t)$, and in the other branch we have a filter matched to $\sin(2\pi f_i t)$, both of which are defined for the time interval $0 \leq t \leq T$. The filter outputs are sampled at time $t = T$, squared, and then added together.

To obtain the second equivalent form of the quadrature receiver, suppose we have a filter that is matched to $s(t) = \cos(2\pi f_i t + \theta)$ for $0 \leq t \leq T$. The envelope of the matched filter output is obviously unaffected by the value of phase θ . Therefore, for convenience, we may simply choose a matched filter with impulse response $\cos[2\pi f_i(T - t)]$, corresponding to $\theta = 0$. The output of such a filter in response to the received signal $x(t)$ is given by

$$\begin{aligned} y(t) &= \int_0^T x(\tau) \cos[2\pi f_i(T - t + \tau)] d\tau \\ &= \cos[2\pi f_i(T - t)] \int_0^T x(\tau) \cos(2\pi f_i \tau) d\tau - \sin[2\pi f_i(T - t)] \int_0^T x(\tau) \sin(2\pi f_i \tau) d\tau \end{aligned} \quad (6.158)$$

The envelope of the matched filter output is proportional to the square root of the sum of the squares of the integrals in Equation (6.158). The envelope, evaluated at time $t = T$, is therefore

$$l_i = \left\{ \left[\int_0^T x(\tau) \cos(2\pi f_i \tau) d\tau \right]^2 + \left[\int_0^T x(\tau) \sin(2\pi f_i \tau) d\tau \right]^2 \right\}^{1/2} \quad (6.159)$$

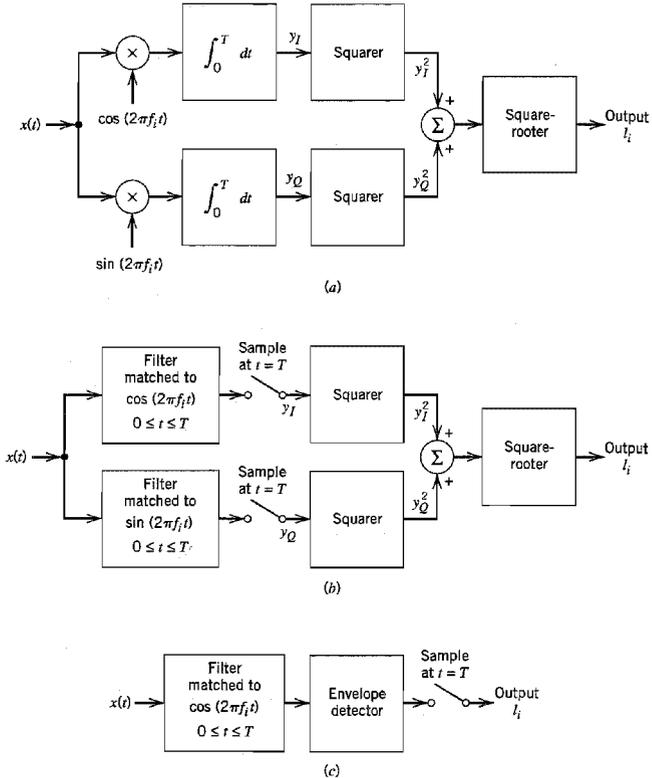


FIGURE 6.37 Noncoherent receivers. (a) Quadrature receiver using correlators. (b) Quadrature receiver using matched filters. (c) Noncoherent matched filter.

But this is just the output of the quadrature receiver. Therefore, the output (at time T) of a filter matched to the signal $\cos(2\pi f_c t + \theta)$, of arbitrary phase θ , followed by an envelope detector is the same as the corresponding output of the quadrature receiver of Figure 6.37a. This form of receiver is shown in Figure 6.37c. The combination of matched filter and envelope detector shown in Figure 6.37c is called a *noncoherent matched filter*.

The need for an envelope detector following the matched filter in Figure 6.37c may also be justified intuitively as follows. The output of a filter matched to a rectangular RF wave reaches a positive peak at the sampling instant $t = T$. If, however, the phase of the filter is not matched to that of the signal, the peak may occur at a time different from the sampling instant. In actual fact, if the phases differ by 180 degrees, we get a negative peak at the sampling instant. Figure 6.38 illustrates the matched filter output for the two limiting conditions: $\theta = 0$ and $\theta = 180$ degrees. To avoid poor sampling that arises in the absence of prior information about the phase θ , it is reasonable to retain only the envelope of the matched filter output, since it is completely independent of the phase mismatch θ .

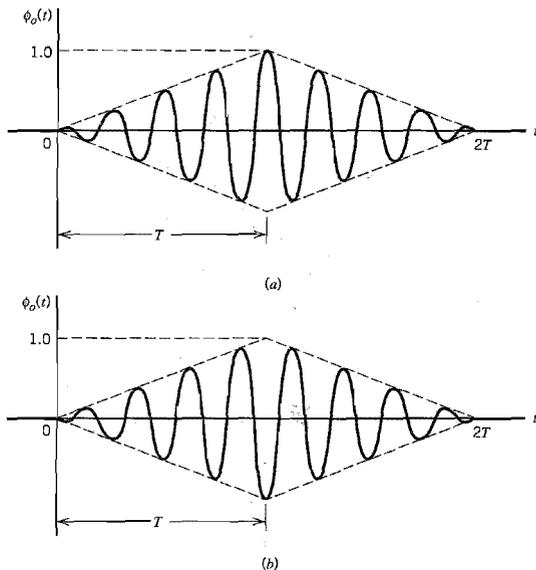


FIGURE 6.38 Output of matched filter for a rectangular RF wave: (a) $\theta = 0$, and (b) $\theta = 180$ degrees.

6.7 Noncoherent Orthogonal Modulation

With the noncoherent receiver structures of Figure 6.37 at our disposal, we may now proceed to study the noise performance of *noncoherent orthogonal modulation* that includes two noncoherent receivers as special cases: noncoherent binary frequency-shift keying and differential phase-shift keying.

Consider a binary signaling scheme that involves the use of two orthogonal signals $s_1(t)$ and $s_2(t)$, which have equal energy. During the interval $0 \leq t \leq T$, one of these two signals is sent over an imperfect channel that shifts the carrier phase by an unknown amount. Let $g_1(t)$ and $g_2(t)$ denote the phase-shifted versions of $s_1(t)$ and $s_2(t)$, respectively. It is assumed that the signals $g_1(t)$ and $g_2(t)$ remain orthogonal and have the same energy E , regardless of the unknown carrier phase. We refer to such a signaling scheme as *noncoherent orthogonal modulation*. Depending on how we define the orthogonal pair of signals $s_1(t)$ and $s_2(t)$, noncoherent binary FSK and DPSK may be treated as special cases of this modulation scheme.

The channel also introduces an additive white Gaussian noise $w(t)$ of zero mean and power spectral density $N_0/2$. We may thus express the received signal $x(t)$ as

$$x(t) = \begin{cases} g_1(t) + w(t), & s_1(t) \text{ sent, } 0 \leq t \leq T \\ g_2(t) + w(t), & s_2(t) \text{ sent, } 0 \leq t \leq T \end{cases} \quad (6.160)$$

The requirement is to use $x(t)$ to discriminate between $s_1(t)$ and $s_2(t)$, regardless of the carrier phase.

For this purpose, we employ the receiver shown in Figure 6.39a. The receiver consists of a pair of filters matched to the transmitted signals $s_1(t)$ and $s_2(t)$. Because the carrier phase is unknown, the receiver relies on amplitude as the only possible discriminant. Accordingly, the matched filter outputs are envelope detected, sampled, and then compared with each other. If the upper path in Figure 6.39a has an output amplitude I_1 greater than the output amplitude I_2 of the lower path, the receiver makes a decision in favor of $s_1(t)$. If the converse is true, it decides in favor of $s_2(t)$. When they are equal, the decision may be made by flipping a fair coin. In any event, a decision error occurs when the matched filter that rejects the signal component of the received signal $x(t)$ has a larger output amplitude (due to noise alone) than the matched filter that passes it.

From the discussion presented in Section 6.6, we note that a noncoherent matched filter (constituting the upper or lower path in the receiver of Figure 6.39a) may be viewed as being equivalent to a *quadrature receiver*. The quadrature receiver itself has two channels. One version of the quadrature receiver is shown in Figure 6.39b. In the upper channel, called the *in-phase channel*, the received signal $x(t)$ is correlated with the function $\psi_i(t)$, which represents a scaled version of the transmitted signal $s_1(t)$ or $s_2(t)$ with zero carrier phase. In the lower channel, called the *quadrature channel*, on the other hand, $x(t)$ is

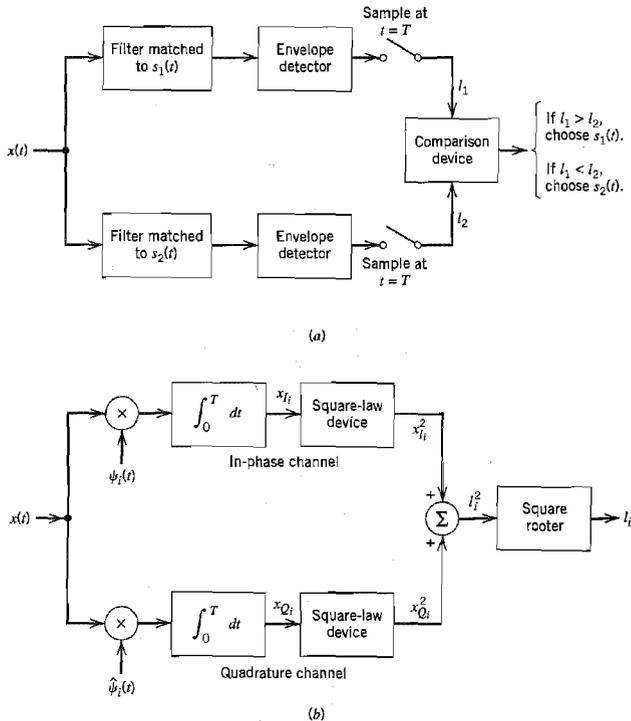


FIGURE 6.39 (a) Generalized binary receiver for noncoherent orthogonal modulation. (b) Quadrature receiver equivalent to either one of the two matched filters in part (a); the index $i = 1, 2$.

correlated with another function $\hat{\psi}_i(t)$, which represents the version of $\psi_i(t)$ that results from shifting the carrier phase by -90 degrees. Naturally, $\psi_i(t)$ and $\hat{\psi}_i(t)$ are orthogonal to each other.

The signal $\hat{\psi}_i(t)$ is in fact the *Hilbert transform* of $\psi_i(t)$; the Hilbert transform is discussed in Appendix 2. To illustrate the nature of this relationship, let

$$\psi_i(t) = m(t) \cos(2\pi f_i t) \quad (6.161)$$

where $m(t)$ is a band-limited message signal. Typically, the carrier frequency f_i is greater than the highest frequency component of $m(t)$. Then (in a manner similar to the discussion on CAP presented in Section 6.4) the Hilbert transform of $\psi_i(t)$ is defined by

$$\hat{\psi}_i(t) = m(t) \sin(2\pi f_i t) \quad (6.162)$$

Since

$$\cos\left(2\pi f_i t - \frac{\pi}{2}\right) = \sin(2\pi f_i t)$$

we see that $\hat{\psi}_i(t)$ is indeed obtained from $\psi_i(t)$ by shifting the carrier $\cos(2\pi f_i t)$ by -90 degrees. An important property of Hilbert transformation is that a signal and its Hilbert transform are orthogonal to each other. Thus, $\psi_i(t)$ and $\hat{\psi}_i(t)$ are orthogonal to each other, as already stated.

The average probability of error for the noncoherent receiver of Figure 6.39a is given by the simple formula

$$P_e = \frac{1}{2} \exp\left(-\frac{E}{2N_0}\right) \quad (6.163)$$

where E is the signal energy per symbol, and $N_0/2$ is the noise spectral density.

✻ DERIVATION OF EQUATION (6.163)*

To derive the formula of Equation (6.163), we make use of the equivalence depicted in Figure 6.39. In particular, we observe that since the carrier phase is unknown, noise at the output of each matched filter in Figure 6.39a has *two degrees of freedom*, namely, in-phase and quadrature. Accordingly, the noncoherent receiver of Figure 6.39a has a total of four noisy parameters that are *conditionally independent* given the phase θ , and also *identically distributed*. These four noisy parameters have sample values denoted by x_{11} , x_{Q1} , x_{12} , and x_{Q2} ; the first two account for degrees of freedom associated with the upper path of Figure 6.39a, and the latter two account for degrees of freedom associated with the lower path.

The receiver of Figure 6.39a has a *symmetric* structure. Hence, the probability of choosing $s_2(t)$, given that $s_1(t)$ was transmitted, is the same as the probability of choosing $s_1(t)$, given that $s_2(t)$ was transmitted. This means that the average probability of error may be obtained by transmitting $s_1(t)$ and calculating the probability of choosing $s_2(t)$, or vice versa, assuming that $s_1(t)$ and $s_2(t)$ are equiprobable.

Suppose that signal $s_1(t)$ is transmitted for the interval $0 \leq t \leq T$. An error occurs if the channel noise $w(t)$ is such that the output l_2 of the lower path in Figure 6.39a is greater than the output l_1 of the upper path. Then the receiver makes a decision in favor of $s_2(t)$

*Readers who are not interested in the formal derivation of Eq. (6.163) may at this point wish to move on to the treatment of noncoherent binary frequency-shift keying (in Section 6.7) and differential phase-shift keying (in Section 6.8) as special cases of noncoherent orthogonal modulation, without loss of continuity.

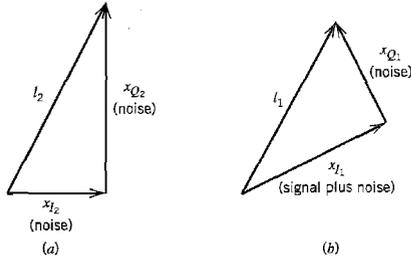


FIGURE 6.40 Geometric interpretations of the two path outputs l_1 and l_2 in the generalized non-coherent receiver.

rather than $s_1(t)$. To calculate the probability of error so made, we must have the probability density function of the random variable L_2 (represented by sample value l_2). Since the filter in the lower path is matched to $s_2(t)$, and $s_2(t)$ is orthogonal to the transmitted signal $s_1(t)$, it follows that the output of this matched filter is due to *noise alone*. Let x_{l2} and x_{Q2} denote the in-phase and quadrature components of the matched filter output in the lower path of Figure 6.39a. Then, from the equivalent structure depicted in Figure 6.39b, we see that (for $i = 2$)

$$l_2 = \sqrt{x_{l2}^2 + x_{Q2}^2} \quad (6.164)$$

Figure 6.40a shows a geometric interpretation of this relation. The channel noise $w(t)$ is both white (with power spectral density $N_0/2$) and Gaussian (with zero mean). Correspondingly, we find that the random variables X_{l2} and X_{Q2} (represented by sample values x_{l2} and x_{Q2}) are both Gaussian-distributed with zero mean and variance $N_0/2$, given the phase θ . Hence, we may write

$$f_{x_{l2}}(x_{l2}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_{l2}^2}{N_0}\right) \quad (6.165)$$

and

$$f_{x_{Q2}}(x_{Q2}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_{Q2}^2}{N_0}\right) \quad (6.166)$$

Next, we use a well-known result in probability theory, namely, the fact that the envelope of a Gaussian process is *Rayleigh-distributed* and independent of the phase θ (see Section 1.12). Specifically, for the situation at hand, we may state that the random variable L_2 [whose sample value l_2 is related to x_{l2} and x_{Q2} by Equation (6.164)] has the following probability density function:

$$f_{L_2}(l_2) = \begin{cases} \frac{2l_2}{N_0} \exp\left(-\frac{l_2^2}{N_0}\right), & l_2 \geq 0 \\ 0, & \text{elsewhere} \end{cases} \quad (6.167)$$

Figure 6.41 shows a plot of this probability density function. The conditional probability that $l_2 > l_1$, given the sample value l_1 , is defined by the shaded area in Figure 6.41. Hence, we have

$$P(l_2 > l_1 | l_1) = \int_{l_1}^{\infty} f_{L_2}(l_2) dl_2 \quad (6.168)$$

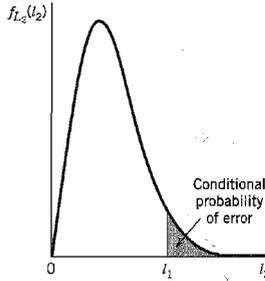


FIGURE 6.41 Calculation of the conditional probability that $l_2 > l_1$, given l_1 .

Substituting Equation (6.167) into Equation (6.168) and integrating, we get

$$P(l_2 > l_1 | l_1) = \exp\left(-\frac{l_1^2}{N_0}\right) \quad (6.169)$$

Consider next the output amplitude l_1 , pertaining to the upper path in Figure 6.39a. Since the filter in this path is matched to $s_1(t)$, and it is assumed that $s_1(t)$ is transmitted, it follows that l_1 is due to *signal plus noise*. Let x_{I1} and x_{Q1} denote the components at the output of the matched filter (in the upper path of Figure 6.39a) that are in phase and in quadrature with respect to the received signal, respectively. Then from the equivalent structure depicted in Figure 6.39b, we see that (for $i = 1$)

$$l_1 = \sqrt{x_{I1}^2 + x_{Q1}^2} \quad (6.170)$$

Figure 6.40b presents a geometric interpretation of this relation. Since a Fourier-transformable signal and its Hilbert transform form an orthogonal pair, it follows that x_{I1} is due to signal plus noise, whereas x_{Q1} is due to noise alone. This means that (1) the random variable X_{I1} represented by the sample value x_{I1} is Gaussian distributed with mean \sqrt{E} and variance $N_0/2$, where E is the signal energy per symbol, and (2) the random variable X_{Q1} represented by the sample value x_{Q1} is Gaussian distributed with zero mean and variance $N_0/2$. Hence, we may express the probability density functions of these two independent random variables as follows:

$$f_{x_{I1}}(x_{I1}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(x_{I1} - \sqrt{E})^2}{N_0}\right) \quad (6.171)$$

and

$$f_{x_{Q1}}(x_{Q1}) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{x_{Q1}^2}{N_0}\right) \quad (6.172)$$

Since the two random variables X_{I1} and X_{Q1} are independent, their joint probability density function is simply the product of the probability density functions given in Equations (6.171) and (6.172).

To find the average probability of error, we have to average the conditional probability of error given in Equation (6.169) over all possible values of l_1 . Naturally, this calculation requires knowledge of the probability density function of random variables L_1 represented by sample value l_1 . The standard method is now to combine Equations (6.171) and (6.172) to find the probability density function of L_1 due to signal plus noise. However, this leads to rather complicated calculations involving the use of Bessel functions. This

analytic difficulty may be circumvented by the following approach. Given x_{I1} and x_{Q1} , an error occurs when, in Figure 6.39a, the lower path's output amplitude l_2 due to noise alone exceeds l_1 due to signal plus noise; from Equation (6.170) we have

$$l_1^2 = x_{I1}^2 + x_{Q1}^2 \quad (6.173)$$

The probability of such an occurrence is obtained by substituting Equation (6.173) into Equation (6.169), as shown by

$$P(\text{error} | x_{I1}, x_{Q1}) = \exp\left(-\frac{x_{I1}^2 + x_{Q1}^2}{N_0}\right) \quad (6.174)$$

This is now a conditional probability of error, conditional on the output of the matched filter in the upper path taking on values X_{I1} and X_{Q1} . This conditional probability multiplied by the joint probability density function of X_{I1} and X_{Q1} is then the *error-density*, given x_{I1} and x_{Q1} . Since X_{I1} and X_{Q1} are statistically independent, their joint probability density function equals the product of their individual probability density functions. The resulting error-density is a complicated expression in x_{I1} and x_{Q1} . However, the average probability of error, which is the issue of interest, may be obtained in a relatively simple manner. We first use Equations (6.171), (6.172), and (6.174) to evaluate the desired error-density as

$$\begin{aligned} & P(\text{error} | x_{I1}, x_{Q1}) f_{X_{I1}}(x_{I1}) f_{X_{Q1}}(x_{Q1}) \\ &= \frac{1}{\pi N_0} \exp\left\{-\frac{1}{N_0} [x_{I1}^2 + x_{Q1}^2 + (x_{I1} - \sqrt{E})^2 + x_{Q1}^2]\right\} \end{aligned} \quad (6.175)$$

Completing the square in the exponent of Equation (6.175), we may rewrite the exponent except for $-1/N_0$ as

$$x_{I1}^2 + x_{Q1}^2 + (x_{I1} - \sqrt{E})^2 + x_{Q1}^2 = 2\left(x_{I1} - \frac{\sqrt{E}}{2}\right)^2 + 2x_{Q1}^2 + \frac{E}{2} \quad (6.176)$$

Next, we substitute Equation (6.176) into Equation (6.175) and integrate the error-density over all x_{I1} and x_{Q1} . We thus evaluate the average probability of error as

$$\begin{aligned} P_e &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(\text{error} | x_{I1}, x_{Q1}) f_{X_{I1}}(x_{I1}) f_{X_{Q1}}(x_{Q1}) dx_{I1} dx_{Q1} \\ &= \frac{1}{\pi N_0} \exp\left(-\frac{E}{2N_0}\right) \int_{-\infty}^{\infty} \exp\left[-\frac{2}{N_0} \left(x_{I1} - \frac{\sqrt{E}}{2}\right)^2\right] dx_{I1} \\ &\quad \cdot \int_{-\infty}^{\infty} \exp\left(-\frac{2x_{Q1}^2}{N_0}\right) dx_{Q1} \end{aligned} \quad (6.177)$$

We now use the following two identities:

$$\int_{-\infty}^{\infty} \exp\left[-\frac{2}{N_0} \left(x_{I1} - \frac{\sqrt{E}}{2}\right)^2\right] dx_{I1} = \sqrt{\frac{N_0\pi}{2}} \quad (6.178)$$

and

$$\int_{-\infty}^{\infty} \exp\left(-\frac{2x_{Q1}^2}{N_0}\right) dx_{Q1} = \sqrt{\frac{N_0\pi}{2}} \quad (6.179)$$

The identity of Equation (6.178) is obtained by considering a Gaussian-distributed variable with mean $\sqrt{E}/2$ and variance $N_0/4$, and recognizing that the total area under the curve of a random variable's probability density function equals unity; the identity of Equation

(6.179) follows as a special case of Equation (6.178). Thus, in light of these two identities, Equation (6.177) simplifies as follows:

$$P_a = \frac{1}{2} \exp\left(-\frac{E}{2N_0}\right)$$

which is the desired result presented previously as Equation (6.163).

With this formula at our disposal, we are ready to consider noncoherent binary FSK and DPSK as special cases, which we do in the next two sections, respectively.¹¹

6.8 Noncoherent Binary Frequency-Shift Keying

In the binary FSK case, the transmitted signal is defined by

$$s_i(t) = \begin{cases} \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_i t), & 0 \leq t \leq T_b \\ 0, & \text{elsewhere} \end{cases} \quad (6.180)$$

where the carrier frequency f_i equals one of two possible values, f_1 and f_2 ; to ensure that the signals representing these two frequencies are orthogonal, we choose $f_i = n_i/T_b$, where n_i is an integer. The transmission of frequency f_1 represents symbol 1, and the transmission of frequency f_2 represents symbol 0. For the noncoherent detection of this frequency-modulated wave, the receiver consists of a pair of matched filters followed by envelope detectors, as in Figure 6.42. The filter in the upper path of the receiver is matched to $\cos(2\pi f_1 t)$, and the filter in the lower path is matched to $\cos(2\pi f_2 t)$, and in both cases $0 \leq t \leq T_b$. The resulting envelope detector outputs are sampled at $t = T_b$, and their values are compared. The envelope samples of the upper and lower paths in Figure 6.42 are shown as l_1 and l_2 , respectively. Then, if $l_1 > l_2$, the receiver decides in favor of symbol 1, and if $l_1 < l_2$, it decides in favor of symbols 0. If $l_1 = l_2$, the receiver simply makes a guess in favor of symbol 1 or 0.

The noncoherent binary FSK described herein is a special case of noncoherent orthogonal modulation with $T = T_b$ and $E = E_b$, where T_b is the bit duration and E_b is the

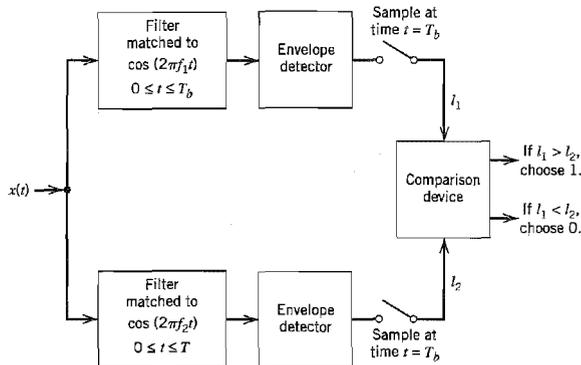


FIGURE 6.42 Noncoherent receiver for the detection of binary FSK signals.

signal energy per bit. Hence, using (Equation (6.163), we find that the *bit error rate* for *noncoherent binary FSK* is

$$P_e = \frac{1}{2} \exp\left(-\frac{E_b}{2N_0}\right) \quad (6.181)$$

The formula of Equation (6.181) is derived as a special case of noncoherent orthogonal modulation. In Problem 6.31 we address the same issue using a direct approach that invokes the application of Rayleigh and Rician distributions; these distributions pertain respectively to the random variables L_2 and L_1 whose sample values are defined by Equations (6.164) and (6.170), respectively.

6.9 Differential Phase-Shift Keying

As remarked earlier in Section 6.1, we may view *differential phase-shift keying* (DPSK) as the noncoherent version of PSK. It eliminates the need for a coherent reference signal at the receiver by combining two basic operations at the transmitter: (1) *differential encoding* of the input binary wave and (2) *phase-shift keying*—hence, the name, *differential phase-shift keying* (DPSK). In effect, to send symbol 0, we phase advance the current signal waveform by 180 degrees, and to send symbol 1 we leave the phase of the current signal waveform unchanged. The receiver is equipped with a *storage* capability, so that it can measure the *relative phase difference* between the waveforms received during two successive bit intervals. Provided that the unknown phase θ contained in the received wave varies slowly (that is, slow enough for it to be considered essentially constant over two bit intervals), the phase difference between waveforms received in two successive bit intervals will be independent of θ .

DPSK is another example of noncoherent orthogonal modulation, when it is considered over two bit intervals. Suppose the transmitted DPSK signal equals $\sqrt{E_b}/2T_b \cos(2\pi f_c t)$ for $0 \leq t \leq T_b$, where T_b is the bit duration and E_b is the signal energy per bit. Let $s_1(t)$ denote the transmitted DPSK signal for $0 \leq t \leq 2T_b$ for the case when we have binary symbol 1 at the transmitter input for the second part of this interval, namely, $T_b \leq t \leq 2T_b$. The transmission of symbol 1 leaves the carrier phase unchanged over the interval $0 \leq t \leq 2T_b$, and so we define $s_1(t)$ as

$$s_1(t) = \begin{cases} \sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t), & 0 \leq t \leq T_b \\ \sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t), & T_b \leq t \leq 2T_b \end{cases} \quad (6.182)$$

Let $s_2(t)$ denote the transmitted DPSK signal for $0 \leq t \leq 2T_b$ for the case when we have binary symbol 0 at the transmitter input for $T_b \leq t \leq 2T_b$. The transmission of 0 advances the carrier phase by 180 degrees, and so we define $s_2(t)$ as

$$s_2(t) = \begin{cases} \sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t), & 0 \leq t \leq T_b \\ \sqrt{\frac{E_b}{2T_b}} \cos(2\pi f_c t + \pi), & T_b \leq t \leq 2T_b \end{cases} \quad (6.183)$$

We readily see from Equations (6.182) and (6.183) that $s_1(t)$ and $s_2(t)$ are indeed orthogonal over the two-bit interval $0 \leq t \leq 2T_b$. In other words, DPSK is a special case of noncoherent orthogonal modulation with $T = 2T_b$ and $E = 2E_b$. Hence, using Equation (6.163), we find that the *bit error rate for DPSK* is given by

$$P_e = \frac{1}{2} \exp\left(-\frac{E_b}{N_0}\right) \quad (6.184)$$

which provides a gain of 3 dB over noncoherent FSK for the same E_b/N_0 .

Generation and Detection of DPSK

The next issue to be considered is the generation of DPSK signals. The differential encoding process at the transmitter input starts with an arbitrary first bit, serving as reference. Let $\{d_k\}$ denote the differentially encoded sequence with this added reference bit. We now introduce the following definitions in the generation of this sequence:

- ▶ If the incoming binary symbol b_k is 1, leave the symbol d_k unchanged with respect to the previous bit.
- ▶ If the incoming binary symbol b_k is 0, change the symbol d_k with respect to the previous bit.

The differentially encoded sequence $\{d_k\}$ thus generated is used to phase-shift a carrier with phase angles 0 and π radians representing symbols 1 and 0, respectively. The differential-phase encoding process is illustrated in Table 6.7. Note that d_k is the complement of the modulo-2 sum of b_k and d_{k-1} .

The block diagram of a DPSK transmitter is shown in Figure 6.43a. It consists, in part, of a logic network and a one-bit delay element interconnected so as to convert the raw binary sequence $\{b_k\}$ into a differentially encoded sequence $\{d_k\}$. This sequence is amplitude-level encoded and then used to modulate a carrier wave of frequency f_c , thereby producing the desired DPSK signal.

Suppose next, in differentially coherent detection of binary DPSK, the carrier phase is unknown. Then, in light of the receiver being equipped with an in-phase and a quadrature channel, we have a signal space diagram where the received signal points are $(A \cos \theta, A \sin \theta)$ and $(-A \cos \theta, -A \sin \theta)$, with θ denoting the unknown phase and A denoting the amplitude. This geometry of possible signals is illustrated in Figure 6.44. The receiver measures the coordinates (x_{I_0}, x_{Q_0}) at time $t = T_b$ and (x_{I_1}, x_{Q_1}) at time $t = 2T_b$. The issue to be resolved is whether these two points map to the same signal point or different ones. Recognizing that the two vectors \mathbf{x}_0 and \mathbf{x}_1 , with end points (x_{I_0}, x_{Q_0}) and (x_{I_1}, x_{Q_1}) are pointed roughly in the same direction if their inner product is positive, we may formulate the hypothesis test as follows:

Is the inner product $\mathbf{x}_0^T \mathbf{x}_1$ positive or negative?

TABLE 6.7 Illustrating the generation of DPSK signal

$\{b_k\}$	1	0	0	1	0	0	1	1
$\{d_{k-1}\}$	1	1	0	1	1	0	1	1
Differentially encoded sequence $\{d_k\}$	1	1	0	1	1	0	1	1
Transmitted phase (radians)	0	0	π	0	0	π	0	0

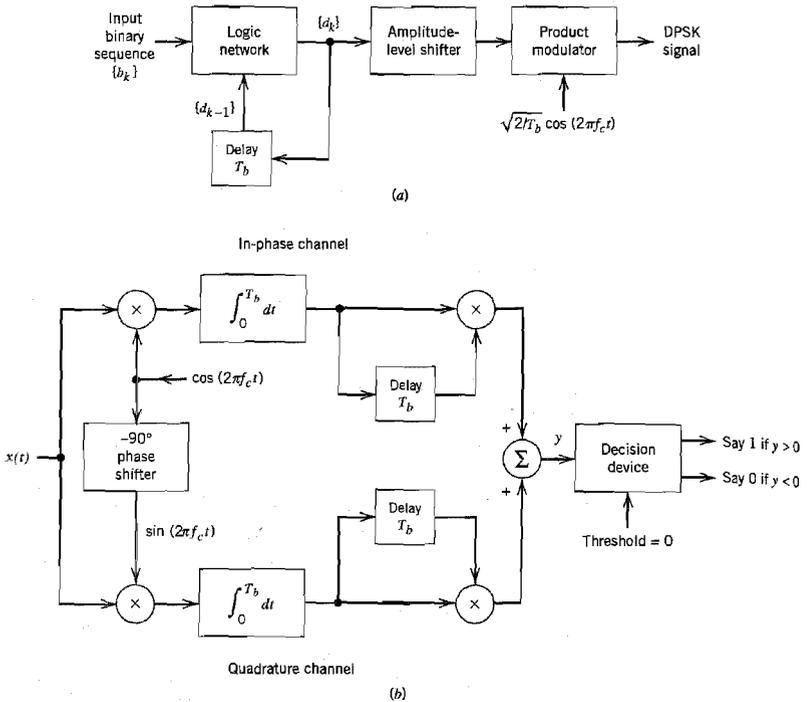


FIGURE 6.43 Block diagrams of (a) DPSK transmitter and (b) DPSK receiver.

Accordingly, we may write

$$x_{I_0}x_{I_1} + x_{Q_0}x_{Q_1} \underset{\text{say 0}}{\overset{\text{say 1}}{\cong}} 0 \tag{6.185}$$

We now note the following identity:

$$x_{I_0}x_{I_1} + x_{Q_0}x_{Q_1} = \frac{1}{4} [(x_{I_0} + x_{I_1})^2 - (x_{I_0} - x_{I_1})^2 + (x_{Q_0} + x_{Q_1})^2 - (x_{Q_0} - x_{Q_1})^2]$$

Hence substituting this identity into Equation (6.185) and multiplying both sides of the test by 4, we get the equivalent test:

$$(x_{I_0} + x_{I_1})^2 + (x_{Q_0} + x_{Q_1})^2 - (x_{I_0} - x_{I_1})^2 - (x_{Q_0} - x_{Q_1})^2 \underset{\text{say 0}}{\overset{\text{say 1}}{\cong}} 0 \tag{6.186}$$

The decision-making process may therefore be thought of as testing whether the point (x_{I_0}, x_{Q_0}) is closer to (x_{I_1}, x_{Q_1}) or its image $(-x_{I_1}, -x_{Q_1})$.

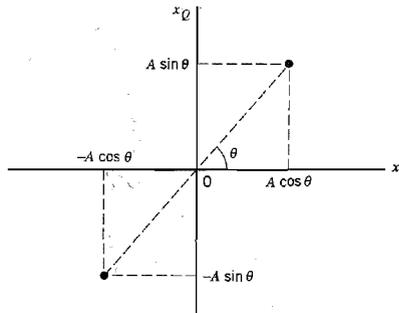


FIGURE 6.44 Signal-space diagram of received DPSK signal.

Thus, the *optimum receiver*¹² for differentially coherent detection of binary DPSK is as shown in Figure 6.43*b*, which follows directly from Equation (6.185). This implementation merely requires that *sample* values be stored, thereby avoiding the need for fancy delay lines that may be needed otherwise. The equivalent receiver implementation that tests squared elements as in Equation (6.186) is more complicated, but its use makes the *analysis* easier to handle in that the two signals to be considered are orthogonal over the interval $(0, 2T_b)$; hence, the noncoherent orthogonal demodulation analysis applies.

6.10 Comparison of Digital Modulation Schemes Using a Single Carrier

■ PROBABILITY OF ERROR

In Table 6.8 we have summarized the expressions for the bit error rate (BER) for coherent binary PSK, conventional coherent binary FSK with one-bit decoding, DPSK, noncoherent binary FSK, coherent QPSK, and coherent MSK, when operating over an AWGN channel. In Figure 6.45 we have used the expressions summarized in Table 6.8 to plot the BER as a function of the signal energy per bit-to-noise spectral density ratio, E_b/N_0 .

TABLE 6.8 Summary of formulas for the bit error rate of different digital modulation schemes

Signaling Scheme	Bit Error Rate
(a) Coherent binary PSK	$\frac{1}{2} \operatorname{erfc}(\sqrt{E_b/N_0})$
Coherent QPSK	
Coherent MSK	
(b) Coherent binary FSK	$\frac{1}{2} \operatorname{erfc}(\sqrt{E_b/2N_0})$
(c) DPSK	$\frac{1}{2} \exp(-E_b/N_0)$
(d) Noncoherent binary FSK	$\frac{1}{2} \exp(-E_b/2N_0)$

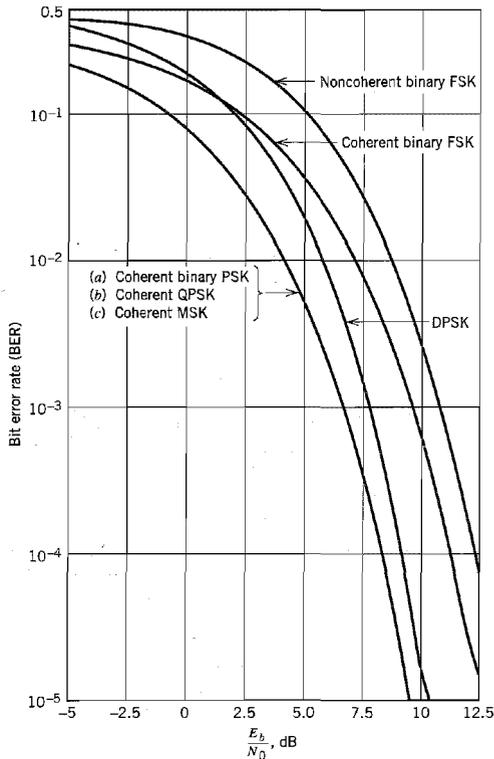


FIGURE 6.45 Comparison of the noise performance of different PSK and FSK schemes.

Based on the performance curves shown in Figure 6.45, the summary of formulas given in Table 6.8, and the defining equations for the pertinent modulation formats, we can make the following statements:

1. The bit error rates for all the systems decrease monotonically with increasing values of E_b/N_0 ; the defining curves have a similar shape in the form of a *waterfall*.
2. For any value of E_b/N_0 , coherent binary PSK, QPSK, and MSK produce a smaller bit error rate than any of the other modulation schemes.
3. Coherent binary PSK and DPSK require an E_b/N_0 that is 3 dB less than the corresponding values for conventional coherent binary FSK and noncoherent binary FSK, respectively, to realize the same bit error rate.
4. At high values of E_b/N_0 , DPSK and noncoherent binary FSK perform almost as well (to within about 1 dB) as coherent binary PSK and conventional coherent binary FSK, respectively, for the same bit rate and signal energy per bit.
5. In coherent QPSK, two orthogonal carriers $\sqrt{2/T} \cos(2\pi f_c t)$ and $\sqrt{2/T} \sin(2\pi f_c t)$ are used, where the carrier frequency f_c is an integer multiple of the symbol rate