INTRODUCTORY CIRCUIT THEORY

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Of particular interest is the result 163 if the network is excited by a single source. Letting this one be E_1, we have in this special case

$$P = P_{av} + \mathrm{Re}\left[\frac{E_1 I_1}{2} e^{j2\omega t}\right] \tag{164}$$

Taking E_1 as phase reference and denoting the input admittance angle by φ, we have

$$P = P_{av} + \frac{|E_1 I_1|}{2} \cos(2\omega t + \varphi) \tag{165}$$

However, noting Eq. 157,

$$\frac{|E_1 I_1|}{2} = \frac{|\bar{E}_1 I_1}{2} = \sqrt{P_{av}^{\ 2} + Q_{av}^{\ 2}} \tag{166}$$

so that Eq. 165 can be written

$$P = P_{av} + \sqrt{P_{av}^{\ 2} + Q_{av}^{\ 2}} \cos(2\omega t + \varphi) \tag{167}$$

a result which shows that the amplitude of the double-frequency sinusoid equals the magnitude of the vector power.

7 Equivalence of Kirchhoff and Lagrange Equations

In this article we wish to show that Lagrange's equations, which express the equilibrium of a system in terms of its associated energy functions, are identical with the Kirchhoff-law equations so far as the end results are concerned. We need first some preliminary relations which can readily be seen from Eqs. 111, 112, 113 for the functions F, T, V in terms of the loop currents. If we differentiate partially with respect to a particular loop current, we find

$$\frac{\partial F}{\partial i_i} = \sum_{k=1}^{l} R_{ik} i_k \tag{168}$$

$$\frac{\partial T}{\partial i_i} = \sum_{k=1}^{l} L_{ik} i_k \tag{169}$$

$$\frac{\partial V}{\partial q_i} = \sum_{k=1}^{l} S_{ik} q_k \tag{170}$$

These results may most easily be obtained if one considers the pertinent function written out completely as T is in Eq. 121. It is then obvious that a particular loop current, say i_2, is contained in all terms of the second row and second column, and only in these terms. Hence, if we differentiate partially with respect to i_2, no other terms are involved, and we find

$$\frac{\partial}{\partial i_2}(2T) = L_{21}i_1 + 2L_{22}i_2 + L_{23}i_3 + \cdots + L_{2l}i_l$$
$$+ L_{12}i_1 + \qquad + L_{32}i_3 + \cdots + L_{l2}i_l \qquad (171)$$

where we note that the term with L_{22} yields a factor 2 because the derivative of i_2^2 is involved. However, since $L_{ik} = L_{ki}$, we can rewrite this result as

$$\frac{\partial}{\partial i_2}(2T) = 2(L_{21}i_1 + L_{22}i_2 + \cdots + L_{2l}i_l) \qquad (172)$$

from which Eq. 169 follows. Equations 168 and 170 are obtained in the same manner. In all three, the summation involved is a simple summation on the index k.

If we differentiate Eq. 169 totally with respect to time, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial i_i}\right) = \sum_{k=1}^{l} L_{ik}\frac{di_k}{dt} \qquad (173)$$

and Eq. 170 can be rewritten as

$$\frac{\partial V}{\partial q_i} = \sum_{k=1}^{l} S_{ik}\int i_k\,dt \qquad (174)$$

so that with Eq. 168 we obtain

$$\frac{d}{dt}\left(\frac{\partial T}{\partial i_i}\right) + \frac{\partial F}{\partial i_i} + \frac{\partial V}{\partial q_i} = \sum_{k=1}^{l}\left(L_{ik}\frac{d}{dt} + R_{ik} + S_{ik}\int dt\right) i_k \qquad (175)$$

Reference to the Kirchhoff voltage-law Eqs. 108 now shows that these may alternatively be written

$$\frac{d}{dt}\left(\frac{\partial T}{\partial i_i}\right) + \frac{\partial F}{\partial i_i} + \frac{\partial V}{\partial q_i} = e_{li}, \qquad i = 1, 2, \cdots, l \qquad (176)$$

This form, in which the voltage equilibrium equations are expressed in terms of the energy functions, is known as the *Lagrangian equations.* From the way in which they are here obtained, it is clear that they are equivalent to the Kirchhoff-law equations although their outward appearance does not place this fact in evidence.