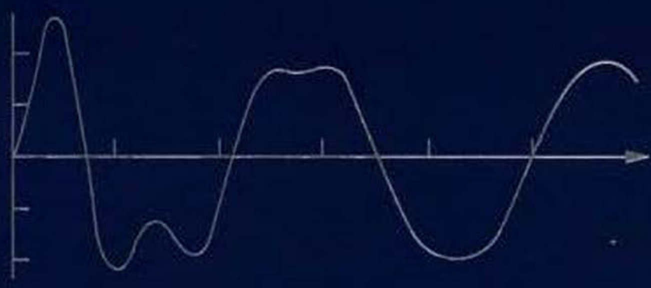
Fundamentals of Acoustics

FOURTH EDITION



Lawrence E. Kinsler

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GLOSSARY OF SYMBOLS

This list identifies some symbols that are not necessarily defined every time they appear in the text.

a	acceleration; absorption coefficient (dB per distance); Sabine absorptivity
a_E	random-incidence energy absorption coefficient
A	sound absorption
AG	array gain
b	loss per bounce; decay parameter
$b(\theta, \phi)$	beam pattern
B	magnetic field; susceptance
BL	bottom loss
$\mathcal{B}$	adiabatic bulk modulus
$\mathcal{B}_T$	isothermal bulk modulus
c	speed of sound
c_g	group speed
c_p	phase speed
C	electrical capacitance; acoustic compliance; heat capacity
$C_{\mathcal{P}}$	heat capacity at constant pressure
$c_{\mathcal{P}}$	specific heat at constant pressure
C_V	heat capacity at constant volume
c_V	specific heat at constant volume
$CNEL$	community noise equivalent level (dBA)
d	detection index
d'	detectability index
D	directivity; dipole strength
DI	directivity index
DNL	detected noise level
DT	detection threshold
$\mathcal{D}$	diffraction factor
e	specific energy
E	total energy
E_k	kinetic energy
E_p	potential energy
EL	echo level
$\mathcal{E}$	time-averaged energy density
$\mathcal{E}_i$	instantaneous energy density
f	instantaneous force; frequency (Hz)
f_r	resonance frequency
f_u, f_l	upper, lower half-power frequencies
F	peak force amplitude; frequency (kHz)
F_e	effective force amplitude
g	spectral density of a transient function; sound-speed gradient; acceleration of gravity; aperture function
G	conductance
$\mathcal{G}$	adiabatic shear modulus
h	specific enthalpy
$H(\theta, \phi)$	directional factor

$H(T_K)$	population function
I	time-averaged acoustic intensity; current, effective current amplitude
I_{ref}	reference acoustic intensity
$I(t)$	instantaneous acoustic intensity
IIC	impact isolation class
IL	intensity level
ISL	intensity spectrum level
$\mathcal{I}$	time-averaged spectral density of intensity
$\mathcal{I}(t)$	instantaneous spectral density of intensity
$\mathcal{J}$	impulse
k	wave number
$\vec{k}$	propagation vector
k_B	Boltzmann's constant
k_c, k_m	coupling coefficients
ℓ	discontinuity distance
L	inductance
L_A	A-weighted sound level (dBA)
L_C	C-weighted sound level (dBC)
L_d	daytime average sound level (dBA)
L_{dn}	day–night averaged sound level (dBA)
L_e	evening average sound level (dBA)
L_{eq}	equivalent continuous sound level (dBA)
L_{ex}	noise exposure level (dBA)
L_{EPN}	effective perceived noise level
L_h	hourly average sound level (dBA)
L_I	intensity level *re* 10^{-12} W/m^2
L_N	loudness level (phon)
L_n	night average sound level (dBA)
L_{TPN}	tone-corrected perceived noise level
L_x	x-percentile-exceeded sound level (dBA, fast)
LNP	noise pollution level (dBA)
m	mass
m_r	radiation mass
M	acoustic inertance; bending moment; molecular weight; acoustic Mach number, flow Mach number
$\mathcal{M}$	microphone sensitivity
$\mathcal{ML}$	microphone sensitivity level
$\mathcal{M}_{ref}$	reference microphone sensitivity
N	loudness (sone)
NCB	balanced noise criterion curves
NEF	noise exposure forecast
NL	noise level
NR	noise reduction
NSL	noise spectrum level
p	acoustic pressure
P	peak acoustic pressure amplitude
P_e	effective acoustic pressure amplitude
P_{ref}	reference effective acoustic pressure amplitude
PR	privacy rating
Pr	Prandtl number
PSL	pressure spectrum level
PTS	permanent threshold shift
$\mathcal{P}$	hydrostatic pressure
$\mathcal{P}_0$	equilibrium hydrostatic pressure
q	charge; source strength density; thermal energy; scaled acoustic pressure ($p/\rho_0 c^2$)
Q	quality factor; source strength (amplitude of volume velocity)

(*continued on back endpapers*)

FUNDAMENTALS OF ACOUSTICS

Fourth Edition

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PREFACE

Credit for the longevity of this work belongs to the original two authors, Lawrence Kinsler and Austin Frey, both of whom have now passed away. When Austin entrusted us with the preparation of the third edition, our goal was to update the text while maintaining the spirit of the first two editions. The continued acceptance of this book in advanced undergraduate and introductory graduate courses suggests that this goal was met. For this fourth edition, we have continued this updating and have added new material.

Considerable effort has been made to provide more homework problems. The total number has been increased from about 300 in the previous editions to over 700 in this edition. The availability of desktop computers now makes it possible for students to investigate many acoustic problems that were previously too tedious and time consuming for classroom use. Included in this category are investigations of the limits of validity of approximate solutions and numerically based studies of the effects of varying the various parameters in a problem. To take advantage of this new tool, we have added a great number of problems (usually marked with a suffix "C") where the student may be expected to use or write computer programs. Any convenient programming language should work, but one with good graphing software will make things easier. Doing these problems should develop a greater appreciation of acoustics and its applications while also enhancing computer skills.

The following additional changes have been made in the fourth edition: (1) As an organizational aid to the student, and to save instructors some time, equations, figures, tables, and homework problems are all now numbered by chapter and section. Although appearing somewhat more cumbersome, we believe the organizational advantages far outweigh the disadvantages. (2) The discussion of transmitter and receiver sensitivity has been moved to Chapter 5 to facilitate early incorporation of microphones in any accompanying laboratory. (3) The chapters on absorption and sources have been interchanged so that the discussion of beam patterns precedes the more sophisticated discussion of absorption effects. (4) Derivations from the diffusion equation of the effects of thermal conductivity on the attenuation of waves in the free field and in pipes have been added to the chapter on absorption. (5) The discussions of normal modes and waveguides have been collected into a single chapter and have been expanded to include normal modes in cylindrical and spherical cavities and propagation in layers. (6) Considerations of transient excitations and orthonormality have been enhanced. (7) Two new chapters have been added to illustrate how the principles of acoustics can be applied to topics that are not normally covered in an undergraduate course. These chapters, on finite-amplitude acoustics and shock waves, are not meant to survey developments in these fields. They are intended to introduce the relevant underlying acoustic principles and to demonstrate how the fundamentals of acoustics can be extended to certain more complicated problems. We have selected these examples from our own areas of teaching and research. (8) The appendixes have been enhanced to provide more information on physical constants, elementary transcendental functions (equations, tables, and figures), elements of thermodynamics, and elasticity and viscosity.

New materials are frequently at a somewhat more advanced level. As in the third edition, we have indicated with asterisks in the Contents those sections in each chapter that can be eliminated in a lower-level introductory course. Such a course can be based on the first five or six chapters with selected topics from the seventh and eighth. Beyond these, the remaining chapters are independent of each other (with only a couple of exceptions that can be dealt with quite easily), so that topics of interest can be chosen at will.

With the advent of the handheld calculator, it was no longer necessary for textbooks to include tables for trigonometric, exponential, and logarithmic functions. While the availability of desktop calculators and current mathematical software makes it unnecessary to include tables of more complicated functions (Bessel functions, etc.), until handheld calculators have these functions programmed into them, tables are still useful. However, students are encouraged to use their desktop calculators to make fine-grained tables for the functions found in the appendixes. In addition, they will find it useful to create tables for such things as the shock parameters in Chapter 17.

From time to time we will be posting updated information on our web site: www.wiley.com/college/kinsler. At this site you will also be able to send us messages. We welcome you to do so.

We would like to express our appreciation to those who have educated us, corrected many of our misconceptions, and aided us: our coauthors Austin R. Frey and Lawrence E. Kinsler; our mentors James Mcgrath, Edwin Ressler, Robert T. Beyer, and A. O. Williams; our colleagues O. B. Wilson, Anthony Atchley, Steve Baker, and Wayne M. Wright; and our many students, including Lt. Thomas Green (who programmed many of the computer problems in Chapters 1–15) and L. Miles.

Finally, we offer out heartfelt thanks for their help, cooperation, advice, and guidance to those at John Wiley & Sons who were instrumental in preparing this edition of the book: physics editor Stuart Johnson, production editor Barbara Russiello, designer Kevin Murphy, editorial program assistants Cathy Donovan and Tom Hempstead, as well as to Christina della Bartolomea who copy edited the manuscript and Gloria Hamilton who proofread the galleys.

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Chapter 9

CAVITIES AND WAVEGUIDES

9.1 INTRODUCTION

In this and the next chapter we concentrate on the confinement of acoustic energy to closed or partly closed regions of space. In completely enclosed spaces, two- and three-dimensional standing waves can be stimulated. The normal modes associated with these standing waves determine the acoustic behavior of rooms, auditoriums, and concert halls. If the space is open in one or two dimensions, it can form a waveguide. Applications of waveguides include surface-wave delay lines, high-frequency electronic systems, folded-horn loudspeakers, and propagation of sound in the oceans and the atmosphere.

9.2 THE RECTANGULAR CAVITY

Consider a rectangular cavity of dimensions L_x, L_y, L_z, as indicated in Fig. 9.2.1. This box could represent a living room or auditorium, a simple model of a concert hall, or any other right-hexahedral space that has few windows or other openings and fairly rigid walls. Such applications will be encountered in Chapter 12. Assume that all surfaces of the cavity are perfectly rigid so that the normal component of the particle velocity vanishes at all boundaries,

$$\begin{aligned}\left(\frac{\partial p}{\partial x}\right)_{x=0} &= \left(\frac{\partial p}{\partial x}\right)_{x=L_x} = 0\\ \left(\frac{\partial p}{\partial y}\right)_{y=0} &= \left(\frac{\partial p}{\partial y}\right)_{y=L_y} = 0\\ \left(\frac{\partial p}{\partial z}\right)_{z=0} &= \left(\frac{\partial p}{\partial z}\right)_{z=L_z} = 0\end{aligned} \tag{9.2.1}$$

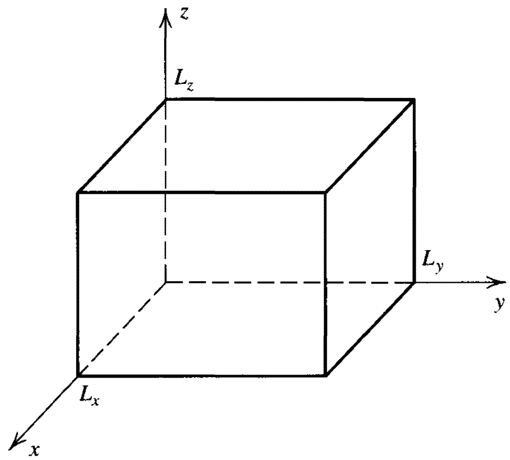


Figure 9.2.1 The rectangular cavity with dimensions L_x, L_y, and L_z.

Since acoustic energy cannot escape from a closed cavity with rigid boundaries, appropriate solutions of the wave equation are standing waves. Substitution of

$$\mathbf{p}(x, y, z, t) = \mathbf{X}(x)\mathbf{Y}(y)\mathbf{Z}(z)e^{j\omega t} \tag{9.2.2}$$

into the wave equation and separation of variables (as performed in Chapter 4) results in the set of equations

$$\begin{aligned} \left(\frac{d^2}{dx^2} + k_x^2\right)\mathbf{X} &= 0 \\ \left(\frac{d^2}{dy^2} + k_y^2\right)\mathbf{Y} &= 0 \\ \left(\frac{d^2}{dz^2} + k_z^2\right)\mathbf{Z} &= 0 \end{aligned} \tag{9.2.3}$$

where the angular frequency must be given by

$$(\omega/c)^2 = k^2 = k_x^2 + k_y^2 + k_z^2 \tag{9.2.4}$$

Application of the boundary conditions (9.2.1) shows that cosines are appropriate solutions, and (9.2.2) becomes

$$\mathbf{p}_{lmn} = \mathbf{A}_{lmn} \cos k_{xl}x \cos k_{ym}y \cos k_{zn}z \, e^{j\omega_{lmn}t} \tag{9.2.5}$$

where the components of k are

$$\begin{aligned} k_{xl} &= l\pi/L_x \qquad & l &= 0, 1, 2, \ldots \\ k_{ym} &= m\pi/L_y \qquad & m &= 0, 1, 2, \ldots \\ k_{zn} &= n\pi/L_z \qquad & n &= 0, 1, 2, \ldots \end{aligned} \tag{9.2.6}$$

Thus, the allowed angular frequencies of vibration are quantized,

$$\omega_{lmn} = c[(l\pi/L_x)^2 + (m\pi/L_y)^2 + (n\pi/L_z)^2]^{1/2} \tag{9.2.7}$$

Each standing wave given by (9.2.5) has its own angular frequency (9.2.7) and can be specified by the ordered integers (l,m,n).

The form (9.2.5) gives three-dimensional standing waves in the cavity with nodal planes parallel to the walls. Between these nodal planes the pressure varies sinusoidally, with the pressure within a given loop in phase, and with adjacent loops 180° out of phase. Comparison of the mathematical developments of this section with those for the rectangular membrane with fixed rim of Section 4.3 reveals similarities and analogs:

1. If only those modes for which $n = 0$ are considered, the z component of the propagation vector vanishes, and the resulting standing wave patterns become two-dimensional, like those for the rectangular membrane.
2. A rigid boundary for a pressure wave in a fluid is analogous to a free boundary for a membrane displacement wave in that both correspond to respective antinodes. The distribution of nodes and antinodes of these respective pressure and displacement waves in planes perpendicular to any axis will be identical for the same dimensions and modal numbers. Similarly, a pressure release boundary for a fluid is analogous to a fixed boundary for a membrane, both requiring nodes in pressure and displacement, respectively.

If a pressure source is located anywhere on a nodal surface of a normal mode of pressure, that mode will not be excited. The closer a source is to an antinode of the mode, the greater the excitation of that mode. Similarly, a pressure-sensitive receiver will have greatest output if it is placed at an antinode of the mode. These effects are used to either emphasize or suppress selected modes or families of modes. For example, if it is desired to excite and detect all the modes of a rectangular room, the source and receiver must be placed in the corners (junctions of three surfaces). (If, in a hard-walled room like a shower, one hums at an eigenfrequency and moves around in the enclosure, strong fluctuations in loudness will be heard, with maxima when the head is close to a corner or any other pressure antinode. In contrast, the hummer will experience difficulty in trying to drive a mode at a pressure node.)

If two or more modes have the same eigenfrequency, they are called *degenerate*. Degenerate modes can be isolated by judicious placement of the source and receiver. A receiver placed on a nodal plane of one of a set of degenerate modes will not respond to that mode. Similarly, a source located at a node of one of the degenerate modes cannot excite that mode.

Just as a standing wave on a string could be considered as two traveling waves moving in opposite directions, the standing waves in the rectangular cavity can be decomposed into traveling plane waves. If the solutions (9.2.5) are represented in complex exponential form and expanded as a sum of products, it is seen that

$$\mathbf{p}_{lmn} = \tfrac{1}{8}\mathbf{A}_{lmn} \sum_{\pm} e^{j(\omega_{lmn} t \pm k_{xl} x \pm k_{ym} y \pm k_{zn} z)} \tag{9.2.8}$$

where the summation is taken over all permutations of plus and minus signs. Each of these eight terms represents a plane wave traveling in the direction of its propagation vector k_{lmn} whose projections on the coordinate axes are $\pm k_{xl}$, $\pm k_{ym}$, and $\pm k_{zn}$. Thus, the standing wave solution can be viewed as a superposition of eight traveling waves (one into each octant) whose directions of propagation are fixed by the boundary conditions.

*9.3 THE CYLINDRICAL CAVITY

Figure 9.3.1 shows a rigid-walled, right circular cavity with radius a and height L. In cylindrical coordinates (Appendix A7), the Helmholtz equation $\nabla^2\mathbf{p} + k^2\mathbf{p} = 0$ with $\mathbf{p} = P\exp(j\omega t)$ becomes

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta^2} + \frac{\partial^2 P}{\partial z^2} + k^2 P = 0 \tag{9.3.1}$$

and the boundary conditions at the rigid walls are

$$\left(\frac{\partial P}{\partial z}\right)_{z=0} = \left(\frac{\partial P}{\partial z}\right)_{z=L} = \left(\frac{\partial P}{\partial r}\right)_{r=a} = 0 \tag{9.3.2}$$

If a solution of the form

$$P(r,\theta,z) = R(r)\Theta(\theta)Z(z) \tag{9.3.3}$$

is assumed, separation of variables results in three equations:

$$\begin{aligned} \frac{d^2 Z}{dz^2} &= -k_{zl}^2 Z \\ \frac{d^2\Theta}{d\theta^2} &= -m^2\Theta \\ r^2\frac{d^2 R}{dr^2} + r\frac{dR}{dr} + (k_{mn}^2 r^2 - m^2)R &= 0 \end{aligned} \tag{9.3.4}$$

where

$$k^2 = k_{mn}^2 + k_{zl}^2 \tag{9.3.5}$$

These equations have solutions

$$\begin{aligned} Z &= \cos k_{zl} z \\ \Theta &= \cos(m\theta + \gamma_{lmn}) \\ R &= J_m(k_{mn} r) \end{aligned} \tag{9.3.6}$$

with $m = 0, 1, 2, \ldots$ (since Θ must be single valued), $k_{zl}L = l\pi$, where $l = 0, 1, 2, \ldots,$ and $k_{mn}a = j'_{mn}$, where j'_{mn} is the nth extremum of the mth Bessel function of the first kind. The normal modes are designated by the three integers (l,m,n), which denote the number of null surfaces in the z, θ, and r directions, respectively. The pressure of the (l,m,n) mode is

$$\mathbf{p}_{lmn} = \mathbf{A}_{lmn} J_m(k_{mn} r)\cos(m\theta + \gamma_{lmn})\cos k_{zl} z\, e^{j\omega_{lmn} t} \tag{9.3.7}$$

where the angular frequencies are determined from

$$(\omega_{lmn}/c)^2 = k_{lmn}^2 = k_{mn}^2 + k_{zl}^2 \tag{9.3.8}$$

Comparison with the discussion of the circular membrane with fixed rim of Section 4.4 shows that the introduction of the third spatial dimension z has, as in the rectangular case, simply introduced a new component of the propagation vector.

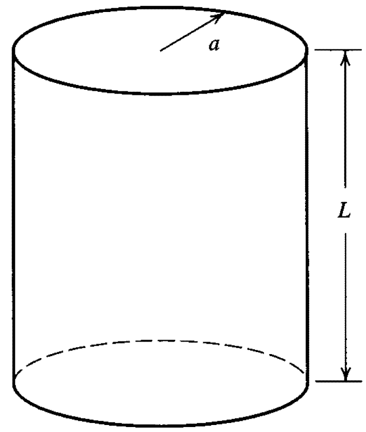


Figure 9.3.1 The right circular cylindrical cavity with height L and radius a.

Just as for the circular membrane, if there were an inner boundary, i.e., a perfectly reflecting cylinder with radius $r = b < a$ and no acoustic field for $r < b$, then the Bessel functions $Y_m(k_{mn}r)$ would also be acceptable solutions and the boundary conditions at $r = a$ and $r = b$ would have to be satisfied by some combination $A_{lmn}J_m(k_{mn}r) + B_{lmn}Y_m(k_{mn}r)$.

As with the rectangular cavity, the standing waves in the cylindrical cavity can be expressed as traveling waves. Expand $\cos(k_z z)$ in terms of exponentials, use $2J_n = H_n^{(1)} + H_n^{(2)}$, and then, purely for ease of interpretation, expand the Hankel functions in their asymptotic approximations. The resulting eight terms in each $\mathbf{p}_{lmn}$ have the forms

$$(2/\pi k_{mn}r)^{1/2}e^{j(\omega_{lmn}t \pm m\theta \pm k_{mn}r \pm k_{zl}z)} \tag{9.3.9}$$

with all permutations of the + and − signs. We have suppressed amplitude factors and the γ's. (See Problem 9.3.1.) These describe eight conical traveling waves whose phases are shaded according to the polar angle θ. In general, the surfaces of constant phase are conical spirals. The intersection of a surface of constant phase with the z plane forms a spiral that propagates with a radial speed ω_{lmn}/k_{mn} outward (or inward), appearing to emanate from (or disappear into) the origin. The propagation vectors have angles of elevation and depression given by $\pm\tan^{-1}(k_{zl}/k_{mn})$.

*9.4 THE SPHERICAL CAVITY

The Helmholtz equation in spherical coordinates (Appendix A7) is

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial P}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 P}{\partial\phi^2} + k^2r^2P = 0 \tag{9.4.1}$$

and the boundary condition for a rigid-walled sphere of radius a is

$$\left(\frac{\partial P}{\partial r}\right)_{r=a} = 0 \tag{9.4.2}$$

For a solution of the form

$$P = R(r)\Theta(\theta)\Phi(\phi) \tag{9.4.3}$$

separation of variables gives

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\eta^2 - \frac{m^2}{sin^2\theta}\right)\Theta = 0 \tag{9.4.4}$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + (k^2r^2 - \eta^2)R = 0$$

where m and η are the separation constants.

Solutions for the Φ dependence are

$$\Phi_m = A\cos(m\phi + \gamma_{lmn}) \tag{9.4.5}$$

Since Φ must be single valued, m must be integral. As discussed in Section 4.4, each phase angle γ_{lmn} must be determined by the initial conditions. If there is no condition to determine γ_{lmn}, then except for $m = 0$ each Φ_m must be considered a pair of degenerate modes. (They can be made orthogonal if, for example, one is chosen to be $\cos m\phi$ and the other $\sin m\phi$.)

The equation for Θ is related to the *Legendre equation*. Solutions to this equation that are continuous, single valued, and finite must have $\eta^2 = l(l+1)$, where $l = 0, 1, 2, \ldots$, and must also have $m \leq l$. These solutions are the *associated Legendre functions of the first kind of order l and degree m*, denoted by $P_l^m(\cos\theta)$. (See Appendix A4 for more details and properties of these functions.)

The equation for the radial dependence can be rewritten as

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \left[k^2r^2 - l(l+1)\right]R = 0 \tag{9.4.6}$$

The solutions to this equation that are finite at the origin of r are the *spherical Bessel functions of order l:*

$$R = j_l(k_{ln}r) \tag{9.4.7}$$

[If the cavity is the space between two perfectly reflecting concentric boundaries of radii a and b, then the spherical Bessel functions of the second kind, $y_l(k_{ln}r)$, are also admissible solutions.] For a rigid-walled cavity, $k_{ln}a = \zeta'_{ln}$ where ζ'_{ln} are the extrema of j_{ln}.

The pressure amplitude in the cavity is then

$$\mathbf{p}_{lmn} = \mathbf{A}_{lmn}j_l(k_{ln}r)P_l^m(\cos\theta)\cos(m\phi + \gamma_{lmn})e^{j\omega_{ln}t} \tag{9.4.8}$$

and the angular frequencies are given by $\omega_{ln} = ck_{ln}$. The lack of any dependence of the propagation constant on m means that all modes having the same values of l and n but different values of m are degenerate.

A study of the spherical Bessel functions shows that the lowest eigenfrequency, found from $k_{11}a = 2.08$, is shared by the single (1, 0, 1) mode and the pair (1, 1, 1). Together they constitute a threefold degeneracy. The spatial pressure distributions of the three are

$$P_{101} = A_{101}\left(\frac{\sin k_{11}r}{(k_{11}r)^2} - \frac{\cos k_{11}r}{k_{11}r}\right)\cos\theta$$

$$P_{111}^{(1)} = A_{111}^{(1)}\left(\frac{\sin k_{11}r}{(k_{11}r)^2} - \frac{\cos k_{11}r}{k_{11}r}\right)\sin\theta\cos\phi \tag{9.4.9}$$

$$P_{111}^{(2)} = A_{111}^{(2)}\left(\frac{\sin k_{11}r}{(k_{11}r)^2} - \frac{\cos k_{11}r}{k_{11}r}\right)\sin\theta\sin\phi$$

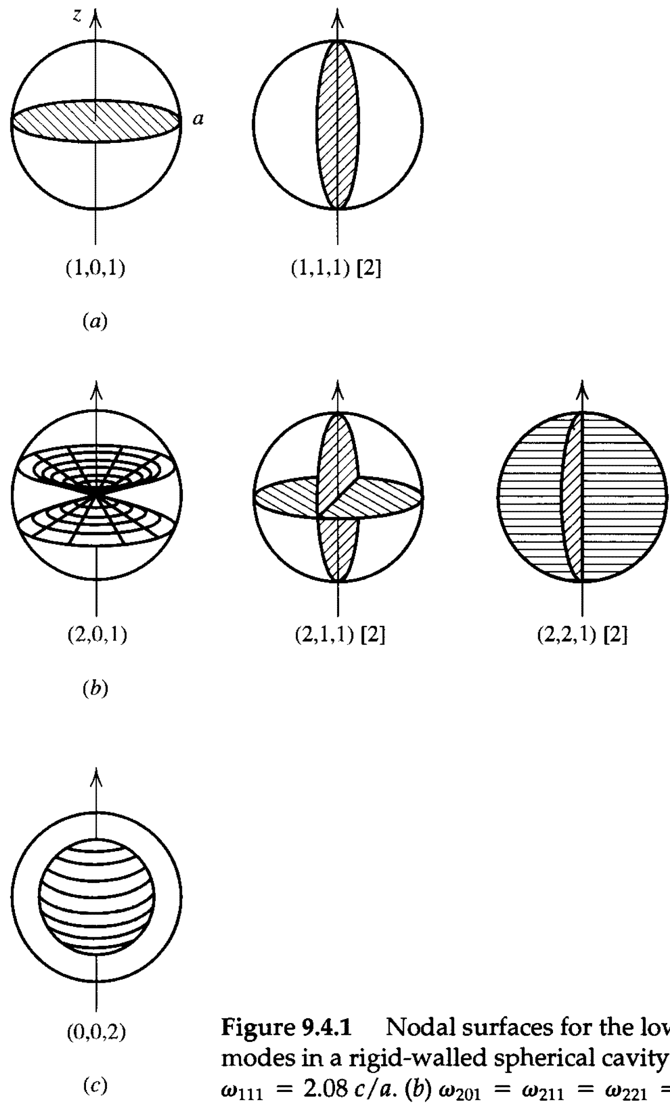


Figure 9.4.1 Nodal surfaces for the lowest three sets of normal modes in a rigid-walled spherical cavity of radius $r = a$. (*a*) $\omega_{101} = \omega_{111} = 2.08\, c/a$. (*b*) $\omega_{201} = \omega_{211} = \omega_{221} = 3.34\, c/a$. (*c*) $\omega_{002} = 4.49\, c/a$.

The next set of modes, forming a fivefold degeneracy, have $k_{21}a = 3.34$. The radial dependence is $j_2(k_{21}r)$ and the angular dependences are one mode with $(l, m, n) = (2, 0, 1)$, two modes with (2, 1, 1), and two with (2, 2, 1). The third set, for which $k_{02}a = 4.49$, has a single member with radial dependence $j_0(k_{02}r) = (\sin k_{02}r)/(k_{02}r)$ and no angular dependence since $P_0(\cos\theta) = 1$. Nodal surfaces for the lowest three sets of normal modes are shown in Fig. 9.4.1.

9.5 THE WAVEGUIDE OF CONSTANT CROSS SECTION

Waveguides having different, but uniform, cross sections and the same boundary conditions will display similar behaviors. We will develop the properties for a waveguide with a rectangular cross section, as shown in Fig. 9.5.1, and then generalize the results to other cross-sectional geometries. Assume the side walls to be rigid and the boundary at $z = 0$ to be a source of acoustic energy. The absence of another boundary on the z axis allows energy to propagate down the waveguide.

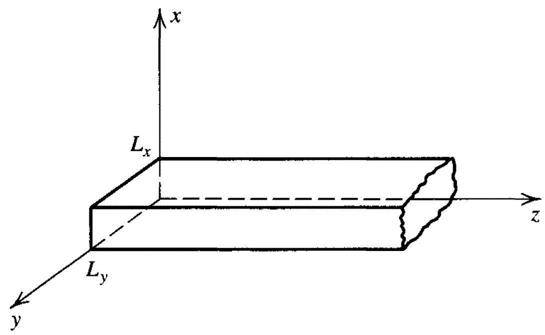


Figure 9.5.1 The rectangular waveguide with dimensions L_x and L_y.

This suggests a wave consisting of standing waves in the transverse directions (x and y) and a traveling wave in the z direction.

Since the cross section is rectangular and the boundaries are rigid, acceptable solutions are

$$\begin{aligned} \mathbf{p}_{lm} &= \mathbf{A}_{lm} \cos k_{xl}x \cos k_{ym}y \, e^{j(\omega t - k_z z)} \\ k_z &= [(\omega/c)^2 - (k_{xl}^2 + k_{ym}^2)]^{1/2} \\ k_{xl} &= l\pi/L_x \qquad l = 0,1,2,\ldots \\ k_{ym} &= m\pi/L_y \qquad m = 0,1,2,\ldots \end{aligned} \tag{9.5.1}$$

Since ω can have any value, k_z is not fixed.

It is convenient to define k_{lm} as the *transverse component* of the propagation vector. For a rectangular cross section,

$$k_{lm} = (k_{xl}^2 + k_{ym}^2)^{1/2} \tag{9.5.2}$$

and the required value of k_z can be written more succinctly as

$$k_z = [(\omega/c)^2 - k_{lm}^2]^{1/2} \tag{9.5.3}$$

When $\omega/c > k_{lm}$, then k_z is real. The wave advances in the $+z$ direction and is called a *propagating mode*. The limiting value of ω/c for which k_{lm} remains real is given by $\omega/c = k_{lm}$, and this defines the *cutoff angular frequency*

$$\boxed{\omega_{lm} = ck_{lm}} \tag{9.5.4}$$

for the (l, m) mode. If the input frequency is lowered below cutoff, the argument of the square root in (9.5.3) becomes negative and k_z must be pure imaginary

$$k_z = \pm j[k_{lm}^2 - (\omega/c)^2]^{1/2} \tag{9.5.5}$$

The minus sign must be taken on physical grounds so that $\mathbf{p} \to 0$ as $z \to \infty$, and (9.5.1) has the form

$$\mathbf{p}_{lm} = \mathbf{A}_{lm} \cos k_{xl}x \cos k_{ym}y \exp\{-[k_{lm}^2 - (\omega/c)^2]^{1/2} z\} \, e^{j\omega t} \tag{9.5.6}$$

This is an *evanescent* standing wave that attenuates exponentially with z. No energy propagates down the waveguide. If the waveguide is excited with a frequency just below the cutoff frequency of some particular mode, then this and higher modes are evanescent and not important at appreciable distances from the source. All modes having cutoff frequencies below the driving frequency may propagate energy and may be detected at large distances.

In a rigid-walled waveguide, only plane waves propagate if the frequency of the sound is sufficiently low. For a waveguide of rectangular cross section of greater dimension L, this frequency is easily shown to be $f = c/2L$.

The *phase speed* of a mode is

$$c_p = \omega/k_z = c/[1 - (k_{lm}/k)^2]^{1/2} = c/[1 - (\omega_{lm}/\omega)^2]^{1/2} \tag{9.5.7}$$

and is greater than c. An understanding of this is obtained by writing the cosines in (9.5.1) in complex exponential form. The solution then consists of the sum

$$\mathbf{p}_{lm} = \tfrac{1}{4}\mathbf{A}_{lm} \sum_{\pm} e^{j(\omega t \pm k_{xl}x \pm k_{ym}y - k_z z)} \tag{9.5.8}$$

(Note that only the minus sign appears before k_z.) The propagation vector $\vec{k}$ for each of the four traveling waves makes an angle θ with the z axis given by

$$\cos\theta = k_z/k = [1 - (\omega_{lm}/\omega)^2]^{1/2} \tag{9.5.9}$$

so that the phase speed (9.5.7) is

$$c_p = c/\cos\theta \tag{9.5.10}$$

This is simply the speed with which a surface of constant phase appears to propagate along the z axis. (See Problems. 9.2.3 and 9.3.2.)

Figure 9.5.2 gives the surfaces of constant phase for the two component waves that represent the (0,1) mode of a rigid-walled rectangular waveguide. The waves exactly cancel each other for $y = L_y/2$, so that there is a nodal plane midway between the walls. At the upper and lower walls the waves are always in phase so that the pressure amplitude is maximized at these (rigid) boundaries. The *apparent wavelength* λ_z measured in the z direction is $\lambda_z = \lambda/(\cos\theta)$.

The lowest mode for a rigid-walled waveguide is the (0, 0) mode. For this case, $k_z = k$ and the four component waves collapse into a single plane wave that travels down the axis of the waveguide with phase speed c. For all other modes, the propagation vectors of the component waves can be at angles to the waveguide axis, one pointing into each of the four forward octants. From (9.5.9) and (9.5.10), at frequencies far above the cutoff of the (l, m) mode, we have $\omega \gg \omega_{lm}$ so that θ tends to zero and the waves are traveling almost straight down the waveguide with $c_p \approx c$. As the input frequency is decreased toward cutoff, the angle θ increases so that the component waves travel in increasingly transverse directions. If we imagine that each component wave carries energy down the waveguide by a process of continual reflection from the walls (much like a bullet ricocheting down a hard-walled corridor), and remember that the energy of a wave is propagated with speed c in the direction of $\hat{k} = \vec{k}/k$, then we see that the speed with which energy moves in the z direction is given by the *group speed* $c_g = c\hat{k}\cdot\hat{z}$, the projection of the component wave velocity along the waveguide axis,

$$c_g = c\cos\theta = c[1 - (\omega_{lm}/\omega)^2]^{1/2} \tag{9.5.11}$$

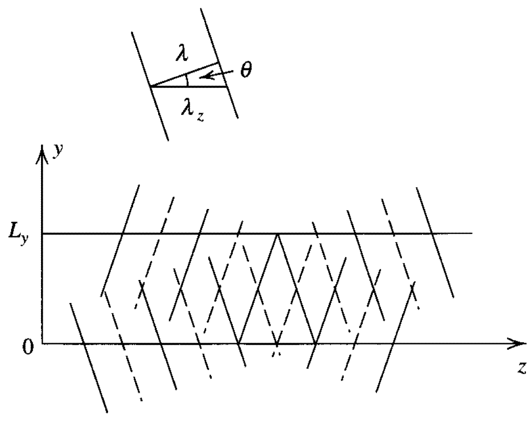


Figure 9.5.2 Component plane waves for the (0,1) mode in a rigid-walled, rectangular cavity. These waves travel with speed c in directions that make angles $\pm\theta$ with the z axis of the waveguide.

For a given angular frequency ω, each modal wave with $\omega_{lm} < \omega$ has its own individual values of c_p and c_g. The behaviors of the group and phase speeds as functions of frequency for three modes in a rigid-walled waveguide are shown in Fig. 9.5.3.

It is straightforward to generalize the above discussion and derive the behavior of a rigid-walled waveguide with a circular cross section of radius $r = a$. Separation of variables and solution results in

$$\begin{aligned} \mathbf{p}_{ml} &= \mathbf{A}_{ml} J_m(k_{ml} r) \cos m\theta \, e^{j(\omega t - k_z z)} \\ k_z &= [(\omega/c)^2 - k_{ml}^2]^{1/2} \end{aligned} \tag{9.5.12}$$

where r, θ, and z are the cylindrical coordinates, J_m is the mth order Bessel function, and the allowed k_{ml} are determined by the boundary condition for the rigid wall,

$$k_{ml} = j'_{ml}/a \tag{9.5.13}$$

where j'_{ml} are the extrema of $J_m(z)$. These values are tabulated in Appendix A5. Once the values of k_{ml} are found, all the salient results developed for rectangular waveguides can be applied simply by substituting the values of k_{ml} for a circular waveguide. For example, the (0, 0) mode is a plane wave that propagates with $c_p = c$ for all $\omega > 0$. The nonplanar mode with the lowest cutoff frequency is the (1, 1) mode (the first "sloshing" mode) with cutoff frequency $\omega_{11} = 1.84\ c/a$ or $f_{11} = 100/a$ for air. It is of great practical importance that for frequencies below f_{11} only plane waves can propagate in a rigid-walled, circular waveguide.

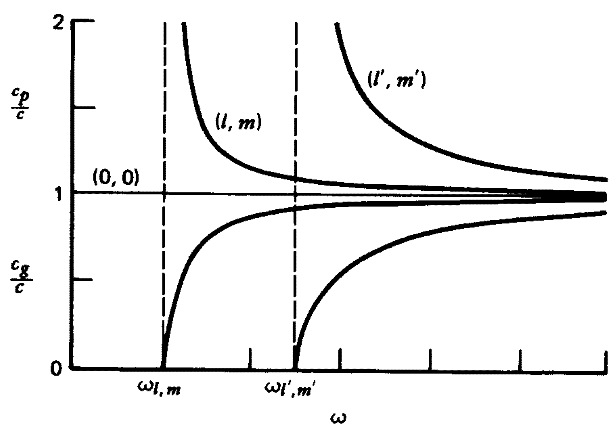


Figure 9.5.3 Group and phase speeds for the lowest three normal modes in a rigid-walled waveguide.

*9.6 SOURCES AND TRANSIENTS IN CAVITIES AND WAVEGUIDES

Up to this point we have not dealt with the acoustic source. If we know the pressure or velocity distribution of the source, then these can be related to the behavior of the pressure or the velocity of the total acoustic field as was done for membranes in Section 4.10. In what follows we will sketch the development of a few special cases to demonstrate the basics.

Assume that a rigid-walled rectangular enclosure is excited by a point impulsive pressure source (like a cap or starter pistol shot). This is the three-dimensional extension and analog of the impulsive point excitation of a rectangular membrane. The source can be described by an initial condition at $t = 0$ of

$$\mathbf{p}(x, y, z, 0) = \delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0) \tag{9.6.1}$$

and this must be matched to (9.2.5). Since the space must be quiescent before the pressure impulse, the particle velocity field throughout the enclosure must be zero at $t = 0$. This requires $\mathbf{A}_{lmn} = A_{lmn}$ so that the real pressure standing waves are cosinusoidal in time and we must have

$$\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0) = \sum_{l,m,n} A_{lmn} \cos k_{xl}x \cos k_{ym}y \cos k_{zn}z \tag{9.6.2}$$

Inversion and use of orthogonality to solve for the coefficients provides the resultant pressure field,

$$p(x, y, z, t) = \frac{8}{L_x L_y L_z} \sum_{l,m,n} \cos k_{xl}x_0 \cos k_{ym}y_0 \cos k_{zn}z_0 \cos k_{xl}x \cos k_{ym}y \cos k_{zn}z \cos \omega_{lmn}t \tag{9.6.3}$$

This result is simply an extension of what has been done before. Application to cylindrical enclosures proceeds similarly with no surprises. If there are losses, each standing wave will decay as $\exp(-\beta_{lmn}t)$.

Excitation of the enclosure by a monofrequency source presents a few more difficulties: losses must be included, and these require introduction of the frequency dependence of the amplitude of each of the driven lossy standing waves. Excitation of the cavity with losses by a monofrequency source is deferred until Section 12.9.

In the case of excitation of a waveguide of uniform cross section on the plane $z = 0$, assume that the source distribution is

$$\mathbf{p}(x, y, 0, t) = \mathbf{P}(x, y)e^{j\omega t} \tag{9.6.4}$$

Again, $\mathbf{p}$ can be written as a superposition of the normal modes of the waveguide as in Section 4.10. For a waveguide with rectangular cross section and rigid walls, we have

$$\mathbf{p}(x, y, z, t) = \sum_{l,m} \mathbf{A}_{lm} \cos k_{xl}x \cos k_{ym}y \; e^{j(\omega t - k_z z)} \tag{9.6.5}$$

Evaluation at $z = 0$ and use of (9.6.4) gives

$$\mathbf{P}(x, y) = \sum_{l,m} \mathbf{A}_{lm} \cos k_{xl}x \cos k_{ym}y \tag{9.6.6}$$

from which we can determine the required values of $\mathbf{A}_{lm}$.

The existence of three speeds c_p, c_g, and c in the description of each traveling wave in a waveguide serves to elucidate the propagation behavior of transient signals. First, we will develop some general results based on the *method of stationary phase* and then examine more exactly the behavior of a particular transient. Consider a well-defined pulse generated at the source and propagating down the waveguide. Recalling the elements of Fourier superposition stated in Section 1.15, we can write the dependence of the pulse on distance and time in the form of a weighted superposition of monofrequency components. The spectral density $g(\omega)$ can be found from the behavior of the source at $z = 0$. If, instead of $\exp(j\omega t)$, the source generates a known signature $f(t)$, then

$$
\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} g(\omega) e^{j\omega t}\, d\omega \\
g(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt
\end{aligned}
\tag{9.6.7}
$$

The extension of (9.6.5) to a transient excitation becomes

$$
\mathbf{p}(x,y,z,t) = \sum_{l,m}\left[\mathbf{A}_{lm}\cos k_{xl}x\cos k_{ym}y\int_{-\infty}^{\infty} g(\omega)e^{j(\omega t - k_z z)}\, d\omega\right] \tag{9.6.8}
$$

(Recall that k_z is a function of ω and is different for each nondegenerate wave.) Because of the distance-dependent phase in the integrand, it is clear that the pulse will evolve in shape as it travels along the z axis. If the pulse is initially well defined, then $g(\omega)$ is a smoothly varying function of frequency and strong over a broad bandwidth. In this case, the portion of the integrand that contributes the most to the pulse is that for which the phase is nearly *stationary* (constant) as a function of frequency. For other frequencies the phase of the integrand is rapidly varying so that adjacent cycles of the integrand tend to cancel. Thus, the major portion of the pulse will begin near the time for which the phase is stationary, and for each mode this time is found from

$$
\begin{aligned}
\frac{d}{d\omega}(\omega t - k_z z) &= 0 \\
t &= \frac{dk_z}{d\omega} z
\end{aligned}
\tag{9.6.9}
$$

The speed with which this major portion of the pulse travels down the waveguide is the group speed,

$$
\boxed{c_g = \frac{d\omega}{dk_z}} \tag{9.6.10}
$$

[It is straightforward to show that this is identical with (9.5.11) for the waveguide of rectangular cross section, but (9.6.10) is more general and can be applied to any lossless dispersive medium.] The phase speed c_p of each frequency component of the signal is, of course, still given by

$$
\boxed{c_p = \omega / k_z} \tag{9.6.11}
$$

Now, let us analyze a simple transient signal exciting a single mode of the waveguide. Write the pressure p_{lm} and the z component u_{zlm} of the particle velocity $\vec{u}_{lm}$ associated with the (l, m) normal mode in the forms

$$
\begin{aligned}
p_{lm}(x,y,z,t) &= P_{lm}(x,y)f(z,t) \\
u_{zlm}(x,y,z,t) &= P_{lm}(x,y)v(z,t)
\end{aligned}
\tag{9.6.12}
$$

If at the source location, $z = 0$, the function $v(0, t)$ is taken to be $1(t)/\rho_0 c$, where $1(t)$ is the unit step function, then with the help of Table 1.15.1 and standard acoustic relations,

$$\begin{aligned} v(z,t) &= \frac{1}{\rho_0 c}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\omega}e^{j(\omega t - k_z z)}\,d\omega \\ f(z,t) &= \frac{1}{2\pi c}\int_{-\infty}^{\infty}\frac{1}{jk_z}e^{j(\omega t - k_z z)}\,d\omega \end{aligned} \tag{9.6.13}$$

Evaluation of $f(z, t)$ from a more general table of Fourier transforms or use of Problems 9.6.5–9.6.7 gives

$$f(z,t) = J_0\left(\omega_{lm}\sqrt{t^2 - T^2}\right)\cdot 1(t - T) \qquad T = z/c \tag{9.6.14}$$

where $T = z/c$ is the time of flight of the leading edge of the signal, which travels with the free field speed of sound c. [The basic mechanism for sound propagation, collisions between molecules, is not changed by the presence of boundaries. Consequently, the first information that the source has been turned on must arrive at z by the shortest path (directly down the z axis) with speed c.]

Recognizing that the Bessel function behaves very much like a cosinusoidal function of the same argument (and with a slightly shifted phase), write the argument of J_0 in the form appropriate for a traveling wave with instantaneous angular frequency ω and propagation constant k_z,

$$\omega_{lm}[t^2 - (z/c)^2]^{1/2} = \omega t - k_z z \tag{9.6.15}$$

Differentiating with respect to t gives ω as a function of z and t, and differentiation with respect to z does the same for k_z,

$$\begin{aligned} \omega &= \omega_{lm} t \big/ [t^2 - (z/c)^2]^{1/2} \\ k_z &= \omega_{lm} z \big/ \left\{c^2[t^2 - (z/c)^2]^{1/2}\right\} \end{aligned} \tag{9.6.16}$$

Examination of the first of (9.6.16) shows that when t is just slightly larger than T, corresponding to the earliest portions of the signal arriving at location z, ω is very much larger than ω_{lm}. For very long elapsed times, $t \gg T$, the cutoff angular frequency appears at z. Higher frequencies arrive much faster than do lower frequencies, and none less than the cutoff value propagate down the waveguide. If this first equation is solved for z/t in terms of ω_{lm}/ω, we get

$$z/t = c[1 - (\omega_{lm}/\omega)^2]^{1/2} \tag{9.6.17}$$

This gives the time t at which the portion of the signal with angular frequency ω will appear at z. Thus z/t is the group speed c_g for energy associated with angular frequency ω,

$$c_g/c = [1 - (\omega_{lm}/\omega)^2]^{1/2} \tag{9.6.18}$$

Taking the ratio of the two equations in (9.6.16) and then eliminating z/t with (9.6.17) gives the phase speed c_p associated with the angular frequency ω,

$$c_p/c = 1 \big/ [1 - (\omega_{lm}/\omega)^2]^{1/2} \tag{9.6.19}$$

These are identical with the earlier results.

*9.7 THE LAYER AS A WAVEGUIDE

Another important case of waveguide propagation is encountered when a source radiates into a horizontally stratified fluid contained between two horizontal planes. This and the following sections will provide a simplified introduction to the normal-mode approach to this subject. For further information and more mathematically sophisticated methods of analysis, start with the references.[1]

In cylindrical coordinates, assume a point source with time dependence $\exp(j\omega t)$ and unit pressure amplitude at a distance of 1 m is located at a depth $z = z_0$ on the axis ($r = 0$) within a layer of fluid that is bounded at two depths by perfectly reflecting planes (Fig. 9.7.1) at $z = 0$ and $z = H$. The speed of sound within the layer of fluid can be a function of depth z, but not of range r. If the pressure field is written as $\mathbf{p}(r, z, t) = \mathbf{P}(r, z)\exp(j\omega t)$, then the appropriate Helmholtz equation is found from (5.16.5) to be

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2\right]\mathbf{P}(r,z) = -\frac{2}{r}\delta(r)\,\delta(z - z_0) \tag{9.7.1}$$

where $\delta(\vec{r} - \vec{r}_0)$ has been expressed in cylindrical coordinates with the help of Problem 5.16.3. Since this is a case of waveguide propagation, we can assume a solution of the form

$$\mathbf{p}(r, z, t) = e^{j\omega t}\sum_n \mathbf{R}_n(r) Z_n(z) \tag{9.7.2}$$

where Z_n satisfies the one-dimensional Helmholtz equation solution

$$\frac{d^2 Z_n}{dz^2} + \left[\left(\frac{\omega}{c}\right)^2 - \kappa_n^2\right] Z_n = 0 \tag{9.7.3}$$

and κ_n is the separation constant. With appropriate normalization, the Z_n form an *orthonormal* set of eigenfunctions,

$$\int_0^H Z_n(z) Z_m(z)\,dz = \delta_{nm} \tag{9.7.4}$$

Substitution of (9.7.2) and (9.7.3) into (9.7.1) yields

$$\sum_n \left[Z_n \frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathbf{R}_n}{dr}\right) + \kappa_n^2 Z_n \mathbf{R}_n\right] = -\frac{2}{r}\,\delta(r)\,\delta(z - z_0) \tag{9.7.5}$$

Multiplication by Z_m, integration over the depth, and use of orthonormality gives an inhomogeneous Helmholtz equation for $\mathbf{R}_n$,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathbf{R}_n}{dr}\right) + \kappa_n^2 \mathbf{R}_n = -\frac{2}{r}\delta(r) Z_n(z_0) \tag{9.7.6}$$

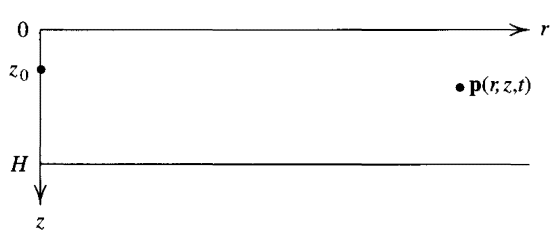


Figure 9.7.1 The fluid layer with a source at depth z_0 between two perfectly reflecting parallel planes.

[1]Officer, *Sound Transmission*, McGraw-Hill (1958). Stephen (ed.), *Underwater Acoustics*, Wiley (1970). Frisk, *Ocean and Seabed Acoustics*, Prentice Hall (1994).

The solution of this equation corresponding to outgoing waves and valid for all r (including the origin) is

$$\mathbf{R_n}(r) = -j\pi Z_n(z_0)H_0^{(2)}(\kappa_n r) \tag{9.7.7}$$

so that the complex pressure field is given by

$$\mathbf{p}(r,z,t) = -j\pi e^{j\omega t}\sum_n Z_n(z_0)Z_n(z)H_0^{(2)}(\kappa_n r) \tag{9.7.8}$$

The allowed values of the κ_n and the form of the orthonormal functions $Z_n(z)$ are found by solving (9.7.3) with the desired speed of sound profile and appropriate boundary conditions.

In many circumstances the separation constants are not all discrete but may also be *continuous* over some interval of κ. The solutions to (9.7.3) for these continuous values of κ form a set of continuous eigenfunctions. Fortunately, these continuous eigenfunctions are associated with untrapped energy or evanescent modes and generate waves significant only close to the source. They can therefore be neglected for our purposes.

At sufficiently large distances the Hankel functions can be replaced with their asymptotic forms and (9.7.8) becomes

$$\mathbf{p}(r,z,t) = -j\sum_n (2\pi/\kappa_n r)^{1/2} Z_n(z_0)Z_n(z)e^{j(\omega t-\kappa_n r+\pi/4)} \tag{9.7.9}$$

Thus, each term of (9.7.8) is a propagating cylindrical wave with phase speed $c_p = \omega/\kappa_n$. The values of the discrete κ_n are fixed, but the magnitude of the propagation vector $k = \omega/c$ can be a function of depth. The angle θ of elevation or depression of the local direction of propagation of the traveling wave is found from $\cos\theta = \kappa_n/k(z)$. Thus, each traveling wave corresponds to a collection of rays traveling in the fluid whose local directions of propagation at each depth z are given by the angles $\pm\theta(z)$.

A convenient analogy can be used to provide further insight for readers having some acquaintance with quantum mechanics. Write the *minimum value* of the speed of sound as c_{min}. Then with the definitions

$$\begin{aligned} E_n &= (\omega/c_{min})^2 - \kappa_n^2 \\ U(z) &= (\omega/c_{min})^2 - (\omega/c)^2 \end{aligned} \tag{9.7.10}$$

the Helmholtz equation (9.7.3) takes on the form

$$\frac{d^2 Z_n}{dz^2} + [E_n - U(z)]Z_n = 0 \tag{9.7.11}$$

This is the one-dimensional time-independent Schroedinger equation with $\hbar^2/2m = 1$. The definition of the minimum speed c_{min} ensures in this analog that $U(z)$ is the potential energy well (with zero minimum value) and E_n is the energy level of the wave function $Z_n(z)$. Now the argument about continuous and discrete values of κ_n can be couched in quantum mechanical terms. If the potential energy $U(z)$ has a finite maximum value, then quantum states having energies E_n large enough that the wave function extends to infinity in either or both directions along the z axis form a continuous set of eigenfunctions so that E_n and therefore κ_n take on continuous values. Thus, unbound quantum states correspond to the untrapped and evanescent modes. When the energy levels lie within the potential well, each wave function has two turning points, E_n and κ_n have discrete values, and these states correspond to the modes trapped in a channel. For a given speed of sound profile, $U(z)$ depends on ω^2. The well becomes more deeply notched with higher walls as frequency increases above cutoff. This means that for a given normal mode the vertical "spread" of the function over depth will tend to be greatest for frequencies close to cutoff and diminish as frequency increases.

For all but a handful of profiles (9.7.3) must be solved by numerical computation. Among those that can be solved analytically, there are a couple of simple cases that provide some physical insight.

*9.8 AN ISOSPEED CHANNEL

Assume that a layer of fluid has constant speed of sound c_0 throughout and is contained by a pressure release surface at $z = 0$ and a rigid bottom at $z = H$. The boundary conditions are $Z(0) = 0$ and $\partial Z/\partial z = 0$ at $z = H$. Solution of (9.7.3) is straightforward,

$$Z_n(z) = \sqrt{2/H}\sin k_{zn}z \qquad k_{zn} = (n - \tfrac{1}{2})\pi/H \tag{9.8.1}$$

and the values of the separation constants κ_n are determined by

$$\kappa_n = [(\omega/c_0)^2 - k_{zn}^2]^{1/2} \tag{9.8.2}$$

For values of k_{zn} exceeding ω/c_0 the associated κ_n values must be imaginary. This yields waves that do not propagate, but decay exponentially with range. Thus, all waves with indices n exceeding the integer N given by

$$N \le (H/\pi)(\omega/c_0) + \tfrac{1}{2} \tag{9.8.3}$$

are evanescent and important only near $r = 0$. At larger distances, the solution is well approximated by

$$\mathbf{p}(r,z,t) \approx -j\frac{2}{H}\sum_{n=1}^{N}\left(\frac{2\pi}{\kappa_n r}\right)^{1/2}\sin k_{zn}z_0 \sin k_{zn}z\, e^{j(\omega t - \kappa_n r + \pi/4)} \tag{9.8.4}$$

The phase speed c_p associated with each mode is given by (9.5.7) with ω_n replacing ω_{lm}:

$$c_p/c = 1/[1 - (\omega_n/\omega)^2]^{1/2} \tag{9.8.5}$$

*9.9 A TWO-FLUID CHANNEL

Let a fluid layer of constant density ρ_1 and sound speed c_1 overlie a fluid bottom of constant density ρ_2 and sound speed $c_2 > c_1$. Let the surface of fluid 1 be a pressure release boundary at $z = 0$ and let the interface between the two fluids be at a depth $z = H$. Figure 9.9.1 shows the geometry. Because the fluid bottom has a greater speed of sound, reflection in fluid 1 from the interface at $z = H$ will be total for grazing angles of incidence less than the grazing critical angle given by $\cos\theta_c = c_1/c_2$.

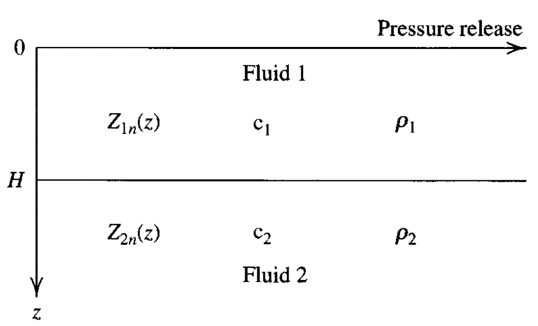


Figure 9.9.1 A channel consisting of a fluid layer of depth H with sound speed c_1 and density ρ_1 overlaying a fluid bottom of infinite depth with sound speed c_2 and density ρ_2.

While c is a function of depth, it is a constant within each layer but changes discontinuously across the interface. We therefore separate the Helmholtz equation

$$\left\{\frac{d^2}{dz^2} + \left[\left(\frac{\omega}{c}\right)^2 - \kappa_n^2\right]\right\} Z_n(z) = 0 \tag{9.9.1}$$

into two equations, one for each region,

$$\begin{aligned} &\left\{\frac{d^2}{dz^2} + \left[\left(\frac{\omega}{c_1}\right)^2 - \kappa_n^2\right]\right\} Z_{1n}(z) = 0 \qquad 0 \le r \le H \\ &\left\{\frac{d^2}{dz^2} + \left[\left(\frac{\omega}{c_2}\right)^2 - \kappa_n^2\right]\right\} Z_{2n}(z) = 0 \qquad H \le r \le \infty \end{aligned} \tag{9.9.2}$$

The boundary conditions are (1) $p_1 = 0$ at $z = 0$, (2) $p_1 = p_2$ and $u_{z1} = u_{z2}$ at $z = H$, and (3) $p_2 \to 0$ as $z \to \infty$. These give us

$$\begin{aligned} Z_{1n}(0) &= 0 \\ Z_{1n}(H) &= Z_{2n}(H) \\ \frac{1}{\rho_1}\left(\frac{dZ_{1n}}{dz}\right)_H &= \frac{1}{\rho_2}\left(\frac{dZ_{2n}}{dz}\right)_H \\ \lim_{z\to\infty} Z_{2n}(z) &= 0 \end{aligned} \tag{9.9.3}$$

Solutions that satisfy the boundary conditions at the surface, at the interface, and at infinite depth are

$$\begin{aligned} Z_{1n}(z) &= \sin k_{zn} z \qquad & 0 \le z \le H \\ Z_{2n}(z) &= \sin k_{zn} H \, e^{-\beta_n(z-H)} \qquad & H \le z \le \infty \\ k_{zn}^2 &= (\omega/c_1)^2 - \kappa_n^2 & \\ \beta_n^2 &= \kappa_n^2 - (\omega/c_2)^2 & \end{aligned} \tag{9.9.4}$$

Both k_{zn} and β_n must be real for trapped normal modes. This restricts κ_n to the interval $\omega/c_2 \le \kappa_n \le \omega/c_1$ and is equivalent to

$$c_1 \le c_{pn} \le c_2 \tag{9.9.5}$$

Manipulation of the boundary conditions at $z = H$ provides a transcendental equation for the allowed values of k_{zn} (and therefore κ_n) at each angular frequency,

$$\tan k_{zn} H = (\rho_2/\rho_1)(k_{zn}/\beta_n) \tag{9.9.6}$$

Definition of

$$y = k_{zn} H \qquad b = \rho_2/\rho_1 \qquad a = \omega H \sqrt{1/c_1^2 - 1/c_2^2} = (\omega/c_1) H \sin\theta_c \tag{9.9.7}$$

allows (9.9.6) to be expressed in a form amenable to graphical or numerical analysis,

$$\tan y = -by/(a^2 - y^2)^{1/2} \tag{9.9.8}$$

See Fig. 9.9.2. The tangent curves have been numbered to designate the associated normal mode. Since a is proportional to frequency, the tangent curves will be intersected

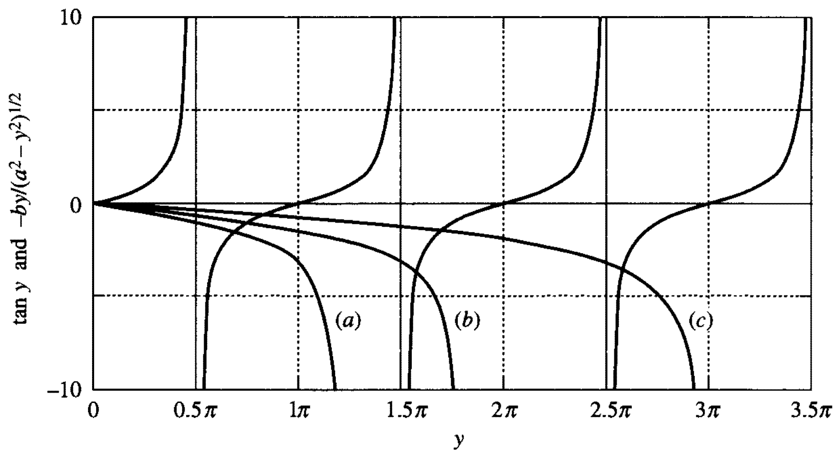


Figure 9.9.2 Graphical solutions for the lower modes of propagation at various frequencies in a shallow-water channel with a fast fluid bottom. The top layer is water with $c_1 = 1500$ m/s, $\rho_1 = 1000$ kg/m^3, and thickness $H = 30$ m. The bottom is quartz sand with $c_2 = 1730$ m/s, $\rho_2 = 2070$ kg/m^3, and infinite thickness. The driving frequencies are (*a*) 60 Hz, (*b*) 90 Hz, and (*c*) 150 Hz.

at different points as the frequency is changed. This is suggested in the figure by the three curves (*a*), (*b*) and (*c*). Since each curve is asymptotic to the appropriate value of $a(\omega)$, it is clear that as a increases with ω the line $y = a$ moves to the right and more normal modes can be excited. The nth normal mode cannot be excited until $a \geq (n - \frac{1}{2})\pi$, and substitution of this into (9.9.7) gives the cutoff angular frequencies,

$$\frac{\omega_n}{c_1} = (n - \tfrac{1}{2})\frac{\pi}{H}\frac{1}{\sin\theta_c} \tag{9.9.9}$$

Once the κ_{zn} have been obtained, κ_n and β_n can be found from (9.9.4). Combination of (9.9.7) and (9.9.9) shows that a is closely related to the ratio of input frequency to the cutoff value,

$$\frac{\omega}{\omega_n} = \frac{a}{(n - \frac{1}{2})\pi} \tag{9.9.10}$$

Figure 9.9.3 reveals the depth dependence of a mode. As the frequency is increased above the cutoff frequency for the nth mode, the value of $k_{zn}H$ increases from $(n - \frac{1}{2})\pi$ at cutoff to $n\pi$ as $\omega \to \infty$. The pressure has an antinode at the interface $z = H$ at cutoff and approaches having a node there at high frequencies. Thus, the interface appears to be rigid at cutoff and pressure release at high frequencies. Evaluation of κ_n at cutoff gives $\kappa_n = \omega_n/c_2$, and (9.9.4) shows that $\beta_n = 0$ so that the normal mode has an extended tail down to infinite depths. As frequency increases above cutoff, κ_n increases, β_n becomes positive real, and the tail decays more rapidly with depth. As frequency becomes arbitrarily large, the tail disappears. This is consistent with the discussion after (9.7.11) about the diminishing vertical extent of the normal mode with increasing frequency.

Trying to form an orthonormal set from Z_n by assuming $Z_n = A_n Z_{1n}$ in the layer and $Z_n = A_n Z_{2n}$ below the layer will not work because of the discontinuity in slope across the boundary at $z = H$. Applying the orthonormality condition to (9.9.1) gives

$$\int_0^\infty \frac{d}{dz}\left(Z_m\frac{dZ_n}{dz} - Z_n\frac{dZ_m}{dz}\right)dz = 0 \tag{9.9.11}$$

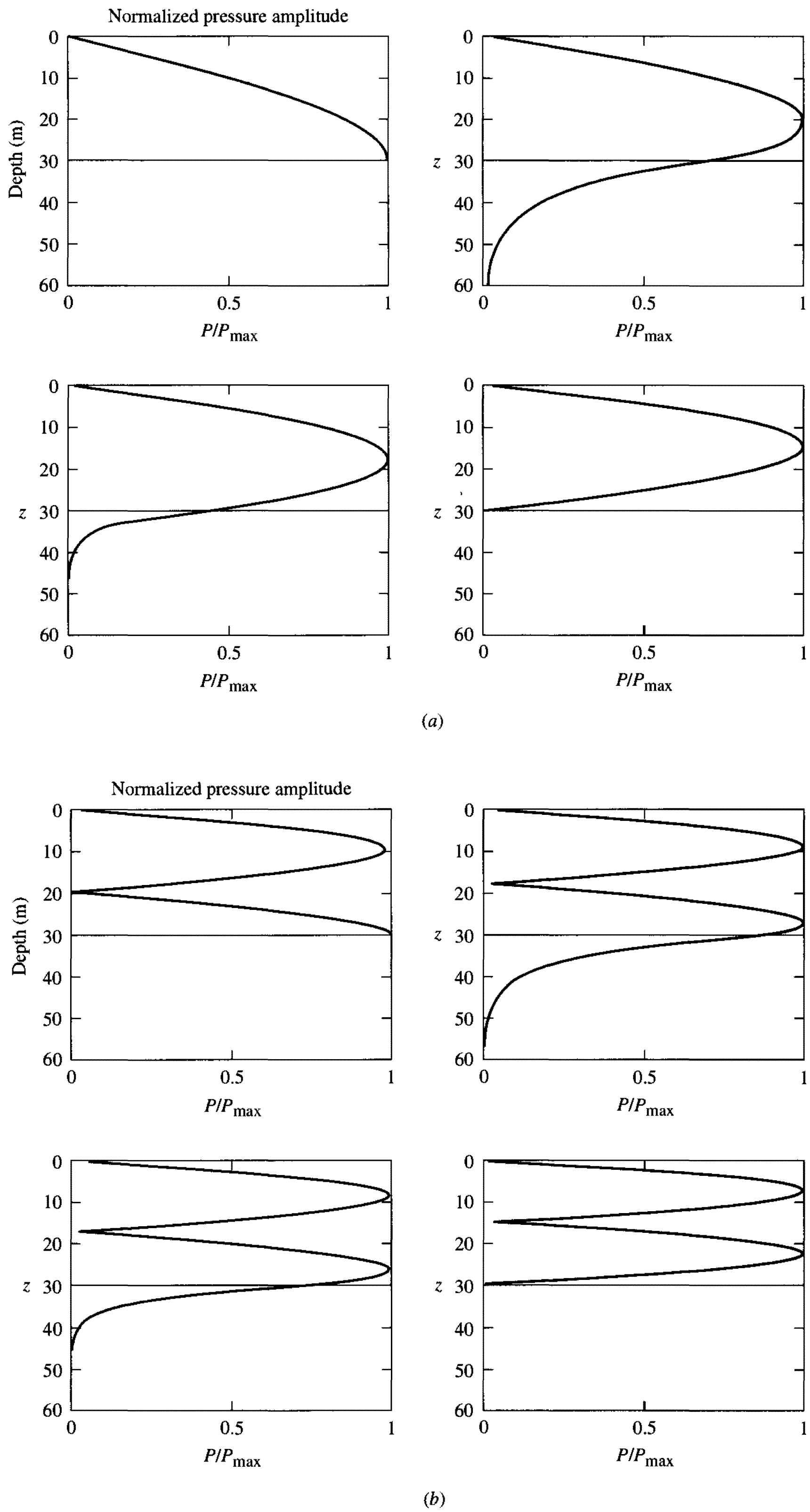


Figure 9.9.3 The depth dependence of the pressure amplitude for various driving frequencies in the shallow-water channel for Figure 9.9.2. (*a*) First propagating mode. (*b*) Second propagating mode. For each mode, the driving frequency increases from just above cutoff (upper left), where the bottom behaves like a rigid surface, to a high frequency (lower right), where the behavior of the bottom approaches that of a free surface.

Integrating and applying the boundary conditions results in

$$\left(Z_m \frac{dZ_n}{dz} - Z_n \frac{dZ_m}{dz}\right)\Bigg|_{H+}^{H-} = 0 \tag{9.9.12}$$

where the upper evaluation is accomplished by approaching the boundary from fluid 1 (z increasing to H) and the lower by approaching the boundary from fluid 2. Direct substitution of the boundary conditions (9.9.3) into (9.9.12) shows that the equation cannot be satisfied by the above assumption. However, the slightly more complicated choice

$$Z_n(z) = \begin{cases} A_n \sin k_{zn} z & 0 \le z \le H \\ A_n(\rho_1/\rho_2)^{1/2} \sin k_{zn} H \, e^{-\beta_n(z-H)} & H \le z \le \infty \end{cases} \tag{9.9.13}$$

does satisfy (9.9.12). Thus, (9.9.13) forms a set of orthogonal eigenfunctions with respect to the *weighting function* $\sqrt{\rho_1/\rho(z)}$. Normalization of the set provides the required values of the A_n for orthonormality,

$$\begin{aligned} \frac{1}{A_n^2} &= \int_0^H Z_{1n}^2(z)\,dz + \frac{\rho_1}{\rho_2}\int_H^\infty Z_{2n}^2(z)\,dz \\ &= (1/2k_{zn})[k_{zn}H - \cos k_{zn}H \sin k_{zn}H - (\rho_1/\rho_2)^2 \sin^2 k_{zn}H \tan k_{zn}H] \end{aligned} \tag{9.9.14}$$

The acoustic pressures in fluid 1 and fluid 2 are found by substituting (9.9.13) into (9.7.8),

$$\begin{aligned} \mathbf{p}_1(r,z,t) &= -j\pi \sum_n A_n^2 \sin k_{zn} z_0 \sin k_{zn} z \, H_0^{(2)}(\kappa_n r) e^{j\omega t} \\ &\to -j \sum_n (2\pi/\kappa_n r)^{1/2} A_n^2 \sin k_{zn} z_0 \sin k_{zn} z \, e^{j(\omega t - \kappa_n r + \pi/4)} \\ \mathbf{p}_2(r,z,t) &= -j\pi \sum_n A_n^2 \sin k_{zn} z_0 \sin k_{zn} z \, e^{-\beta_n(z-H)} H_0^{(2)}(\kappa_n r) e^{j\omega t} \\ &\to -j \sum_n (2\pi/\kappa_n r)^{1/2} A_n^2 \sin k_{zn} z_0 \sin k_{zn} z \, e^{-\beta_n(z-H)} e^{j(\omega t - \kappa_n r + \pi/4)} \end{aligned} \tag{9.9.15}$$

Calculation of the group and phase speeds is a little tricky and the details are treated in Problem 9.9.8. Results can be expressed in implicit forms,

$$\begin{aligned} \left(\frac{c_1}{c_{pn}}\right)^2 &= 1 - \frac{\sin^2 \theta_c}{1 + (b \cot y)^2} \\ \frac{c_1}{c_{gn}} \frac{c_1}{c_{pn}} &= 1 - \frac{(\sin \theta_c \sin y)^2}{\sin^2 y + b^2(\cos^2 y - y \cot y)} \end{aligned} \tag{9.9.16}$$

where $(n - \frac{1}{2})\pi \le y \le n\pi$. Since y increases monotonically with frequency, certain properties of the group and phase speeds can be determined. (1) At cutoff $\cos y = 0$ and (9.9.16) shows that $c_{pn} = c_{gn} = c_2$. With increasing frequency, (2) the phase speed falls monotonically toward an asymptotic value of c_1 and (3) the group speed also approaches the value c_1, but from below, so that (4) the group speed has a minimum value that is less than c_1 at some intermediate frequency. See Fig. 9.9.4 and Problem 9.9.14C.

The fact that for each mode the group speed has a minimum and approaches c_2 for frequencies near cutoff leads to a complicated waveform for a transient excitation. The following general features, sketched in Fig. 9.9.5, can be identified with propagation in each mode.

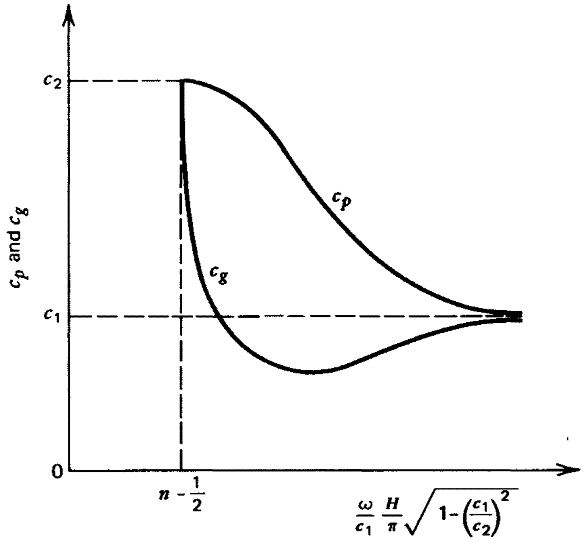


Figure 9.9.4 Group and phase speeds for a normal mode propagating in an isospeed shallow-water channel of depth H with a fast fluid bottom. The speed of sound in the channel is c_1 and that in the bottom is c_2.

1. The *first arrival* reaches the receiver at a time $t = r/c_2$. It consists of Fourier components of the transient having frequencies very close to the cutoff value for the mode and propagating with the group speeds near cutoff. As time increases, slightly higher frequencies traveling with lower group speeds will arrive. This portion of the signal is the *ground wave* and corresponds to energy propagating along the boundary in fluid 2 and radiating back into the layer.
2. At a later time $t = r/c_1$, the highest frequency components arrive with group speeds at and slightly below c_1 and are superimposed on the trailing portion of the ground wave. This high-frequency portion is the *water wave* and corresponds to the high-frequency energy that is propagated radially outward in the channel with angles of elevation and depression very close to zero.
3. For still later times the increasing frequencies in the ground wave and the decreasing frequencies in the water wave become similar and merge into a signal traveling at group speeds slightly above the minimum group speed for the mode. This *Airy phase* comes to a relatively abrupt termination when the energy traveling at the minimum group speed arrives.

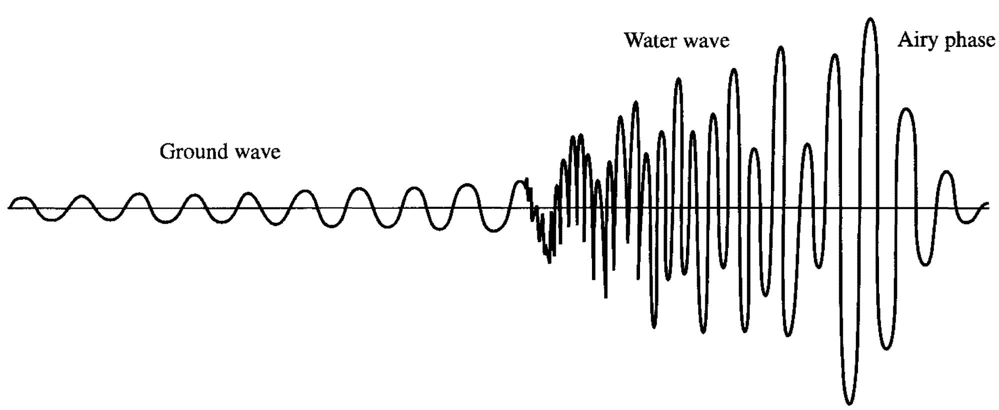


Figure 9.9.5 Sketch of the signal received from a transient propagated in a shallow-water channel with a fast fluid bottom. (After Ewing, Jardetzky, and Press, *Elastic Waves in Layered Media*, McGraw-Hill, 1957.)

A1 CONVERSION FACTORS AND PHYSICAL CONSTANTS

(a) Conversions between SI and CGS

Quantity	*Multiply SI*	*by*	*to obtain CGS*
Length	meter (m)	10^2	centimeter (cm)
Mass	kilogram (kg)	10^3	gram (g)
Time	second (s)	1	second (s)
Force	newton (N)	10^5	dyne
Energy	joule (J)	10^7	erg
Power	watt (W)	10^7	erg/s
Volume density	kg/m^3	10^{-3}	g/cm^3
Pressure	pascal (Pa)	10	$dyne/cm^2$
Speed	m/s	10^2	cm/s
Energy density	J/m^3	10	erg/cm^3
Elastic modulus	Pa	10	$dyne/cm^2$
Coefficient of viscosity	Pa · s	10	$dyne \cdot s/cm^2$
Volume velocity	m^3/s	10^6	cm^3/s
Acoustic intensity	W/m^2	10^3	$erg/(s \cdot cm^2)$
Mechanical impedance	N · s/m	10^3	dyne · s/cm
Specific acoustic impedance	Pa · s/m	10^{-1}	$dyne \cdot s/cm^3$
Acoustic impedance	$Pa \cdot s/m^3$	10^{-5}	$dyne \cdot s/cm^5$
Mechanical stiffness	N/m	10^3	dyne/cm
Magnetic flux density	tesla (T)	10^4	gauss

(b) Other Conversions (≡ designates exact conversions)

1 lb (mass) ≡ 0.45359237 kg
1 in. ≡ 2.54 cm
1 ft ≡ 0.3048 m
1 yd ≡ 0.9144 m
1 fathom ≡ 1.8288 m
1 mi (U.S. statute) ≡ 1.609344 km
1 mi (international and U.S. nautical) ≡ 1 nm ≡ 1.852 km = 6076 ft
1 mph ≡ 0.44704 m/s ≡ 1.609344 km/h
1 knot ≡ 1 nm/h ≡ 1.852 km/h = 0.5144 m/s = 1.1508 mph
1 bar ≡ 1×10^5 Pa ≡ 1×10^6 $dyne/cm^2$ = 14.5037 psi
1 kgf/m^2 ≡ 9.80665 Pa
1 ft H_2O (39.2°F) = 2.98898×10^3 Pa
1 in. Hg (32°F) = 3.38639×10^3 Pa
1 $lbf/in.^2$ (psi) = 6.89476×10^3 Pa
1 atm ≡ 1.01325 bar = 14.6959 psi ($lbf/in.^2$) = 1.03323×10^4 kgf/m^2
= 33.8995 ft H_2O (39.2°F) = 29.9213 in. Hg (32°F)
°C = K − 273.15 = $\frac{5}{9}$(°F − 32)

(c) Physical Constants

Acceleration of gravity	g	9.80665 (standard)	m/s^2
Avogadro constant	A	6.022×10^{26}	$kmol^{-1}$
Boltzmann constant	k_B	1.3807×10^{-23}	J/K
Gas constant	$\mathfrak{R}$	8.3145	$J/(mol \cdot K)$
		8.3145×10^3	$J/(kmol \cdot K)$
Molecular weight	M		
Dry air		28.964	kg/kmol
H_2O		18.016	kg/kmol
Specific gas constant	r		
Dry air		287.06	$J/(kg \cdot K)$
H_2O (gas)		461.50	$J/(kg \cdot K)$

A2 COMPLEX NUMBERS

Let x and y be real functions and define $j = \sqrt{-1}$. Then from

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

we obtain *Euler's identity*

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

and thus

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

If

$$\mathbf{f} = x + jy = Ae^{j\theta}$$

then

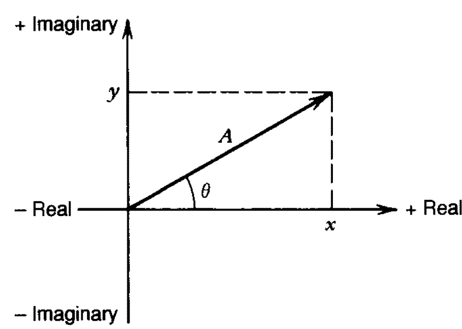
$$\mathrm{Re}\{\mathbf{f}\} = x = A\cos\theta$$

$$\mathrm{Im}\{\mathbf{f}\} = y = A\sin\theta$$

$$|\mathbf{f}| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

$$\mathbf{f}^* = x - jy = Ae^{-j\theta}$$



If

$$\mathbf{f} = Fe^{j(n\omega t + \theta)} \qquad \mathbf{g} = Ge^{j(n\omega t + \phi)}$$

then

$$|\mathbf{fg}| = |\mathbf{f}||\mathbf{g}| = FG$$

$$\langle \mathrm{Re}\{\mathbf{f}\}\mathrm{Re}\{\mathbf{g}\}\rangle_T = \tfrac{1}{2}\mathrm{Re}\{\mathbf{fg}^*\} = \tfrac{1}{2}\mathrm{Re}\{\mathbf{f}^*\mathbf{g}\} = \tfrac{1}{2}FG\cos(\theta - \phi)$$

In the last expression, $T = 2\pi/\omega$. If $n = 0$ then the factors of $\frac{1}{2}$ must be deleted.

A3 CIRCULAR AND HYPERBOLIC FUNCTIONS

Let $z = x + jy$.

$$\sinh z = (e^z - e^{-z})/2 \qquad \cosh z = (e^z + e^{-z})/2$$

$$\tanh z = \sinh z / \cosh z \qquad \coth z = 1/\tanh z$$

$$\frac{d}{dz}\sinh z = \cosh z \qquad \frac{d}{dz}\cosh z = \sinh z$$

$$\sin(jy) = j\sinh y \qquad \sinh(jy) = j\sin y$$

$$\cos(jy) = \cosh y \qquad \cosh(jy) = \cos y$$

$$\sin z = \sin x \cosh y + j\cos x \sinh y \qquad \sinh z = \sinh x \cos y + j\cosh x \sin y$$

$$\cos z = \cos x \cosh y - j\sin x \sinh y \qquad \cosh z = \cosh x \cos y + j\sinh x \sin y$$

$$\sin^2 z + \cos^2 z = 1$$

$$\cosh^2 z - \sinh^2 z = 1$$

In the following equations, the companion relationships can be obtained by differentiation with respect to z_1 and/or z_2.

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$

$$2\sin z_1 \sin z_2 = \cos(z_1 - z_2) - \cos(z_1 + z_2)$$

$$2\sinh z_1 \sinh z_2 = \cosh(z_1 + z_2) - \cosh(z_1 - z_2)$$

Useful summations are

$$\sum_{n=0}^{N-1} \cos(n\theta) = \frac{\sin(N\theta/2)\cos[(N-1)\theta/2]}{\sin(\theta/2)}$$

$$\sum_{n=0}^{N-1} \sin(n\theta) = \frac{\sin(N\theta/2)\sin[(N-1)\theta/2]}{\sin(\theta/2)}$$

A4 SOME MATHEMATICAL FUNCTIONS

In this appendix, $z = x + jy$ with x and y real, the index ν is a real number, and l, m, and n are real integers. Any other restrictions on these quantities will be explicitly noted. We will quote relationships pertinent to the text. For complete properties, see Abramowitz and Stegun, *Handbook of Mathematical Functions*, Dover (1965).

(a) Gamma Function

The gamma function, while not necessary, is convenient. For arguments with positive real part, it is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad z = 0$$

Useful equalities are

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \qquad \Gamma\left(n + \tfrac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}\sqrt{\pi}$$

$$\Gamma(1) = 1 \qquad \Gamma(n+1) = n!$$

$$\Gamma(z+1) = z\Gamma(z)$$

(b) Bessel Functions, Modified Bessel Functions, and Struve Functions

The differential equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2)w = \frac{4(z/2)^{\nu+1}}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)}$$

has *homogeneous* solutions that are any linear combinations $\mathbf{A}J_\nu(z) + \mathbf{B}Y_\nu(z)$ of the Bessel function of the first kind $J_\nu(z)$ and the Bessel function of the second kind $Y_\nu(z)$ [also known as Weber's function or Neumann's function and sometimes notated as $N_\nu(z)$]. The specific combinations

$$H_\nu^{(1)}(z) = J_\nu(z) + jY_\nu(z) \qquad H_\nu^{(2)}(z) = J_\nu(z) - jY_\nu(z)$$

are the Bessel functions of the third kind, the Hankel functions. The *particular* solution of the differential equation is the Struve function $\mathbf{H}_\nu(z)$. The index ν designates the *order* of the functions.

In the rest of this appendix, all functions are understood to be of argument z unless otherwise written.

For integral orders,

$$J_{-n} = (-1)^n J_n \qquad Y_{-n} = (-1)^n Y_n$$

The Wronskian for J_ν and Y_ν is

$$W\{J_\nu, Y_\nu\} = J_{\nu+1}Y_\nu - J_\nu Y_{\nu+1} = 2/\pi z$$

Series expansions for orders 0 and 1, useful for small z, are

$$J_0 = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 \cdot 4^2} - \frac{z^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$J_1 = \frac{z}{2} + \frac{2z^3}{2 \cdot 4^2} - \frac{3z^5}{2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$Y_0 = \frac{2}{\pi}\left\{\left[\ln\left(\frac{z}{2}\right) + \gamma\right]J_0 + \frac{z^2}{2^2} - \frac{z^4}{2^2 \cdot 4^2}\left(1 + \frac{1}{2}\right) + \cdots\right\}$$

$$Y_1 = -\frac{2}{\pi}\frac{1}{z} + \cdots$$

$$\mathbf{H}_0 = \frac{2}{\pi}\left(z - \frac{z^3}{1^2 \cdot 3^2} + \frac{z^5}{1^2 \cdot 3^2 \cdot 5^2} - \cdots\right)$$

$$\mathbf{H}_1 = \frac{2}{\pi}\left(\frac{z^2}{1^2\cdot 3} - \frac{z^4}{1^2\cdot 3^2\cdot 5} + \frac{z^6}{1^2\cdot 3^2\cdot 5^2\cdot 7} - \cdots\right)$$

where $\gamma = 0.57721\ldots$ is Euler's constant.

Approximations useful for large z and $|\arg z| < \pi$ are

$$\begin{aligned}
J_\nu &\to \sqrt{2/\pi z}\cos(z - \nu\pi/2 - \pi/4)\\
Y_\nu &\to \sqrt{2/\pi z}\sin(z - \nu\pi/2 - \pi/4)\\
H_\nu^{(1)} &\to \sqrt{2/\pi z}\exp[j(z - \nu\pi/2 - \pi/4)]\\
H_\nu^{(2)} &\to \sqrt{2/\pi z}\exp[-j(z - \nu\pi/2 - \pi/4)]
\end{aligned}$$

$$\mathbf{H}_\nu - Y_\nu \to \frac{1}{\pi}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^{\nu-1}$$

Let C_ν represent any linear combination of Bessel functions of order ν. Then some recurrence and differential relations for the Bessel and Struve functions are

$$C_{\nu-1} + C_{\nu+1} = \frac{2\nu}{z}C_\nu \qquad C_{\nu-1} - C_{\nu+1} = 2\frac{d}{dz}C_\nu$$

$$\frac{d}{dz}C_0 = -C_1 \qquad \frac{d}{dz}(z^\nu C_\nu) = z^\nu C_{\nu-1} \qquad \frac{d}{dz}\left(\frac{1}{z^\nu}C_\nu\right) = -\frac{1}{z^\nu}C_{\nu+1}$$

$$\frac{d}{dz}\mathbf{H}_0 = \frac{2}{\pi} - \mathbf{H}_1 \qquad \frac{d}{dz}(z^\nu\mathbf{H}_\nu) = z^\nu\mathbf{H}_{\nu-1}$$

Useful integral representations are

$$J_0(z) = \frac{2}{\pi}\int_0^{\pi/2}\cos(z\cos\theta)\,d\theta \qquad \mathbf{H}_0(z) = \frac{2}{\pi}\int_0^{\pi/2}\sin(z\cos\theta)\,d\theta$$

$$J_n(z) = \frac{(z/2)^n}{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}\int_0^\pi \cos(z\cos\theta)\sin^{2n}\theta\,d\theta = \frac{(-j)^n}{2\pi}\int_0^{2\pi} e^{jz\cos\theta}\cos n\theta\,d\theta$$

The arguments of J_ν for which the function has real zeros and extrema are real and defined as $j_{\nu n}$ and $j'_{\nu n}$. Relevant evaluations include

$$\begin{aligned}
J_\nu(j_{\nu n}) &= 0 \qquad J'_\nu(j_{\nu n}) = J_{\nu-1}(j_{\nu n}) = -J_{\nu+1}(j_{\nu n})\\
J'_\nu(j'_{\nu n}) &= 0 \qquad J_\nu(j'_{\nu n}) = \frac{j'_{\nu n}}{\nu}J_{\nu-1}(j'_{\nu n}) = \frac{j'_{\nu n}}{\nu}J_{\nu+1}(j'_{\nu n})
\end{aligned}$$

With these defined arguments, normalizations for orthogonal Bessel functions of the first kind are facilitated with

$$\int_0^1 J_\nu(j_{\nu m}t)J_\nu(j_{\nu n}t)t\,dt = \tfrac{1}{2}[J'_\nu(j_{\nu n})]^2\delta_{nm}$$

$$\int_0^1 J_\nu(j'_{\nu m}t)J_\nu(j'_{\nu n}t)t\,dt = \frac{1}{2}\frac{(j'_{\nu n})^2 - \nu^2}{(j'_{\nu n})^2}[J_\nu(j'_{\nu n})]^2\delta_{nm}$$

The *modified* Bessel function I_ν satisfies the differential equation

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2 + \nu^2)w = 0$$

and is related to J_ν by

$$I_n(z) = j^{-n}J_n(jz)$$

All other needed relationships follow from the substitution jz for the argument z in the preceding equations for $J_\nu(z)$. For example,

$$I_{\nu-1} - I_{\nu+1} = \frac{2\nu}{z}I_\nu \qquad I_{\nu-1} + I_{\nu+1} = 2\frac{d}{dz}I_\nu$$

$$\frac{d}{dz}I_0 = I_1 \qquad \frac{d}{dz}(z^\nu I_\nu) = z^\nu I_{\nu-1} \qquad \frac{d}{dz}\left(\frac{1}{z^\nu}I_\nu\right) = \frac{1}{z^\nu}I_{\nu+1}$$

(c) Spherical Bessel Functions

The *spherical* Bessel functions $j_n(z)$ of the first kind, $y_n(z)$ of the second kind, and $h_n(z)$ of the third kind, and any linear combination of them satisfy the differential equation

$$z^2\frac{d^2w}{dz^2} + 2z\frac{dw}{dz} + [z^2 - n(n+1)]w = 0$$

They are related to the Bessel functions by

$$\begin{aligned} j_n &= \sqrt{\pi/2z}\,J_{n+1/2} \\ y_n &= \sqrt{\pi/2z}\,Y_{n+1/2} \\ h_n^{(1,2)} &= \sqrt{\pi/2z}\,H_{n+1/2}^{(1,2)} \end{aligned}$$

Explicit forms for j_n are

$$j_0 = \frac{\sin z}{z} \qquad j_1 = \frac{\sin z}{z^2} - \frac{\cos z}{z}$$

$$j_2 = \left(\frac{3}{z^3} - \frac{1}{z}\right)\sin z - \frac{3}{z^2}\cos z \qquad j_{n+1} = \frac{2n+1}{z}j_n - j_{n-1}$$

(d) Legendre Functions

The Legendre function $P_l^m(z)$ of *degree* l and *order* m is a solution of the differential equation

$$(1 - z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left(l(l+1) - \frac{m^2}{1 - z^2}\right)w = 0$$

The greatest order for any degree is limited by $m \leq l$. While the Legendre functions are in general rather complicated functions of z, our interest is restricted to their behavior for real arguments x lying in the interval $|x| \leq 1$.

For spherical standing waves, $x = \cos\theta$. For the zeroth order Legendre functions, $m = 0$ and the order superscript is suppressed. These are the *Legendre polynomials.* They can be obtained from

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

and the lowest four integral degrees are

$$P_0 = 1 \qquad P_1 = \cos\theta$$
$$P_2 = \tfrac{1}{2}(3\cos^2\theta - 1) \qquad P_3 = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

Higher orders, the *associated* Legendre functions, can be obtained from

$$P_l^m(x) = (-1)^m(1 - x^2)^{m/2}\frac{d^m}{dx^m}P_l(x)$$

so that

$$P_1^1 = -\sin\theta \qquad P_2^1 = -3\sin\theta\cos\theta \qquad P_3^1 = -\tfrac{3}{2}\sin\theta(5\cos^2\theta - 1)$$
$$P_2^2 = 3\sin^2\theta \qquad P_3^2 = 15\sin^2\theta\cos\theta$$
$$P_3^3 = -15\sin^3\theta$$

Two recurrence relations are

$$(l - m + 1)P_{l+1}^m = (2l + 1)xP_l^m - (l + m)P_{l-1}^m$$
$$P_l^{m+1} = (1 - x^2)^{-1/2}[(l - m)xP_l^m - (l + m)P_{l-1}^m]$$

A5 BESSEL FUNCTIONS: TABLES, GRAPHS, ZEROS, AND EXTREMA

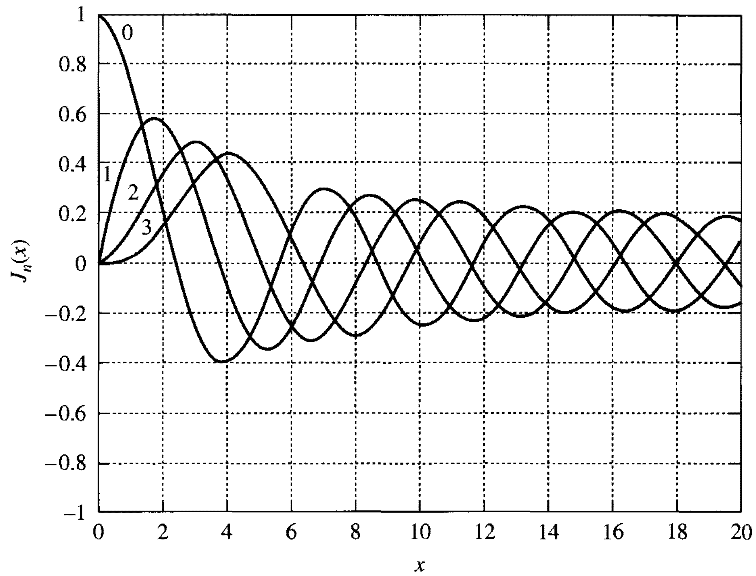
(a) Table: Bessel and Modified Bessel Functions of the First Kind of Orders 0, 1, and 2

x	$J_0(x)$	$J_1(x)$	$J_2(x)$	$Y_0(x)$	$Y_1(x)$	$Y_2(x)$	$I_0(x)$	$I_1(x)$	$I_2(x)$
0	1.0000	0	0	$-\infty$	$-\infty$	$-\infty$	1.0000	0	0
0.1000	0.9975	0.0499	0.0012	−1.5342	−6.4590	−127.6448	1.0025	0.0501	0.0013
0.2000	0.9900	0.0995	0.0050	−1.0811	−3.3238	−32.1571	1.0100	0.1005	0.0050
0.3000	0.9776	0.1483	0.0112	−0.8073	−2.2931	−14.4801	1.0226	0.1517	0.0113
0.4000	0.9604	0.1960	0.0197	−0.6060	−1.7809	−8.2983	1.0404	0.2040	0.0203
0.5000	0.9385	0.2423	0.0306	−0.4445	−1.4715	−5.4414	1.0635	0.2579	0.0319
0.6000	0.9120	0.2867	0.0437	−0.3085	−1.2604	−3.8928	1.0920	0.3137	0.0464
0.7000	0.8812	0.3290	0.0588	−0.1907	−1.1032	−2.9615	1.1263	0.3719	0.0638
0.8000	0.8463	0.3688	0.0758	−0.0868	−0.9781	−2.3586	1.1665	0.4329	0.0844
0.9000	0.8075	0.4059	0.0946	0.0056	−0.8731	−1.9459	1.2130	0.4971	0.1083
1.0000	0.7652	0.4401	0.1149	0.0883	−0.7812	−1.6507	1.2661	0.5652	0.1357
1.1000	0.7196	0.4709	0.1366	0.1622	−0.6981	−1.4315	1.3262	0.6375	0.1671
1.2000	0.6711	0.4983	0.1593	0.2281	−0.6211	−1.2633	1.3937	0.7147	0.2026
1.3000	0.6201	0.5220	0.1830	0.2865	−0.5485	−1.1304	1.4693	0.7973	0.2426
1.4000	0.5669	0.5419	0.2074	0.3379	−0.4791	−1.0224	1.5534	0.8861	0.2875
1.5000	0.5118	0.5579	0.2321	0.3824	−0.4123	−0.9322	1.6467	0.9817	0.3378
1.6000	0.4554	0.5699	0.2570	0.4204	−0.3476	−0.8549	1.7500	1.0848	0.3940
1.7000	0.3980	0.5778	0.2817	0.4520	−0.2847	−0.7870	1.8640	1.1963	0.4565

(continued)

x	$J_0(x)$	$J_1(x)$	$J_2(x)$	$Y_0(x)$	$Y_1(x)$	$Y_2(x)$	$I_0(x)$	$I_1(x)$	$I_2(x)$
1.8000	0.3400	0.5815	0.3061	0.4774	−0.2237	−0.7259	1.9896	1.3172	0.5260
1.9000	0.2818	0.5812	0.3299	0.4968	−0.1644	−0.6699	2.1277	1.4482	0.6033
2.0000	0.2239	0.5767	0.3528	0.5104	−0.1070	−0.6174	2.2796	1.5906	0.6889
2.1000	0.1666	0.5683	0.3746	0.5183	−0.0517	−0.5675	2.4463	1.7455	0.7839
2.2000	0.1104	0.5560	0.3951	0.5208	0.0015	−0.5194	2.6291	1.9141	0.8891
2.3000	0.0555	0.5399	0.4139	0.5181	0.0523	−0.4726	2.8296	2.0978	1.0054
2.4000	0.0025	0.5202	0.4310	0.5104	0.1005	−0.4267	3.0493	2.2981	1.1342
2.5000	−0.0484	0.4971	0.4461	0.4981	0.1459	−0.3813	3.2898	2.5167	1.2765
2.6000	−0.0968	0.4708	0.4590	0.4813	0.1884	−0.3364	3.5533	2.7554	1.4337
2.7000	−0.1424	0.4416	0.4696	0.4605	0.2276	−0.2919	3.8417	3.0161	1.6075
2.8000	−0.1850	0.4097	0.4777	0.4359	0.2635	−0.2477	4.1573	3.3011	1.7994
2.9000	−0.2243	0.3754	0.4832	0.4079	0.2959	−0.2038	4.5027	3.6126	2.0113
3.0000	−0.2601	0.3391	0.4861	0.3769	0.3247	−0.1604	4.8808	3.9534	2.2452
3.1000	−0.2921	0.3009	0.4862	0.3431	0.3496	−0.1175	5.2945	4.3262	2.5034
3.2000	−0.3202	0.2613	0.4835	0.3071	0.3707	−0.0754	5.7472	4.7343	2.7883
3.3000	−0.3443	0.2207	0.4780	0.2691	0.3879	−0.0340	6.2426	5.1810	3.1027
3.4000	−0.3643	0.1792	0.4697	0.2296	0.4010	0.0063	6.7848	5.6701	3.4495
3.5000	−0.3801	0.1374	0.4586	0.1890	0.4102	0.0454	7.3782	6.2058	3.8320
3.6000	−0.3918	0.0955	0.4448	0.1477	0.4154	0.0831	8.0277	6.7927	4.2540
3.7000	−0.3992	0.0538	0.4283	0.1061	0.4167	0.1192	8.7386	7.4357	4.7193
3.8000	−0.4026	0.0128	0.4093	0.0645	0.4141	0.1535	9.5169	8.1404	5.2325
3.9000	−0.4018	−0.0272	0.3879	0.0234	0.4078	0.1858	10.3690	8.9128	5.7983
4.0000	−0.3971	−0.0660	0.3641	−0.0169	0.3979	0.2159	11.3019	9.7595	6.4222
4.1000	−0.3887	−0.1033	0.3383	−0.0561	0.3846	0.2437	12.3236	10.6877	7.1100
4.2000	−0.3766	−0.1386	0.3105	−0.0938	0.3680	0.2690	13.4425	11.7056	7.8684
4.3000	−0.3610	−0.1719	0.2811	−0.1296	0.3484	0.2916	14.6680	12.8219	8.7043
4.4000	−0.3423	−0.2028	0.2501	−0.1633	0.3260	0.3115	16.0104	14.0462	9.6258
4.5000	−0.3205	−0.2311	0.2178	−0.1947	0.3010	0.3285	17.4812	15.3892	10.6415
4.6000	−0.2961	−0.2566	0.1846	−0.2235	0.2737	0.3425	19.0926	16.8626	11.7611
4.7000	−0.2693	−0.2791	0.1506	−0.2494	0.2445	0.3534	20.8585	18.4791	12.9950
4.8000	−0.2404	−0.2985	0.1161	−0.2723	0.2136	0.3613	22.7937	20.2528	14.3550
4.9000	−0.2097	−0.3147	0.0813	−0.2921	0.1812	0.3660	24.9148	22.1993	15.8538
5.0000	−0.1776	−0.3276	0.0466	−0.3085	0.1479	0.3677	27.2399	24.3356	17.5056
5.1000	−0.1443	−0.3371	0.0121	−0.3216	0.1137	0.3662	29.7889	26.6804	19.3259
5.2000	−0.1103	−0.3432	−0.0217	−0.3313	0.0792	0.3617	32.5836	29.2543	21.3319
5.3000	−0.0758	−0.3460	−0.0547	−0.3374	0.0445	0.3542	35.6481	32.0799	23.5425
5.4000	−0.0412	−0.3453	−0.0867	−0.3402	0.0101	0.3439	39.0088	35.1821	25.9784
5.5000	−0.0068	−0.3414	−0.1173	−0.3395	−0.0238	0.3308	42.6946	38.5882	28.6626
5.6000	0.0270	−0.3343	−0.1464	−0.3354	−0.0568	0.3152	46.7376	42.3283	31.6203
5.7000	0.0599	−0.3241	−0.1737	−0.3282	−0.0887	0.2970	51.1725	46.4355	34.8794
5.8000	0.0917	−0.3110	−0.1990	−0.3177	−0.1192	0.2766	56.0381	50.9462	38.4704
5.9000	0.1220	−0.2951	−0.2221	−0.3044	−0.1481	0.2542	61.3766	55.9003	42.4273
6.0000	0.1506	−0.2767	−0.2429	−0.2882	−0.1750	0.2299	67.2344	61.3419	46.7871
6.1000	0.1773	−0.2559	−0.2612	−0.2694	−0.1998	0.2039	73.6628	67.3194	51.5909
6.2000	0.2017	−0.2329	−0.2769	−0.2483	−0.2223	0.1766	80.7179	73.8859	56.8838
6.3000	0.2238	−0.2081	−0.2899	−0.2251	−0.2422	0.1482	88.4616	81.1000	62.7155
6.4000	0.2433	−0.1816	−0.3001	−0.1999	−0.2596	0.1188	96.9616	89.0261	69.1410
6.5000	0.2601	−0.1538	−0.3074	−0.1732	−0.2741	0.0889	106.2929	97.7350	76.2205
6.6000	0.2740	−0.1250	−0.3119	−0.1452	−0.2857	0.0586	116.5373	107.3047	84.0208
6.7000	0.2851	−0.0953	−0.3135	−0.1162	−0.2945	0.0283	127.7853	117.8208	92.6150
6.8000	0.2931	−0.0652	−0.3123	−0.0864	−0.3002	−0.0019	140.1362	129.3776	102.0839
6.9000	0.2981	−0.0349	−0.3082	−0.0563	−0.3029	−0.0315	153.6990	142.0790	112.5167
7.0000	0.3001	−0.0047	−0.3014	−0.0259	−0.3027	−0.0605	168.5939	156.0391	124.0113
7.1000	0.2991	0.0252	−0.2920	0.0042	−0.2995	−0.0885	184.9529	171.3834	136.6759
7.2000	0.2951	0.0543	−0.2800	0.0339	−0.2934	−0.1154	202.9213	188.2503	150.6296
7.3000	0.2882	0.0826	−0.2656	0.0628	−0.2846	−0.1407	222.6588	206.7917	166.0035
7.4000	0.2786	0.1096	−0.2490	0.0907	−0.2731	−0.1645	244.3410	227.1750	182.9424
7.5000	0.2663	0.1352	−0.2303	0.1173	−0.2591	−0.1864	268.1613	249.5844	201.6055
7.6000	0.2516	0.1592	−0.2097	0.1424	−0.2428	−0.2063	294.3322	274.2225	222.1684
7.7000	0.2346	0.1813	−0.1875	0.1658	−0.2243	−0.2241	323.0875	301.3124	244.8246
7.8000	0.2154	0.2014	−0.1638	0.1872	−0.2039	−0.2395	354.6845	331.0995	269.7872
7.9000	0.1944	0.2192	−0.1389	0.2065	−0.1817	−0.2525	389.4063	363.8539	297.2914
8.0000	0.1717	0.2346	−0.1130	0.2235	−0.1581	−0.2630	427.5641	399.8731	327.5958

(b) Graphs: Bessel Functions of the First Kind of Orders 0, 1, 2, and 3



(c) Zeros: Bessel Functions of the First Kind, $J_m(j_{mn}) = 0$

m \ n	0	1	2	3	4	5
	j_{mn}					
0	—	2.40	5.52	8.65	11.79	14.93
1	0	3.83	7.02	10.17	13.32	16.47
2	0	5.14	8.42	11.62	14.80	17.96
3	0	6.38	9.76	13.02	16.22	19.41
4	0	7.59	11.06	14.37	17.62	20.83
5	0	8.77	12.34	15.70	18.98	22.22

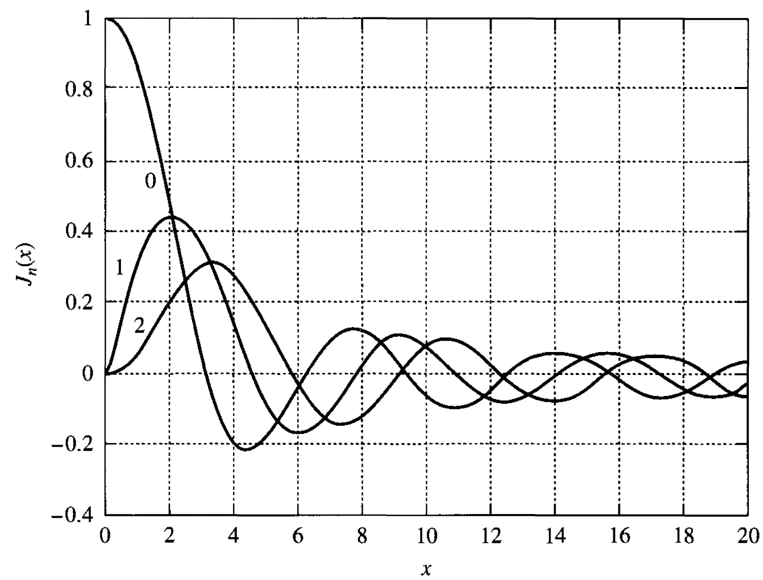
(d) Extrema: Bessel Functions of the First Kind, $J'_m(j'_{mn}) = 0$

m \ n	1	2	3	4	5
	j'_{mn}				
0	0	3.83	7.02	10.17	13.32
1	1.84	5.33	8.54	11.71	14.86
2	3.05	6.71	9.97	13.17	16.35
3	4.20	8.02	11.35	14.59	17.79
4	5.32	9.28	12.68	15.96	19.20
5	6.41	10.52	13.99	17.31	20.58

(e) Table: Spherical Bessel Functions of the First Kind of Orders 0, 1, and 2

x	$j_0(x)$	$j_1(x)$	$j_2(x)$	x	$j_0(x)$	$j_1(x)$	$j_2(x)$
0	1.000	0	0	4.0000	−0.1892	0.1161	0.2763
0.1000	0.9983	0.0333	0.0007	4.1000	−0.1996	0.0915	0.2665
0.2000	0.9933	0.0664	0.0027	4.2000	−0.2075	0.0673	0.2556
0.3000	0.9851	0.0991	0.0060	4.3000	−0.2131	0.0437	0.2435
0.4000	0.9735	0.1312	0.0105	4.4000	−0.2163	0.0207	0.2304
0.5000	0.9589	0.1625	0.0164	4.5000	−0.2172	−0.0014	0.2163
0.6000	0.9411	0.1929	0.0234	4.6000	−0.2160	−0.0226	0.2013
0.7000	0.9203	0.2221	0.0315	4.7000	−0.2127	−0.0426	0.1855
0.8000	0.8967	0.2500	0.0408	4.8000	−0.2075	−0.0615	0.1691
0.9000	0.8704	0.2764	0.0509	4.9000	−0.2005	−0.0790	0.1521
1.0000	0.8415	0.3012	0.0620	5.0000	−0.1918	−0.0951	0.1347
1.1000	0.8102	0.3242	0.0739	5.1000	−0.1815	−0.1097	0.1170
1.2000	0.7767	0.3453	0.0865	5.2000	−0.1699	−0.1228	0.0991
1.3000	0.7412	0.3644	0.0997	5.3000	−0.1570	−0.1342	0.0811
1.4000	0.7039	0.3814	0.1133	5.4000	−0.1431	−0.1440	0.0631
1.5000	0.6650	0.3962	0.1273	5.5000	−0.1283	−0.1522	0.0453
1.6000	0.6247	0.4087	0.1416	5.6000	−0.1127	−0.1586	0.0277
1.7000	0.5833	0.4189	0.1560	5.7000	−0.0966	−0.1634	0.0106
1.8000	0.5410	0.4268	0.1703	5.8000	−0.0801	−0.1665	−0.0060
1.9000	0.4981	0.4323	0.1845	5.9000	−0.0634	−0.1679	−0.0220
2.0000	0.4546	0.4354	0.1984	6.0000	−0.0466	−0.1678	−0.0373
2.1000	0.4111	0.4361	0.2120	6.1000	−0.0299	−0.1661	−0.0518
2.2000	0.3675	0.4345	0.2251	6.2000	−0.0134	−0.1629	−0.0654
2.3000	0.3242	0.4307	0.2375	6.3000	0.0027	−0.1583	−0.0780
2.4000	0.2814	0.4245	0.2492	6.4000	0.0182	−0.1523	−0.0896
2.5000	0.2394	0.4162	0.2601	6.5000	0.0331	−0.1452	−0.1001
2.6000	0.1983	0.4058	0.2700	6.6000	0.0472	−0.1368	−0.1094
2.7000	0.1583	0.3935	0.2789	6.7000	0.0604	−0.1275	−0.1175
2.8000	0.1196	0.3792	0.2867	6.8000	0.0727	−0.1172	−0.1244
2.9000	0.0825	0.3633	0.2933	6.9000	0.0838	−0.1061	−0.1299
3.0000	0.0470	0.3457	0.2986	7.0000	0.0939	−0.0943	−0.1343
3.1000	0.0134	0.3266	0.3027	7.1000	0.1027	−0.0820	−0.1373
3.2000	−0.0182	0.3063	0.3054	7.2000	0.1102	−0.0692	−0.1391
3.3000	−0.0478	0.2848	0.3067	7.3000	0.1165	−0.0561	−0.1396
3.4000	−0.0752	0.2622	0.3066	7.4000	0.1214	−0.0429	−0.1388
3.5000	−0.1002	0.2389	0.3050	7.5000	0.1251	−0.0295	−0.1369
3.6000	−0.1229	0.2150	0.3021	7.6000	0.1274	−0.0163	−0.1338
3.7000	−0.1432	0.1905	0.2977	7.7000	0.1283	−0.0033	−0.1296
3.8000	−0.1610	0.1658	0.2919	7.8000	0.1280	0.0095	−0.1244
3.9000	−0.1764	0.1409	0.2847	7.9000	0.1264	0.0218	−0.1182
				8.0000	0.1237	0.0336	−0.1111

(f) Graphs: Spherical Bessel Functions of the First Kind of Orders 0, 1, and 2



(g) Zeros: Spherical Bessel Functions of the First Kind, $j_m(\zeta_{mn}) = 0$

	ζ_{mn}					
$m \backslash n$	0	1	2	3	4	5
0	—	3.14	6.28	9.42	12.57	15.71
1	0	4.49	7.73	10.90	14.07	17.22
2	0	5.76	9.10	12.32	15.51	18.69
3	0	6.99	10.42	13.70	16.92	20.12
4	0	8.18	11.70	15.04	18.30	21.53
5	0	9.36	12.97	16.35	19.65	22.90

(h) Extrema: Spherical Bessel Functions of the First Kind, $j'_m(\zeta'_{mn}) = 0$

	ζ'_{mn}				
$m \backslash n$	1	2	3	4	5
0	0	4.49	7.73	10.90	14.07
1	2.08	5.94	9.21	12.40	15.58
2	3.34	7.29	10.61	13.85	17.04
3	4.51	8.58	11.97	15.24	18.47
4	5.65	9.84	13.30	16.61	19.86
5	6.76	11.07	14.59	17.95	21.23

A6 TABLE OF DIRECTIVITIES AND IMPEDANCE FUNCTIONS FOR A PISTON

x	*Directivity Functions* ($x = ka \sin\theta$)		*Impedance Functions* ($x = 2ka$)	
	Pressure	*Intensity*	*Resistance*	*Resistance*
	$\frac{2J_1(x)}{x}$	$\left(\frac{2J_1(x)}{x}\right)^2$	$R_1(x)$	$X_1(x)$
0.0	1.0000	1.0000	0.0000	0.0000
0.2	0.9950	0.9900	0.0050	0.0847
0.4	0.9802	0.9608	0.0198	0.1680
0.6	0.9557	0.9134	0.0443	0.2486
0.8	0.9221	0.8503	0.0779	0.3253
1.0	0.8801	0.7746	0.1199	0.3969
1.2	0.8305	0.6897	0.1695	0.4624
1.4	0.7743	0.5995	0.2257	0.5207
1.6	0.7124	0.5075	0.2876	0.5713
1.8	0.6461	0.4174	0.3539	0.6134
2.0	0.5767	0.3326	0.4233	0.6468
2.2	0.5054	0.2554	0.4946	0.6711
2.4	0.4335	0.1879	0.5665	0.6862
2.6	0.3622	0.1326	0.6378	0.6925
2.8	0.2927	0.0857	0.7073	0.6903
3.0	0.2260	0.0511	0.7740	0.6800
3.2	0.1633	0.0267	0.8367	0.6623
3.4	0.1054	0.0111	0.8946	0.6381
3.6	0.0530	0.0028	0.9470	0.6081
3.8	+0.0068	0.00005	0.9932	0.5733
4.0	−0.0330	0.0011	1.0330	0.5349
4.5	−0.1027	0.0104	1.1027	0.4293
5.0	−0.1310	0.0172	1.1310	0.3232
5.5	−0.1242	0.0154	1.1242	0.2299
6.0	−0.0922	0.0085	1.0922	0.1594
6.5	−0.0473	0.0022	1.0473	0.1159
7.0	−0.0013	0.00000	1.0013	0.0989
7.5	+0.0361	0.0013	0.9639	0.1036
8.0	0.0587	0.0034	0.9413	0.1219
8.5	0.0643	0.0041	0.9357	0.1457
9.0	0.0545	0.0030	0.9455	0.1663
9.5	0.0339	0.0011	0.9661	0.1782
10.0	+0.0087	0.00008	0.9913	0.1784
10.5	−0.0150	0.0002	1.0150	0.1668
11.0	−0.0321	0.0010	1.0321	0.1464
11.5	−0.0397	0.0016	1.0397	0.1216
12.0	−0.0372	0.0014	1.0372	0.0973
12.5	−0.0265	0.0007	1.0265	0.0779
13.0	−0.0108	0.0001	1.0108	0.0662
13.5	+0.0056	0.00003	0.9944	0.0631
14.0	0.0191	0.0004	0.9809	0.0676
14.5	0.0267	0.0007	0.9733	0.0770
15.0	0.0273	0.0007	0.9727	0.0880
15.5	0.0216	0.0005	0.9784	0.0973
16.0	0.0113	0.0001	0.9887	0.1021

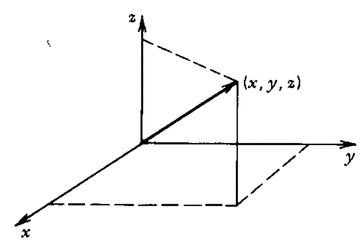
A7 VECTOR OPERATORS

In these relations the scalars f and g and the vectors $\vec{A}$ and $\vec{B}$ can be functions of time as well as space. The magnitude of $\vec{A}$ is A.

$$\nabla^2 f = \nabla \cdot (\nabla f)$$
$$\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A}$$
$$\nabla \times (\nabla f) = 0$$
$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

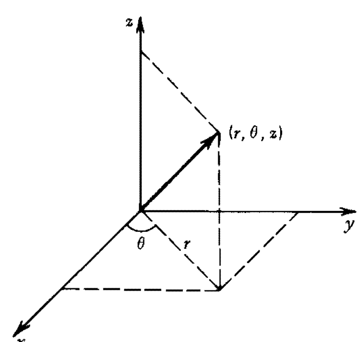
$$\nabla(fg) = f\nabla g + g\nabla f$$
$$\nabla \cdot (f\vec{A}) = f\nabla \cdot \vec{A} + \vec{A} \cdot \nabla f$$
$$\nabla \times (f\vec{A}) = (\nabla f) \times \vec{A} + f(\nabla \times \vec{A})$$
$$\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A})$$
$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$
$$(\vec{A} \cdot \nabla)\vec{A} = \tfrac{1}{2}\nabla(\vec{A} \cdot \vec{A}) - \vec{A} \times \nabla \times \vec{A}$$
$$\vec{A} \cdot \frac{d\vec{A}}{dt} = A\frac{dA}{dt}$$

(a) Cartesian Coordinates



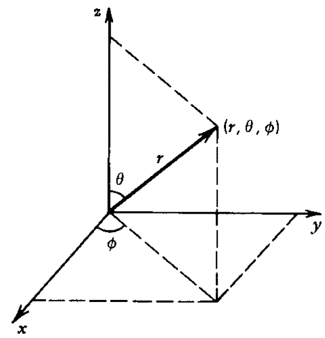
$$dV = dx\,dy\,dz$$
$$\nabla f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z}$$
$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
$$\nabla \times \vec{A} = \hat{x}\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right) + \hat{y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{z}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)$$

(b) Cylindrical Coordinates



$$dV = r\,dr\,d\theta\,dz$$
$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{z}\frac{\partial f}{\partial z}$$
$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}A_\theta + \frac{\partial}{\partial z}A_z$$
$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

(c) Spherical Coordinates



$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

A8 GAUSS'S THEOREM AND GREEN'S THEOREM

(a) Gauss's Theorem in Two- and Three-Dimensional Coordinate Systems

Gauss's theorem is a special case of the transport theorem. In three dimensions, it is

$$\int_V \nabla \cdot \vec{F} \, dV = \int_S \vec{F} \cdot \hat{n} \, dS$$

where $\hat{n}$ is the outward unit vector normal to the surface S of the volume V. In words, it states that the total outward flux of $\vec{F}$ through a surface S is equal to the totality of the divergence of $\vec{F}$ in the enclosed volume V. In electromagnetism it relates the integral over a closed surface of the normal component of the electrostatic field to the total enclosed charge.

A special case is the two-dimensional form,

$$\int_S \nabla \cdot \vec{F} \, dS = \int_C \vec{F} \cdot \hat{n} \, dl$$

where $\hat{n}$ is the unit vector normal to the perimeter C of the surface S in a two-dimensional coordinate system.

(b) Green's Theorem

Green's theorem

$$\int_V (U\nabla^2 V - V\nabla^2 U) \, dV = \int_S (U\nabla V - V\nabla U) \cdot \hat{n} \, dS$$

is a consequence of the vector identities

$$\nabla \cdot (U\nabla V) = \nabla U \cdot \nabla V + U\nabla^2 V$$
$$\nabla \cdot (V\nabla U) = \nabla V \cdot \nabla U + V\nabla^2 U$$

and Gauss's theorem. To prove this, take the difference of the above identities and integrate it over the volume V within the surface S,

$$\int_V (U\nabla^2 V - V\nabla^2 U) \, dV = \int_V \nabla \cdot (U\nabla V - V\nabla U) \, dV$$

and then apply Gauss's theorem to the right side with $\vec{F} = U\nabla V - V\nabla U$ to convert the volume integral of $\nabla \cdot \vec{F}$ into the surface integral over S of $\vec{F} \cdot \hat{n}$.

A9 A LITTLE THERMODYNAMICS AND THE PERFECT GAS

(a) Energy, Work, and the First Law

For notational simplicity, the absolute temperature T_K will be written without the subscript. The First Law of Thermodynamics states that heat, like work, is a form of energy, *thermal energy*. Consequently, the change in internal energy dE of a thermodynamic system can be expressed as the sum of the thermal energy δQ added to the system and the work $\delta W = -\mathcal{P}\, dV$ done on the system,

$$\boxed{dE = \delta Q + \delta W = \delta Q - \mathcal{P}\, dV} \tag{A9.1}$$

This is a statement of the conservation of energy, which was originally postulated from empirical observations. The minus sign appears because for negative dV the system has been compressed and must have gained energy from the work done on it. The work δW done on a system and the thermal energy δQ given to the system as it is taken from an initial to a final state both depend on the specifics of the process. They are *path dependent*. For example, if a system is taken from state 1 with pressure $\mathcal{P}_1$, volume V_1, and temperature T_1 to state 2 with $\mathcal{P}_2$, V_2, T_2, the amounts of work and thermal energy required will depend on whether the system is first compressed and then heated, or first heated and then compressed, even though the final internal energy must be the same in both cases. The internal energy E is a *state function*; its value depends only on the state ($\mathcal{P}$, V, T) of the system and not how it got there.

Let the thermodynamic system have a mass M, where M is the *molecular weight in grams*. This amount of material is defined as 1 *mole*. (If the mass is the molecular weight in kilograms, then the amount is 1 *kmol*.) If thermal energy ΔQ is added to a 1 mole system whose volume is held constant and the temperature is changed by ΔT, then the molar *heat capacity at constant volume* is defined as

$$C_V = \lim_{\Delta T \to 0} \left(\frac{\Delta Q}{\Delta T} \right)_V \tag{A9.2}$$

with units J/(mol · K). Since $\Delta V = 0$ and no work is done on the system, $dE = \delta Q$ over each step of the process and (A9.1) gives

$$\begin{aligned} \Delta E &= C_V\, \Delta T \qquad (\Delta V = 0) \\ C_V &= \lim_{\Delta T \to 0} \left(\frac{\Delta E}{\Delta T} \right)_V = \left(\frac{\partial E}{\partial T} \right)_V \end{aligned} \tag{A9.3}$$

Analogously, introduce ΔQ into a 1 mole system under the constraint that the pressure remain fixed. The associated temperature change ΔT is then related to ΔQ by the (molar) *heat capacity at constant pressure*,

$$C_{\mathcal{P}} = \lim_{\Delta T \to 0} \left(\frac{\Delta Q}{\Delta T} \right)_{\mathcal{P}} \tag{A9.4}$$

Use of (A9.1) now gives

$$\Delta E = C_{\mathcal{P}}\Delta T - \mathcal{P}\Delta V \qquad (\Delta\mathcal{P} = 0)$$
$$C_{\mathcal{P}} = \left(\frac{\partial E}{\partial T}\right)_{\mathcal{P}} + \mathcal{P}\left(\frac{\partial V}{\partial T}\right)_{\mathcal{P}} \tag{A9.5}$$

For a system containing 1 mole of a single substance, the *equation of state* relates the pressure, volume, and temperature so that the internal energy is a function of only two of the variables. Thus, the energy can be considered as a function of T and V so that

$$\Delta E = \left(\frac{\partial E}{\partial T}\right)_V \Delta T + \left(\frac{\partial E}{\partial V}\right)_T \Delta V \tag{A9.6}$$

If some process now changes the temperature of a system, but holds the pressure constant, we can write

$$\left(\frac{\partial E}{\partial T}\right)_{\mathcal{P}} = \left(\frac{\partial E}{\partial T}\right)_V + \left(\frac{\partial E}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_{\mathcal{P}} \tag{A9.7}$$

We can relate C_V and $C_{\mathcal{P}}$ by combining (A9.3), (A9.5), and (A9.7),

$$C_{\mathcal{P}} - C_V = \mathcal{P}\left(\frac{\partial V}{\partial T}\right)_{\mathcal{P}} + \left(\frac{\partial E}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_{\mathcal{P}} \tag{A9.8}$$

This will yield a simple result when applied to a perfect gas.

(b) Enthalpy, Entropy, and the Second Law

Two other thermodynamic quantities important in acoustics and fluid flow are the *enthalpy* H and the *entropy* S. (There are two more, the *Gibbs function* $G = H - TS$ and the *Helmholtz function* $A = E - TS$, that need not concern us here.) The enthalpy is defined as

$$H = E + \mathcal{P}V \tag{A9.9}$$

Since E is a state function and the product $\mathcal{P}V$ is a function only of the state of the system, the enthalpy is also a state function. Taking the differential of H and using the First Law gives $dH = \delta Q + Vd\mathcal{P}$. For an *isobaric* process, $d\mathcal{P} = 0$ at each step of the process so that $dH = \delta Q$ at each step of the process and $\Delta H = \Delta Q$. Then (A9.4) gives

$$C_{\mathcal{P}} = \lim_{\Delta T \to 0}\left(\frac{\Delta H}{\Delta T}\right)_{\mathcal{P}} = \left(\frac{\partial H}{\partial T}\right)_{\mathcal{P}} \tag{A9.10}$$

In making a transition between an initial and a final state, a system can proceed in a way that cannot be undone, an *irreversible* process. Examples are the free expansion of a gas, a nuclear detonation, and the diffusion of one gas into another. If, however, a system in equilibrium is acted upon so slowly that it remains essentially in equilibrium as it transits from initial to final states, the process is *reversible*. The slow compression of a gas in a perfectly insulated container is an example. Reversible processes can be described in terms of the entropy S. If δQ_{rev} is the thermal energy added to a system during an infinitesimal reversible process at temperature T, then the change in the entropy is defined by

$$dS = \frac{\delta Q_{rev}}{T} \tag{A9.11}$$

The Second Law of Thermodynamics, also postulated from empirical observations, states that any heat engine must operate between two thermal reservoirs of different temperatures. It is equivalent to asserting that the entropy of a system is a state function. Thus, for example, if a system starts in state 1 and by whatever process ends up in state 2, the entropy change can be calculated by ignoring the actual process and instead finding a reversible way of accomplishing the same change in state (e.g., first moving reversibly at constant T_1 to V_2 and then reversibly to T_2 at constant V_2). The entropy and internal energy are related by

$$dE = T\,dS - \mathcal{P}\,dV \tag{A9.12}$$

(c) The Perfect Gas

A perfect gas can be considered as a collection of infinitesimally small, rigid particles that exert forces on each other only when they collide (e.g., a collection of perfectly elastic, rapidly moving billiard balls). The absence of interparticle forces means that there can be no potential energy. The internal energy of the system is then just the sum of the kinetic energies of all the particles, and application of the *kinetic theory of gases* reveals two important results.

1. The energy of the gas is a function *only* of temperature, which has the immediate consequence

$$\left(\frac{\partial E}{\partial V}\right)_T = 0 \tag{A9.13}$$

so that (A9.8) becomes

$$C_{\mathcal{P}} - C_V = \mathcal{P}\left(\frac{\partial V}{\partial T}\right)_{\mathcal{P}} \tag{A9.14}$$

2. The pressure, volume, and temperature of a mole of a perfect gas are related by the equation of state,

$$\boxed{\mathcal{P}V = \mathcal{R}T} \tag{A9.15}$$

where $\mathcal{R}$ is the *universal gas constant*

$$\mathcal{R} = 8.3145\ \text{J}/(\text{mol}\cdot\text{K}) \tag{A9.16}$$

If there are n moles, then $\mathcal{P}V = n\mathcal{R}T$. In terms of the density ρ, we have $\rho V = M$ so that

$$\mathcal{P} = \rho r T \qquad r = \mathcal{R}/M \tag{A9.17}$$

where r is the gas constant for the particular gas in question.

Two important conclusions about the thermodynamic behavior of perfect gases can now be drawn.

1. Since E is a function only of T, the heat capacities are related by (A9.14) and use of the equation of state (A9.15) reveals

$$\left(\frac{\partial V}{\partial T}\right)_{\mathcal{P}} = \left(\frac{\partial}{\partial T}\frac{\mathcal{R}T}{\mathcal{P}}\right)_{\mathcal{P}} = \frac{\mathcal{R}}{\mathcal{P}} \tag{A9.18}$$

so that

$$\boxed{C_{\mathcal{P}} - C_V = \mathcal{R}} \tag{A9.19}$$

2. For an *adiabatic process* there is no gain or loss of thermal energy so that $\Delta Q = 0$. We then have

$$\Delta E = -\mathcal{P}\Delta V \tag{A9.20}$$

Now, (A9.6), (A9.13), and (A9.3) yield $\Delta E = (\partial E/\partial T)_V \Delta T = C_V\ \Delta T$ so that $-\mathcal{P}\Delta V = C_V\Delta T$ for a perfect gas. Use of (A9.15) gives

$$-\mathcal{R}\Delta V/V = C_V\Delta T/T \tag{A9.21}$$

Integration of both sides gives $-\mathcal{R}\ln(V/V_0) = C_V \ln(T/T_0)$ or

$$(V_0/V)^{\mathcal{R}} = (T/T_0)^{C_V} \tag{A9.22}$$

Use of (A9.15) yields the *adiabat*

$$\boxed{\begin{gathered}\mathcal{P}/\mathcal{P}_0 = (\rho/\rho_0)^{\gamma} \\ \gamma = C_{\mathcal{P}}/C_V\end{gathered}} \tag{A9.23}$$

where γ is defined as the *ratio of heat capacities*.

For a perfect gas with constant heat capacities between states 1 and 2, use of $\Delta E = C_V\Delta T$ and (A9.1) gives $\delta Q_{rev} = C_{\mathcal{P}}\, dT - V\, d\mathcal{P}$. Division by temperature to obtain the entropy, use of (A9.15) to eliminate V, and then direct integration of temperature and pressure between the two states gives

$$S_2 - S_1 = C_V\left[\ln\left(\frac{\mathcal{P}_2}{\mathcal{P}_1}\right) + \gamma \ln\left(\frac{V_2}{V_1}\right)\right] \tag{A9.24}$$

(a) Solids

Solid	*Density* (kg/m³) ρ_0	*Young's Modulus* (Pa) Y	*Shear Modulus* (Pa) $\mathcal{G}$	*Adiabatic Bulk Modulus* (Pa) $\mathcal{B}$	*Poisson's Ratio* σ	*Speed* (m/s) c		*Characteristic Impedance* (Pa·s/m) $\rho_0 c$	
						Bar	Bulk	Bar	Bulk
		$\times 10^{10}$	$\times 10^{10}$	$\times 10^{10}$				$\times 10^{6}$	$\times 10^{6}$
Aluminum	2700	7.1	2.4	7.5	0.33	5150	6300	13.9	17.0
Brass	8500	10.4	3.8	13.6	0.37	3500	4700	29.8	40.0
Copper	8900	12.2	4.4	16.0	0.35	3700	5000	33.0	44.5
Iron (cast)	7700	10.5	4.4	8.6	0.28	3700	4350	28.5	33.5
Lead	11300	1.65	0.55	4.2	0.44	1200	2050	13.6	23.2
Nickel	8800	21.0	8.0	19.0	0.31	4900	5850	43.0	51.5
Silver	10500	7.8	2.8	10.5	0.37	2700	3700	28.4	39.0
Steel	7700	19.5	8.3	17.0	0.28	5050	6100	39.0	47.0
Glass (Pyrex)	2300	6.2	2.5	3.9	0.24	5200	5600	12.0	12.9
Quartz (X-cut)	2650	7.9	3.9	3.3	0.33	5450	5750	14.5	15.3
Lucite	1200	0.4	0.14	0.65	0.4	1800	2650	2.15	3.2
Concrete	2600	—	—	—	—	—	3100	—	8.0
Ice	920	—	—	—	—	—	3200	—	2.95
Cork	240	—	—	—	—	—	500	—	0.12
Oak	720	—	—	—	—	—	4000	—	2.9
Pine	450	—	—	—	—	—	3500	—	1.57
Rubber (hard)	1100	0.23	0.1	0.5	0.4	1450	2400	1.6	2.64
Rubber (soft)	950	0.0005	—	0.1	0.5	70	1050	0.065	1.0
Rubber (rho-c)	1000	—	—	0.24	—	—	1550	—	1.55

(b) Liquids

Liquid	*Temperature* (°C) T	*Density* (kg/m³) ρ_0	*Isothermal Bulk Modulus* (Pa) $\mathscr{B}_T$	*Ratio of Specific Heats* γ	*Speed* (m/s) c	*Characteristic Impedance* (Pa·s/m) $\rho_0 c$	*Coefficient of Shear Viscosity* (Pa·s) η	*Specific Heat* [J/(kg·K)] $c_{\mathscr{P}}$	*Thermal Conductivity* [W/(m·K)] κ	*Prandtl Number* Pr
			$\times 10^9$			$\times 10^6$	$\times 10^{-3}$	$\times 10^3$		
Water (fresh)	20	998	2.18	1.004	1481	1.48	1.00	4.19	0.603	6.95
Water (sea)	13	1026	2.28	1.01	1500	1.54	1.07			
Alcohol (ethyl)	20	790	—	—	1150	0.91	1.20			
Castor (oil)	20	950	—	—	1540	1.45	960			
Mercury	20	13600	25.3	1.13	1450	19.7	1.56	0.14	8.21	0.0266
Turpentine	20	870	1.07	1.27	1250	1.11	1.50			
Glycerin	20	1260	—	—	1980	2.5	1490			
Fluid-like sea bottoms										
Red clay		1340	—	—	1460	1.96				
Calcareous ooze		1570	—	—	1470	2.31				
Coarse silt		1790	—	—	1540	2.76				
Quartz sand		2070	—	—	1730	3.58				

(c) Gases

Gas (*at 1 atm*)	*Temperature* (°C) T	*Density* (kg/m^3) ρ_0	*Ratio of Specific Heats* γ	*Speed* (m/s) c	*Characteristic Impedance* (Pa·s/m) $\rho_0 c$	*Coefficient of Shear Viscosity* (Pa·s) η	*Specific Heat* [J/(kg·K)] $c_{\mathcal{P}}$	*Thermal Conductivity* [W/(m·K)] κ	*Prandtl Number* Pr
						$\times 10^{-5}$	$\times 10^{3}$		
Air	0	1.293	1.402	331.5	429	1.72			
Air	20	1.21	1.402	343	415	1.85	1.01	0.0263	0.710
O_2	0	1.43	1.40	317.2	453	2.00	0.912	0.0245	0.744
CO_2 ($f \ll f_M$)	0	1.98	1.304	258	512	1.45	0.836	0.0145	0.836
CO_2 ($f \gg f_M$)	0	1.98	1.40	268.6	532				
H_2	0	0.090	1.41	1269.5	114	0.88	14.18	0.168	0.743
Steam	100	0.6	1.324	404.8	242	1.3			

A11 ELASTICITY AND VISCOSITY

(a) Solids

Application of an external force to a body distorts the body until internal forces arise to counterbalance the applied force and the body assumes a new shape. The applied force per unit area is the *stress* and the fractional changes in dimensions are the *strains*. For small strains of an isotropic solid, we can make two pivotal assumptions: (1) the stress and strains are linearly related (*Hooke's law*), and (2) individual stresses cause individual strains and the results combine linearly. The various stress–strain relationships can all be expressed in terms of *Young's modulus* Y and *Poisson's ratio* σ. [For a complete development, see Feynman, Leighton, and Sands, *The Feynman Lectures on Physics*, Vol. 2, Chap. 38, Addison-Wesley (1965).]

(1) LONGITUDINAL COMPRESSION OF A THIN ROD The stress–strain relationship for a thin rod under longitudinal compression or extension is developed in Section 3.3. When a compressive force f is applied to the ends of a bar of cross-sectional area S and length l, the bar will shorten by a small amount Δl. The strain $\Delta l/l$ is proportional to the applied stress f/S,

$$f/S = -Y\,\Delta l/l \tag{A11.1}$$

where the constant of proportionality Y is *Young's modulus*. (The minus sign is consistent with a positive pressure leading to a reduction in volume.)

(2) POISSON'S RATIO The change in length of the thin bar is accompanied by a change in the transverse dimensions. If all transverse dimensions are labeled r, the reaction to the change in length Δl is a proportionate change in the transverse dimensions

$$\Delta r/r = -\sigma\,\Delta l/l \tag{A11.2}$$

where the proportionality constant σ is *Poisson's ratio*.

(3) UNIFORM VOLUME COMPRESSION Uniform compression is developed in Chapter 5 and gives the stress–strain relationship (5.2.6)

$$p = -\mathcal{B}\,\Delta V/V \tag{A11.3}$$

where $s = \Delta\rho/\rho = -\Delta V/V$ and $\mathcal{B}$ is the *bulk modulus*. We can easily relate $\mathcal{B}$, Y, and σ. Impose a uniform compression sequentially on a block of material of length l, width w, and thickness t. If $p = f/S$ is applied to the sides delimiting l, the fractional change in length will be given by (A11.1). If the same compressive stress is applied to the sides delimiting w, then l will be pushed back out a little and the magnitude of Δl will be diminished by a factor $(1 - \sigma)$. When the final stress is applied to the sides delimiting t, the length will again be pushed back a little more and Δl will be diminished by a total factor of $(1 - 2\sigma)$, so that the actual change Δl is

$$\Delta l/l = -(1 - 2\sigma)f/YS \tag{A11.4}$$

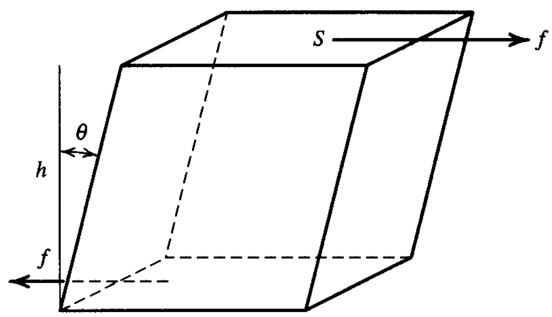


Figure A11.1

Thus, the direct compression is reduced linearly by each of the compressions in the two transverse directions. By symmetry, exactly the same must occur for Δw and Δt. The total change in fractional volume is just the sum of the fractional changes in the dimensions,

$$\Delta V/V = \Delta l/l + \Delta w/w + \Delta t/t = -3(1 - 2\sigma)p/Y \qquad \text{(A11.5)}$$

where p has replaced f/S. Direct comparison of (A11.3) and (A11.5) shows that

$$\mathcal{B} = Y/3(1 - 2\sigma) \qquad \text{(A11.6)}$$

(4) Shear If an antiparallel couple of forces f are applied to the top and bottom surfaces, each of area S, of a cube of height h, the cube will deform as shown in Fig. A11.1. Dividing the displacement of the upper surface with respect to the lower by the height gives the strain. For small strains this is well approximated by the angle of deformation θ. The stress is f/S. The applied stress is proportional to the strain,

$$f/S = \mathcal{G}\theta \qquad \text{(A11.7)}$$

where $\mathcal{G}$ is the *modulus of rigidity* or *shear modulus*. This is identical with (3.13.2), where $\theta = r(d\phi/dx)$ and $S = dw\,dr$. The element does not rotate, so the net torque is zero. This requires induced vertical forces of the same magnitude f. Vector combination of these direct and induced four forces shows that they are equivalent to a pair of compressional forces acting along one diagonal L and a pair of extension forces acting along the other. Geometry shows that these compression and extension stresses have the same magnitude as the shearing stresses. Just as before, the change in length of each diagonal is the sum of the strains from the direct compression along the diagonal and the lateral extension perpendicular to it. The effects enhance each other, so

$$\Delta L/L = -(1 + \sigma)f/YS \qquad \text{(A11.8)}$$

for the compressed diagonal. A little geometry reveals that $\Delta L/L = \theta/2$ and thus

$$\theta = -2(1 + \sigma)f/YS \qquad \text{(A11.9)}$$

Comparison of (A11.7) and (A11.9) gives us

$$\mathcal{G} = Y/2(1 + \sigma) \qquad \text{(A11.10)}$$

(5) LONGITUDINAL BULK COMPRESSION If a compressive stress is applied to the ends of a bar that is restricted so that the transverse dimensions cannot change, then the bar cannot "relax" against the stress by partially compensating with an increase in cross-sectional area. Under this constraint, a greater stress will be required to achieve the same longitudinal strain. The result is a larger constant of proportionality between stress and strain. We can generalize the previous case of bulk compression to consider *different* values of compressive pairs of forces in the three directions. The discussion is straightforward if we assume that the material element is a cube of dimension h. Let us compress the cube in the x direction with a pair of forces f_x and require no expansion or contraction in the y and z directions. By symmetry, the transverse forces in the y and z directions necessary to prevent any transverse expansion or contraction must equal each other, and we label them f_T. In the x direction we must have

$$\Delta h/h = -(f_x - 2\sigma f_T)/YS \tag{A11.11}$$

and in each of the transverse dimensions, for no change of length,

$$0 = -[f_T - \sigma(f_T + f_x)]/YS \tag{A11.12}$$

Solving this equation for f_T/S and substituting into (A11.11) gives us the relationship between the applied stress and the resultant strain for longitudinal bulk compression,

$$\Delta h/h = -[(1+\sigma)(1-2\sigma)/(1-\sigma)]f_x/YS \tag{A11.13}$$

This can be expressed in simpler form with the help of the bulk modulus $\mathcal{B}$ and modulus of rigidity $\mathcal{G}$. Using (A11.6) and (A11.10) results in

$$f_x/S = \left(\mathcal{B} + \tfrac{4}{3}\mathcal{G}\right)\Delta h/h \tag{A11.14}$$

(b) Fluids

The molecules of a fluid have sufficient kinetic energies to migrate from current nearest neighboring molecules to others. This mobility manifests itself in the appearance of additional forces that generate macroscopic effects.

(1) SHEAR VISCOSITY If a fluid is in a state of nonuniform macroscopic motion (acoustic propagation, laminar or turbulent flow, etc.) there can be a diffusion of momentum among neighboring fluid elements caused by the migration of molecules from one element to another. This diffusion of momentum gives rise to internal forces that reduce the relative motions of adjacent elements and bring the fluid back to a state of uniform motion or to rest. The diffusion of momentum occurs whether the fluid is in shear or in longitudinal relative motion, or in both. Shear viscosity is an important mechanism of energy transformation from collective (acoustic) to random (thermal). If a fluid is subjected to a *shear stress* τ, it will respond by developing an attendant shearing motion. The diffusion of momentum between neighboring lamina of the fluid will lead to a local steady-state velocity, or a *rate of deformation* when the frictional forces counterbalance the shearing forces. In the case of *simple shear*, such as the flow between two infinite parallel plates, the velocity of the fluid is parallel to the stress. If the parallel plates extend in the y and z directions and one is moving uniformly in the y direction,

then both the shear stress τ and the fluid velocity u are in the y direction and the magnitude of the stress is proportional to $\partial u/\partial x$, the spatial change in u with respect to the (transverse) x direction,

$$\tau = \eta \frac{\partial u}{\partial x} \tag{A11.15}$$

where the proportionality constant η is the *coefficient of shear viscosity* (Pa · s).

For more complicated motion, obtaining the relationship between the force arising from shear viscosity and the motion is nontrivial. Developing the stress and rate of strain tensors within a body possessing shear viscosity and then calculating the internal body force per unit mass results in

$$\vec{F}_S(\vec{r}, t) = \tfrac{4}{3}\eta\nabla(\nabla \cdot \vec{u}) - \eta\nabla \times \nabla \times \vec{u} \tag{A11.16}$$

as the contribution to the inhomogeneous wave equation of Section 5.14. The interested reader is referred to Temkin, *Elements of Acoustics*, Wiley (1981).

(2) Volume Viscosity For a fluid at rest, the bulk modulus $\mathcal{B}$ is defined exactly the same as for a solid. This modulus expresses the purely compressive (spring-like) properties of the fluid. The frictional effects arising from the transport of momentum between adjacent fluid elements are accounted for by the shear viscosity. In addition to this, however, there are other mechanisms separate from the diffusion of momentum that can lead to losses in some fluids. (1) Under new thermodynamic conditions (higher pressure and temperature for a compression), some groups of molecules may accommodate a different nearest neighbor configuration. This takes time, so that the equilibrium volume (for a constant pressure change) lags behind the instantaneous volume. This, although arising from a totally separate mechanism, introduces the same kind of an adjustment toward equilibrium, requiring a finite duration of time as does shear viscosity. (2) The changing conditions may lead to changes in the equilibrium between ionized and un-ionized concentrations of a compound (e.g., magnesium sulfate in sea water). This can result in a relaxation effect similar to the above, particularly because of the association and dissociation of ionized and un-ionized compounds with the adjacent ionized and un-ionized water complexes. These provide examples of processes that depend on the instantaneous thermodynamic state of the fluid element, and that require some time to adjust to new conditions. In effect, all act like *structural* changes that require a finite time during which the fluid attempts to find a new equilibrium volume in response to the external stimulus. These adjustments, like the momentum diffusion, exhibit themselves as internal friction-like forces but depend only on the externally generated temporal changes in the local density. Consequently, by the linearized equation of continuity, they are functions only of $\nabla \cdot \vec{u}$, the rate of strain of the fluid element, but not on the flow. Since the forces on fluid elements arise from the gradient of the pressure, and thus the gradient of the density, it is reasonable to introduce them into Euler's equation as a body force per unit volume (see Section 5.14),

$$\vec{F}_B(\vec{r}, t) = \eta_B\nabla(\nabla \cdot \vec{u}) \tag{A11.17}$$

where η_B is the *coefficient of bulk viscosity* (also known as the coefficient of volume viscosity, expansion coefficient of viscosity, and so forth).

A12 THE GREEK ALPHABET

Α	**Α**	α	$\boldsymbol{\alpha}$	alpha	Ν	**Ν**	ν	$\boldsymbol{\nu}$	nu
Β	**Β**	β	$\boldsymbol{\beta}$	beta	Ξ	**Ξ**	ξ	$\boldsymbol{\xi}$	xi
Γ	**Γ**	γ	$\boldsymbol{\gamma}$	gamma	Ο	**Ο**	ο	**ο**	omicron
Δ	**Δ**	δ	$\boldsymbol{\delta}$	delta	Π	**Π**	π	$\boldsymbol{\pi}$	pi
Ε	**Ε**	ε	$\boldsymbol{\varepsilon}$	epsilon	Ρ	**Ρ**	ρ	$\boldsymbol{\rho}$	rho
Ζ	**Ζ**	ζ	$\boldsymbol{\zeta}$	zeta	Σ	**Σ**	σ	$\boldsymbol{\sigma}$	sigma
Η	**Η**	η	$\boldsymbol{\eta}$	eta	Τ	**Τ**	τ	$\boldsymbol{\tau}$	tau
Θ	**Θ**	θ	$\boldsymbol{\theta}$	theta	Υ	**Υ**	υ	$\boldsymbol{\upsilon}$	upsilon
Ι	**Ι**	ι	$\boldsymbol{\iota}$	iota	Φ	**Φ**	ϕ	$\boldsymbol{\phi}$	phi
Κ	**Κ**	κ	$\boldsymbol{\kappa}$	kappa	Χ	**Χ**	χ	$\boldsymbol{\chi}$	chi
Λ	**Λ**	λ	$\boldsymbol{\lambda}$	lambda	Ψ	**Ψ**	ψ	$\boldsymbol{\psi}$	psi
Μ	**Μ**	μ	$\boldsymbol{\mu}$	mu	Ω	**Ω**	ω	$\boldsymbol{\omega}$	omega

INDEX

GLOSSARY OF SYMBOLS

(continued from front endpapers)

r	specific gas constant; characteristic acoustic impedance; specific acoustic resistance
r_t	transition range
r_s	skip distance
R	resistance (acoustic, electrical, mechanical); reflection coefficient; radius of curvature
Re	Reynolds number
$\dot{R}$	range rate
R_m	mechanical resistance
R_r	radiation resistance
R_I	intensity reflection coefficient
R_Π	power reflection coefficient
RL	reverberation level
ROC	receiver operating characteristic
$\mathcal{R}$	universal gas constant
s	spring constant; condensation
sL	apparent source level
S	cross-sectional area; surface area; salinity
S_A, S_B	scattering strengths per unit area
$SENEL$	single event noise exposure level L_{ex} (dBA)
SIL	speech interference level
SL	source level
SPL	sound pressure level
SS	sea state
SSL	source spectrum level
STC	sound transmission class
S_V	scattering strength per unit volume
SWR	standing wave ratio
$\mathcal{S}$	transmitter sensitivity
$\mathcal{SL}$	transmitter sensitivity level
$\mathcal{S}_{ref}$	reference transmitter sensitivity
T	period of motion; temperature; tension; transmission coefficient; reverberation time
T_I	intensity transmission coefficient
T_K	temperature in kelvin
T_{em}, T_{me}	transduction coefficients
T_Π	power transmission coefficient
TL	transmission loss
TNI	traffic noise index (dBA)
TS	target strength
TS_R	target strength for reverberation
TTS	temporary threshold shift
$\mathcal{T}$	membrane tension per unit length
$\vec{u}$	particle velocity
u	particle speed
U	peak particle velocity amplitude; volume velocity
U_e	effective particle velocity amplitude

$\vec{v}$ scaled particle velocity ($\vec{u}/c$)

V volume; voltage; effective voltage amplitude; volume displacement

VL voltage level; voice level (dBA)

V_{ref} reference effective voltage amplitude

w angular frequency; bandwidth

W explosive yield

x specific acoustic reactance

X electrical reactance

X_m mechanical reactance

X_r radiation reactance

y transverse displacement

Y admittance; Young's modulus

z specific acoustic impedance

Z impedance (acoustic, electrical, mechanical)

Z_m mechanical impedance

Z_r radiation impedance

α spatial absorption coefficient

β temporal absorption coefficient; one-half the flair constant; a nonlinearity coefficient

γ ratio of heat capacities; attenuation coefficient; a nonlinearity coefficient

δ boundary layer thickness; skin depth

η coefficient of shear viscosity; efficiency

η_B coefficient of bulk viscosity

η_e effective coefficient of viscosity

θ angle of incidence; phase angle; grazing angle; horizontal beam width; angle of elevation or depression

Θ phase angle of impedance

κ thermal conductivity; radius of gyration; transverse component of the propagation vector

λ wavelength

Γ c_0 times the eikonal; Goldberg number

ξ longitudinal particle displacement

Ξ amplitude of ξ

Π time-averaged power

Π_i instantaneous power

ρ instantaneous density (kg/m^3); probability density function

ρ_0 equilibrium density

ρ_L linear density (kg/m)

ρ_S surface density (kg/m^2)

σ Poisson's ratio; standard deviation; extinction cross section

σ_S scattering cross section

τ relaxation time; pulse duration; processing time

ϕ transformation factor; phase angle; torsion angle; turns ratio; inverse turns ratio

Φ velocity potential

ω angular frequency (rad/s)

ω_0 natural angular frequency

ω_d damped angular frequency

ω_u, ω_l upper, lower half-power angular frequencies

Ω solid angle

Ω_{eff} effective solid angle

$\delta(v)$ Dirac delta function of argument v

δ_{nm} Kronecker delta

$1(v)$ Heaviside unit function of argument v (unit step function)